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The origin of discrete symmetries in F-theory models

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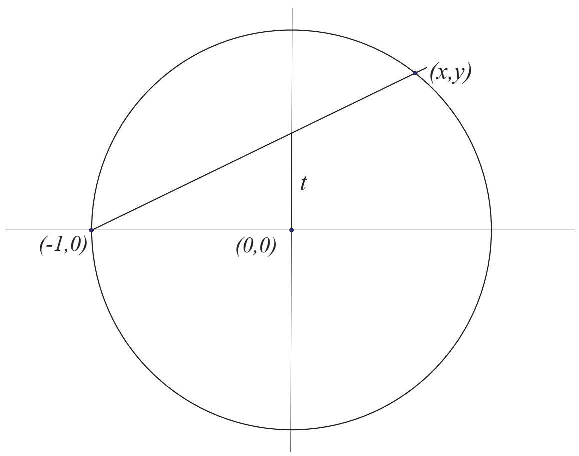
Outline of the Talk

- ▲ $\mathcal{A}$ Rational Points on Elliptic Curves
- ▲ $\mathcal{B}$ F-theory and Elliptic Fibration
- ▲ $\mathcal{C}$ F-GUTs with discrete symmetries
- ▲ $\mathcal{D}$ Mordell-Weil $U(1)$ and GUTs
- ▲ $\mathcal{E}$ Remarks

$\mathcal{A}$

Rational Points on Elliptic Curves

Rational Points (R.P.) on Conics



- Choose one R.P. on conic - taken here to be $(-1, 0)$.
- Project all others on a line (here axis y):

$$x = \frac{1 - t^2}{1 + t^2} \qquad y = \frac{2t}{1 + t^2}$$

$$\Downarrow$$

R.P. on line 1-1 with R.P. on circle

★ Real Rational Elliptic Curves

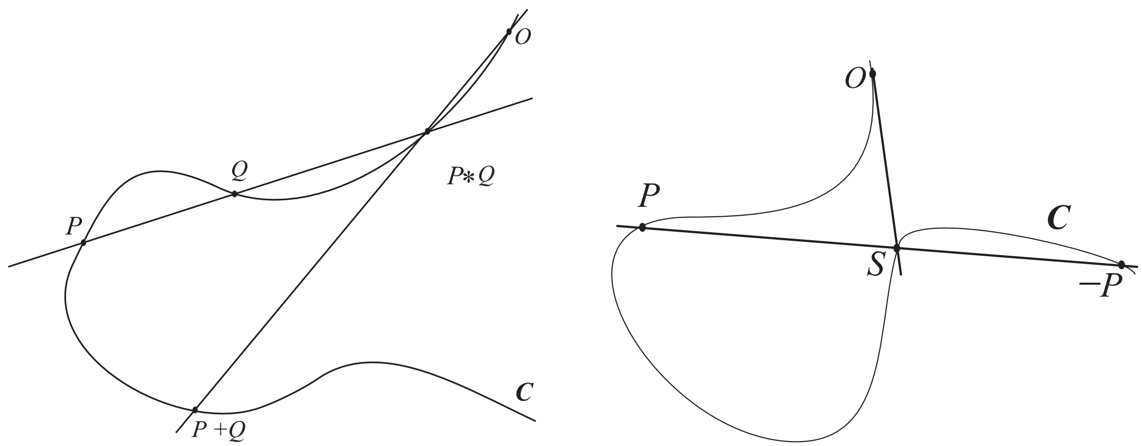
▲ General cubic equation with rational coefficients $f(x, y) = 0$:

$$f = a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3 + a_5 x^2 + a_6 x y + a_7 y^2 + a_8 x + a_9 y + a_{10}$$

▲ *rational points on elliptic curve?* Non-trivial to find **but:**

They obey a group law!

The Group Law on Elliptic Curves



The addition law: $P + Q$ (left).

$(P, Q = rational \to P + Q\ rational.)$

The opposite element $P + (-P) = \mathcal{O}$ (right)

Mordell Theorem

$\Downarrow$

The Rational Points on Elliptic Curve constitute a finitely generated Abelian Group

$\Downarrow$

Mordell - Weil Group

Any cubic equation with a rational point can be written in:

★ Weierstrass form:

$$y^2 = x^3 + fx + g$$

▲ Two important quantities characterising elliptic curves:

1. The Discriminant:

$$\Delta = 4f^3 + 27g^2$$

   ... classifies the curves with respect to its singularities

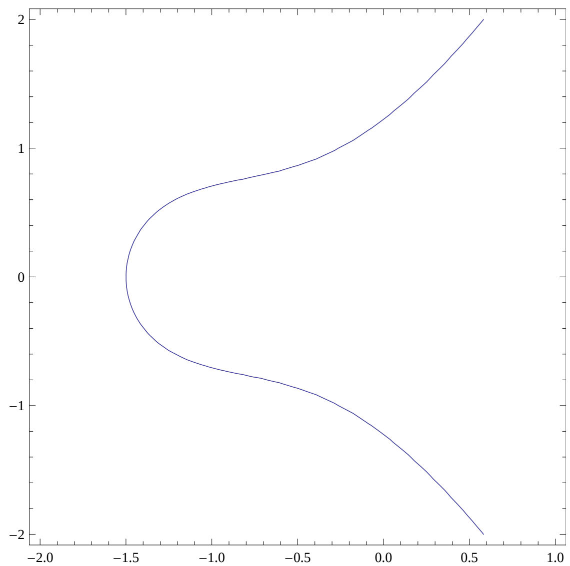
2. The j-invariant function:

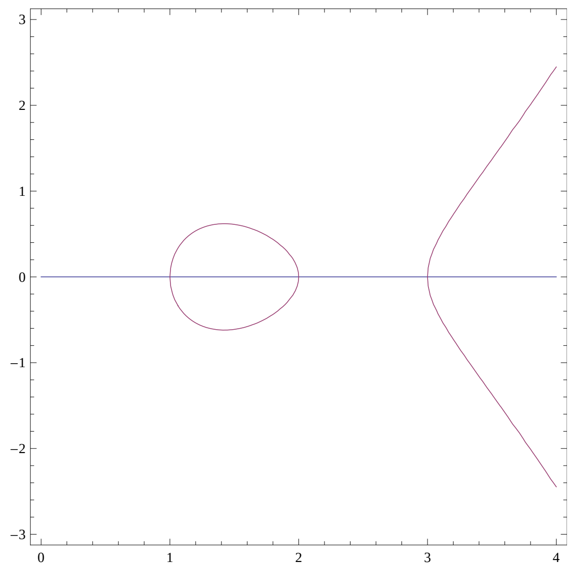
$$j = 4\,\frac{(24f)^3}{4f^3 + 27g^2}$$

   ... takes the same value for equivalent elliptic curves

The role of the Discriminant

▲ $\mathcal{A}$: Non-singular curves: $\Delta \neq 0$.





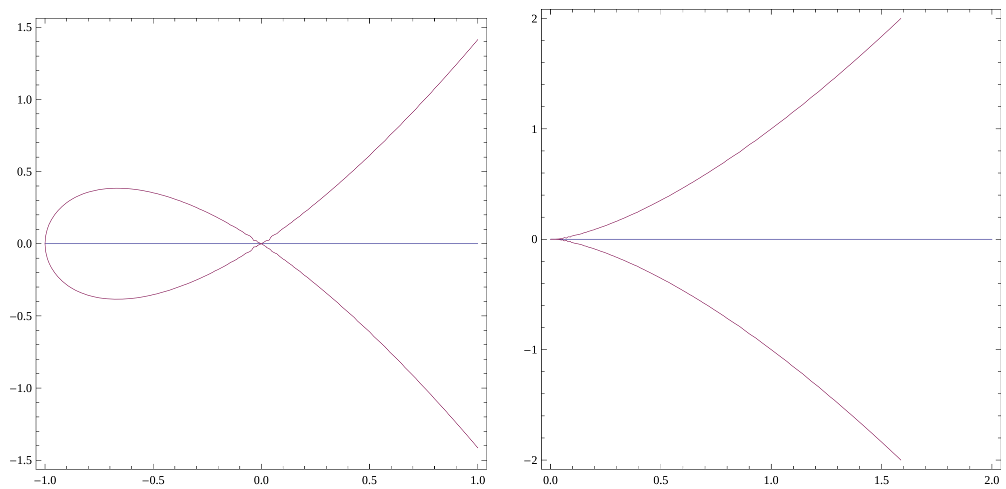
examples of non-singular curves ($\Delta \neq 0$) :

1 real root (left), 3 real roots (right).

▲ $\mathcal{B}$: Singular cases: Discriminant: $\Delta = 0$

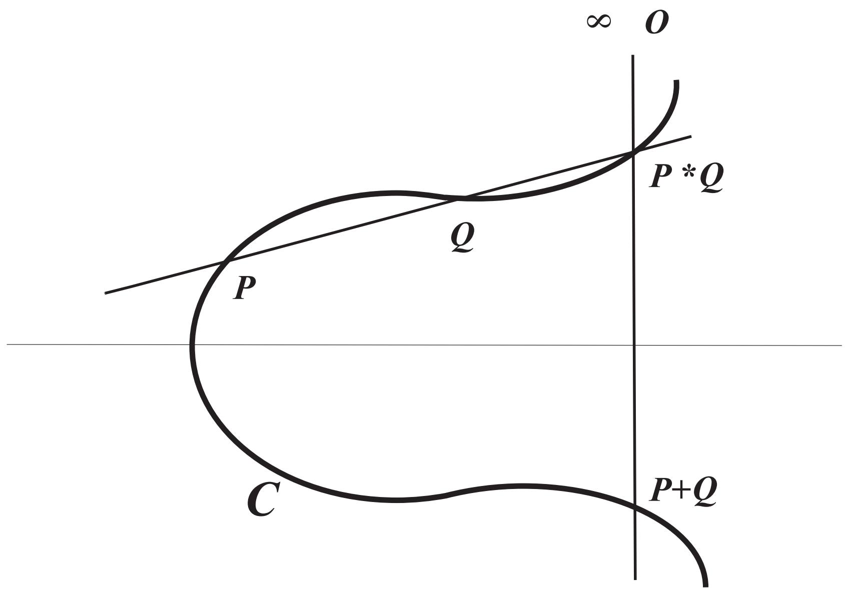
$$y^2 = (x-a)^2(x+b)$$

$$y^2 = x^3$$



Singular curves ($\Delta = 0$) :
double root (left), cusp (right)

★ **Weierstrass form ... x- symmetric curve**:



Addition on Weierstrass form: The zero element $\mathcal{O}$ is at infinity.

★ **Weierstrass** equation with complex coefficients

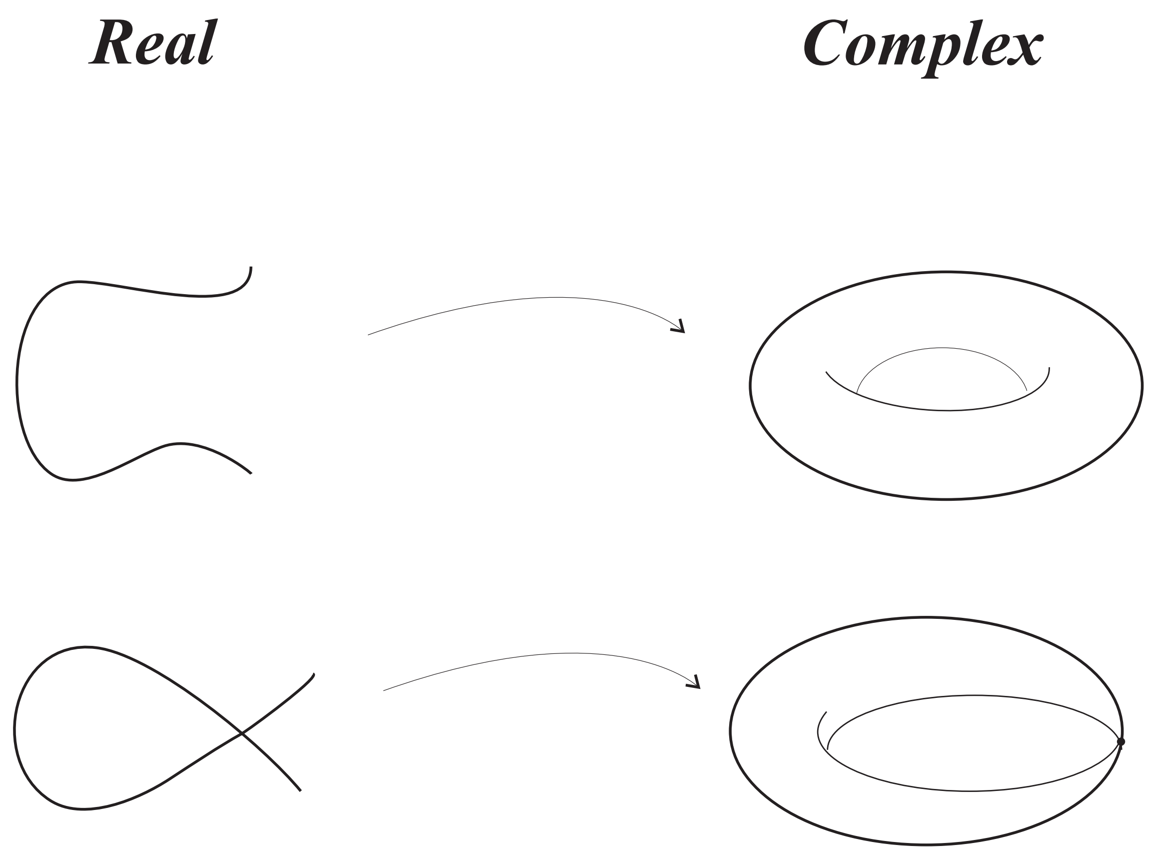


Figure 1: Complex coefficients:→ topology of torus.

Non-singular curve "upgrades" to normal torus

Singular curve corresponds to torus with a pinched radius.

$\mathcal{B}$

F-theory and Elliptic Fibration

★ F-theory ★

(*Vafa 1996*)

$\Downarrow$

Geometrisation of Type II-B superstring

II-B: *closed string spectrum obtained by combining left and right moving open strings with NS and R-boundary conditions:*

$$(NS_+, NS_+),\ (R_-, R_-),\ (NS_+, R_-),\ (R_-, NS_+)$$

Bosonic spectrum:

(NS_+, NS_+): graviton, dilaton and 2-form KB-field:

$$g_{\mu\nu},\ \phi,\ B_{\mu\nu} \to B_2$$

(R_-, R_-): scalar, 2- and 4-index fields (*p-form potentials*)

$$C, C_{\mu\nu}, C_{\kappa\lambda\mu\nu} \to C_p,\ p = 0, 2, 4$$

Definitions (*F-theory bosonic part*)

1. *String coupling:* $g_s = e^{-\phi}$

2. *Combining the two scalars* C_0, ϕ *to one modulus:*

$$\tau = C_0 + i\, e^{\phi} \rightarrow C_0 + \frac{i}{g_s}$$

IIB - *action leading to equs of motion:*

$$S_{IIB} \quad \propto \quad \int d^{10}x\sqrt{-g}\, R - \frac{1}{2}\int \frac{1}{(\mathrm{Im}\tau)^2} d\tau \wedge * d\bar{\tau}$$

$$+ \quad \frac{1}{\mathrm{Im}\tau} G_3 \wedge * \overline{G}_3 + \frac{1}{2}\tilde{F}_5 \wedge * \tilde{F}_5 + C_4 \wedge H_3 \wedge F_3$$

Property:

Invariant under $SL(2, Z)$ S-duality:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

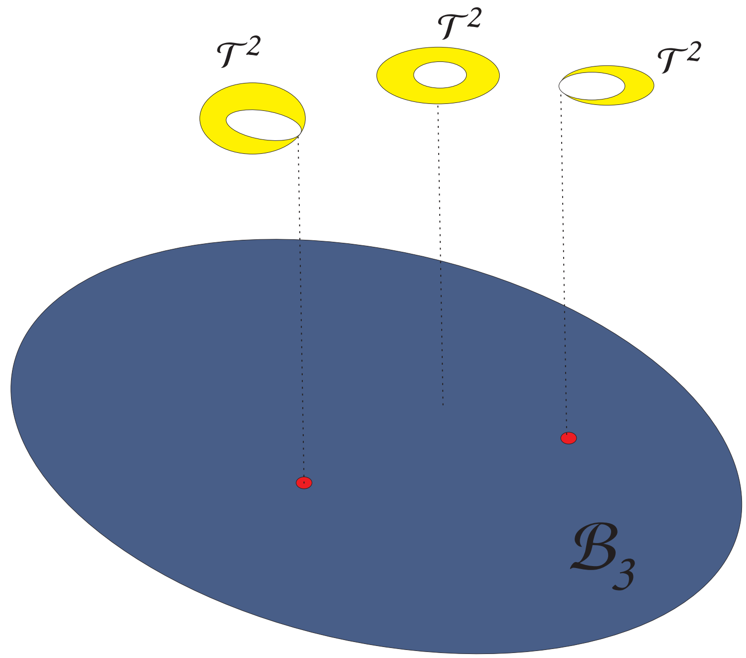
Geometrical Picture:

F-theory $\mathcal{R}^{3,1} \times \mathcal{X}$

$\rightrightarrows$ $\mathcal{X}$, elliptically fibered **CY** 4-fold over B_3 $\leftleftarrows$

$\Downarrow$

▲ a torus $\tau = C_0 + \imath/g_s$ at each point of B_3 ▲



CY 4-fold. Red points: pinched torus $\Rightarrow$ 7-branes $\perp B_3$

Elliptic Fibration

described by $\mathcal{W}$eierstraß $\mathcal{E}$quation

$$y^2 \quad = \quad x^3 + f(w)\, x z^4 + g(w) z^6$$

For each point of B_3 equation describes a torus

1. $x, y, z = [2 : 3 : 1] \to \mathbb{P}_{(2,3,1)}$-weighted projective space
2. $f(w)$, $g(w) \to 8^{th}$ and 12^{th} degree polynomials.
3. Fiber Singularities at zeros of Discriminant

$$\Delta(w) = 4\, f^3 + 27\, g^2 = 0 \to 24 \text{ roots } w_i$$

$j(\tau)$ *invariant at* w_i *vicinity* :

$$j(\tau) \approx \frac{1}{w - w_i} \to \tau(w) \ \approx \frac{1}{2\pi i} \log(w - w_i)$$

$$\Downarrow$$

Kodaira classification:

- Type of Manifold **singularity** is specified by the vanishing order of $f(w)$, $g(w)$ polynomials
- **Singularities** are classified in terms of $\mathcal{A}\,\mathcal{D}\,\mathcal{E}$ Lie groups.

Interpretation of geometric singularities

$$\Downarrow$$

$$\boxed{CY_4\text{-}\textbf{Singularities} \rightleftarrows \textbf{gauge symmetries}}$$

$$\textbf{Groups} \quad \rightarrow \quad \begin{cases} SU(n) \\ SO(m) \\ \mathcal{E}_n \end{cases}$$

Tate's Algorithm

$$y^2 + a_1 x\, y\, z + a_3 y\, z^3 = x^3 + a_2\, x^2 z^2 + a_4 x\, z^4 + a_6 z^6$$

Table: *Classification of Elliptic Singularities w.r.t. vanishing order of Tate's form coefficients* $a_i = a_i(w)$*: (partial results)*

Group	a_1	a_2	a_3	a_4	a_6	Δ
$SU(2n)$	0	1	n	n	$2n$	$2n$
$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
$SU(5)$	0	1	2	3	5	5
$SO(10)$	1	1	2	3	5	7
$\mathcal{E}_6$	1	2	3	3	5	8
$\mathcal{E}_7$	1	2	3	3	5	9
$\mathcal{E}_8$	1	2	3	4	5	10

$\mathcal{C}$

F-GUTs with discrete symmetries

Basic ingredient in F-theory:

$D7$ - brane

GUTs are associated to 7-branes wrapping certain classes of *'internal'* **2-complex dim.** surface $\mathbf{S} \subset B_3$

▲ Gauge symmetry:

$$\mathcal{E}_8 \to \mathbf{G_{GUT}} \times \mathcal{C}$$

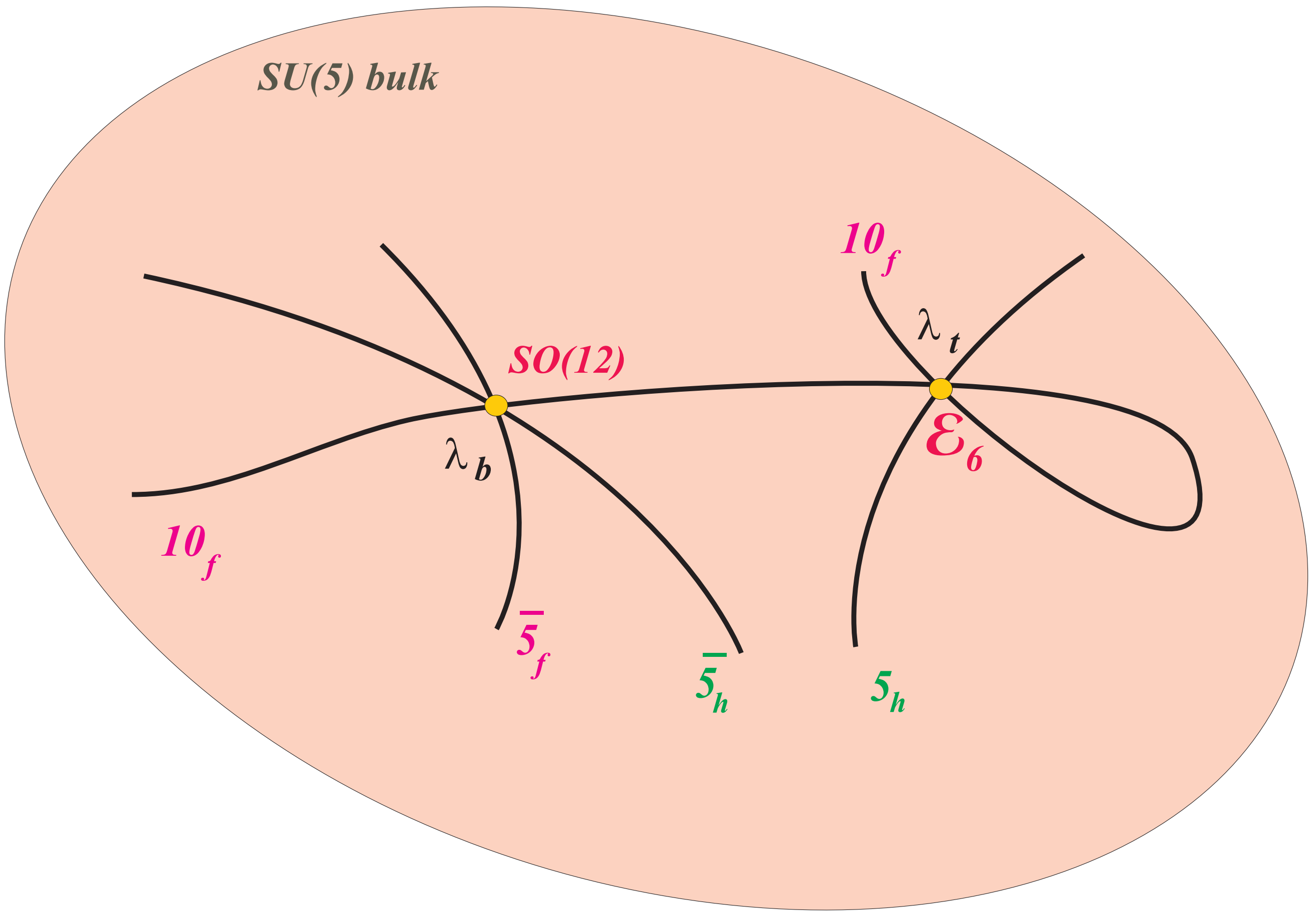
▲ $G_{GUT} = SU(5),\ SO(10),\ \ldots$

★ $\mathcal{C}$ *Commutant ...* $\Rightarrow$ *broken by fluxes to:*

$$U(1)^n, \text{or discrete symmetry } S_n,\ A_n,\ D_n,\ Z_n$$

... acting as family or discrete symmetries (11406.6290) (**see Andrew's Talk**)

Example: $SU(5)$: Matter along intersections with other 7-branes



$\lambda_{t,b}$-Yukawas at intersections and gauge symmetry enhancements

$\mathcal{D}$

Mordell-Weil $U(1)$ and GUTs

▲ *Discrete Symmetries from Mordell-Weil* Group ▲

▲ Non-Abelian group: well understood (*Kodaira classification*)

▲ Abelian sector: not *sufficiently known*

Mordell Weil Theorem

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathcal{G}$$

$r \to$ *Rank of Abelian group* (*unknown*)

$\mathcal{G} \to$ *Torsion* (*endows effective theory with discrete symmetries*):

$$\mathcal{G} = \begin{cases} \mathbb{Z}_n & n = 1, 2, \ldots, 10, 12 \\ \mathbb{Z}_k \times \mathbb{Z}_2 & k = 2, 4, 6, 8 \end{cases}$$

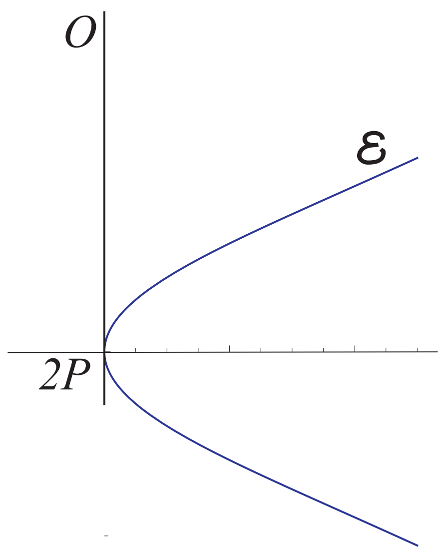
Example: $\mathcal{Z}_2$

If $\exists$ *Point* $P \in \mathcal{E}$ *such that* $P \equiv -P$ **or** $P + P = O$.

$$y^2 = x(x^2 + x + 1)$$

... *tangent connecting* $P = (0,0)$ *to zero element* $O = (0,\infty)$

$$\left(\frac{dy}{dx}\right)_{(0,0)} = \infty$$



★ **An $\mathcal{E}_6 \times U(1)_{MW}$ Model** ★

.. starting with Rank-1 Mordell-Weil

Sections required: $[u : v : w] = [1 : 1 : 2] \rightarrow$

$\mathbb{P}_{(1,1,2)}$-weighted projective space

... *described by the equation:* (hep-th/1208.2695)

$$w^2 + a_2 v^2 w = u(b_0 u^3 + b_1 u^2 v + b_2 u v^2 + b_3 v^3)$$

▲ *non-abelian group: conversion of* $\mathbb{P}_{1,1,2}$ *to* $\mathbb{P}_{2,3,1}$ *required* ▲

★ *Weierstrass model obtained using birational map:*

$$v = \frac{a_2 y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)} \quad (1)$$

$$w = \frac{b_3 u y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)} - \frac{x}{a_2} \quad (2)$$

$$u = z \quad (3)$$

Substitution to $\mathbb{P}_{1,1,2}$ eq. gives Weierstraß equation in Tate's form

$$
\begin{aligned}
y^2 + 2\frac{b_3}{a_2}xyz \pm b_1 a_2 yz^3 \quad &= \quad x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2}\right) x^2 z^2 \\
&\quad -b_0 a_2^2 xz^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2}\right) z^6
\end{aligned}
$$

but now Tate's coefficients are not all independent !

$$y^2 + 2\frac{b_3}{a_2}xyz \pm b_1 a_2 yz^3 \quad = \quad x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2}\right) x^2 z^2$$

$$-b_0 a_2^2 x z^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2}\right) z^6$$

... comparing with standard general Tate's form:

$$y^2 + \alpha_1 xyz + \alpha_3 yz^3 = x^3 + \alpha_2 x^2 z^2 - \alpha_4 x z^4 - \alpha_6 z^6$$

Observation:

$$\boxed{\alpha_6 = \alpha_2 \alpha_4}$$

Implications on the non-abelian structure (1404.6720)

Assume local expansion of Tate's coefficients

$$\alpha_k = a_{k,0} + \alpha_{k,1}\xi + \cdots$$

Vanishing orders for $SU(2n)$:

$$\alpha_2 = a_{2,1}\,\xi + \cdots$$

$$\alpha_4 = \alpha_{4,n}\xi^n + \cdots$$

$$\alpha_6 = \alpha_{6,2n}\xi^{2n} + \cdots$$

$$\alpha_6 = \alpha_2\alpha_4 \to \alpha_{2,1}\alpha_{4,n}\xi^{n+1} = \alpha_{6,2n}\xi^{2n} \;\Rightarrow\; \boxed{\text{n=1}}$$

... from $SU(2n)$ and $SU(2n+1)$ series ... compatible ONLY:

$SU(2)$ and $SU(3)$

. . . continuing the analysis for the exceptional groups ...

$\Rightarrow$ Viable non-Abelian GUTs with $U(1)_{MW}$:

$\mathcal{E}_6$ & $\mathcal{E}_7$

Procedure determines also the vanishing order of $\mathbb{P}_{1,1,2}$ coefficients $b_k \sim b_{k,n}\xi^n$

Group	a_2	b_0	b_1	b_2	b_3
$\mathcal{E}_6$	1	1	1	2	2
	0	3	1	2	1
$\mathcal{E}_7$	1	1	2	2	2
	0	3	3	2	1

$\mathcal{E}$

Remarks

- *viable GUT symmetries with one abelian Mordell-Weil factor:*

$$\mathcal{E}_6 \times U(1)_{MW},\ \mathcal{E}_7 \times U(1)_{MW}$$

- novelty: $U(1)_{MW}$-charges: not necessarily embedded in $\mathcal{E}_8$
- $U(1)_{MW}$'s ... might have interesting implications to Model building ...
- Torsion group: natural explanation of discrete symmetries? (*...preventing fast proton decay?*)

THANK YOU