Is there a reason for the *linearity* of QM?

→ superpos. principle → interference → entanglement
Def.: linearity $\iff$ dynamics maps states linearly on states.

Theorem: QM is linear.

proofs: E.P. Wigner, V. Bargmann
– assumption: dynamics does not change $|\langle \psi' | \psi \rangle|$

proof: T.F. Jordan
– assumption: no influences without interactions

“... that the system we are considering can be described as part of a larger system without interaction with the rest of the larger system.”
... we shall replace Jordan’s “No Influences Without Interactions”

Based on synthesis of three ingredients ...

- **deterministic discrete mechanics** – T.D. Lee *et al.*

- **sampling theory** for discrete structures on / of spacetime
  – A. Kempf *et al.*

- **QM in terms of classical notions of observables & phase space**
  – A. Heslot; HTE; N. Buric *et al.*
... three ingredients, which lead us to:

- deterministic discrete mechanics
  \[\rightarrow\] Hamiltonian cellular automata (CA), action principle

- sampling theory for discreteness on spacetime
  \[\rightarrow\] map: CA $\leftrightarrow$ continuum QM + corrections

- QM in terms of classical notions of observables & phase space
  \[\rightarrow\] “oscillator representation”

... $\rightarrow$ from CA perspective linearity of QM unavoidable.
Blue Cellular Automaton – Laura Pesce, fused glass (2013)
Assumptions

- There is a fundamental length or time scale $l$
- and dynamics is fundamentally discrete
- (and time is dynamical)
Hamiltonian Cellular Automata (CA) – “bit machines”

- classical CA with denumerable degrees of freedom
- state described by \textbf{integer valued} coordinates \( x_\alpha^n, \tau_n \) and momenta \( p_\alpha^n, \pi_n \)
  \( \alpha \in \mathbb{N}_0 \): different degrees of freedom
  \( n \in \mathbb{Z} \): successive states
- finite differences, \( \Delta f_n := f_n - f_{n-1} \)

No infinitesimals anywhere!
The CA Action Principle

\[ A_n := \Delta \tau_n (H_n + H_{n-1}) + c_n \pi_n, \]
\[ H_n := \frac{1}{2} S_{\alpha \beta} (p_{\alpha n}^\beta + x_{\alpha n}^\beta) + A_{\alpha \beta} p_{\alpha n}^\beta x_{\beta n}^\beta + R_n(x, p), \]

constants \( c_n \), sym. \( \hat{S} \equiv \{ S_{\alpha \beta} \} \), antisym. \( \hat{A} \equiv \{ A_{\alpha \beta} \} \), remainder \( R_n \),
all integer valued parameters

\[ S := \sum_n [(p_{\alpha n}^\alpha + p_{\alpha n-1}^\alpha) \Delta x_{\alpha n}^\alpha + (\pi_n + \pi_{n-1}) \Delta \tau_n - A_n]. \]

Postulate (Action Principle): \( \delta S \overset{!}{=} 0 \Rightarrow \) CA updating rules,
for arbitrary integer valued variations of the dynam. variables,
\[ \delta g(f_n) := \frac{[g(f_n + \delta f_n) - g(f_n - \delta f_n)]}{2}. \] remarks ...
\( R_n \equiv 0. \)
CA equations of motion (e.o.m.)

- \( S := \sum_n [(p_n^\alpha + p_{n-1}^\alpha) \Delta x_n^\alpha + (\pi_n + \pi_{n-1}) \Delta \tau_n - A_n] \),
  
  with \( \delta S = 0 \Rightarrow \) CA finite difference e.o.m.:
  
  \[
  \dot{x}_n^\alpha = \dot{\tau}_n (S_{\alpha\beta} p_n^\beta + A_{\alpha\beta} x_n^\beta), \]
  
  \[
  \dot{p}_n^\alpha = -\dot{\tau}_n (S_{\alpha\beta} x_n^\beta - A_{\alpha\beta} p_n^\beta), \]
  
  \[
  \dot{\tau}_n = c_n, \quad \dot{\pi}_n = \dot{H}_n, \quad \text{with} \quad \dot{O}_n := O_{n+1} - O_{n-1}. \]

- e.o.m. time reversal invariant, \( (n \mp 1, n) \rightarrow (n \pm 1) \)

\[ \Rightarrow \quad \dot{\psi}_n^\alpha = -i \dot{\tau}_n \hat{H}_{\alpha\beta} \psi_n^\beta, \quad \text{discrete “Schrödinger equation”} \]

with \( \hat{H} := \hat{S} + i \hat{A} \), self-adjoint, \( \psi_n^\alpha := x_n^\alpha + i p_n^\alpha \), CA “time” \( n \)
CA conservation laws

- discrete “Schrödinger equation”, \( \dot{\psi}_n^\alpha = -i \tau_n \hat{H}_{\alpha \beta} \psi_n^\beta \) \( \Rightarrow \)

- Theorem: For any \( \hat{G} \) with \([\hat{G}, \hat{H}] = 0\) there is a discrete conservation law: \( \psi_n^* \hat{G}_{\alpha \beta} \dot{\psi}_n^\beta + \dot{\psi}_n^* \hat{G}_{\alpha \beta} \psi_n^\beta = 0 \).

For self-adjoint \( \hat{G} \), with complex integer elements \( \rightarrow \) real integer quantities.

For \( \hat{G} := \hat{1} \) \( \Rightarrow \) constraint: \( \psi_n^* \dot{\psi}_n^\alpha + \dot{\psi}_n^* \psi_n^\alpha = 0 \).

For \( \hat{G} := \hat{H} \) \( \Rightarrow \) “energy conservation”.

- conservation laws not “integrable”, since Leibniz rule modified, e.g.: \( (O_n \dot{O}_n') = \frac{1}{2} (\dot{O}_n [O'_{n+1} + O'_{n-1}] + [O_{n+1} + O_{n-1}] \dot{O}_n') \).
incorporate $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$

$A_n := \Delta \tau_n (H_n + H_{n-1}) + c_n \pi_n$, as before

$\rightarrow$ alternative action:

$S := \sum_n [\text{Im}(\psi_n^\alpha \psi_{n-1}^\alpha) + (\pi_n + \pi_{n-1}) \Delta \tau_n - A_n]$

with: $H_n := \frac{1}{2} H_{\alpha\beta} \psi_n^\alpha \psi_n^\beta$

$\rightarrow$ action invariant under $\psi_n \rightarrow \psi'_n = \hat{U} \psi_n$, $[\hat{U}, \hat{H}] = 0$, $n$-independent unitary transformations; admissible ones?

$\rightarrow$ conservation laws without continuous symmetries!
How to obtain more of QM ...

- recall $\psi_n^\alpha := x_n^\alpha + i p_n^\alpha$, CA “time” $n$

introduce fundamental scale $l \rightarrow n \cdot l$, physical time?

Problem: continuum limit, $l \rightarrow 0$, does not work
- integer valuedness $\Rightarrow$ time derivatives diverge!

- Idea: invertible MAP between discrete integer valued and continuous (differentiable ... ) quantities. – G. ’t Hooft

"... aha! digital audio and video!" – information can be simultaneously continuous & discrete – C.E. Shannon
The Sampling Theorem

- Consider square integrable bandlimited functions $f$:
  \[ f(t) = (2\pi)^{-1} \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} d\omega \ e^{-i\omega t} \tilde{f}(\omega), \text{ bandwidth } \omega_{\text{max}}. \]

- Shannon’s Theorem:

  Given \( \{f(t_n)\} \) for set \( \{t_n\} \) of equidistantly spaced times (spacing \( \pi/\omega_{\text{max}} \)), function \( f \) is obtained for all \( t \) by:
  \[ f(t) = \sum_n f(t_n) \frac{\sin[\omega_{\text{max}}(t-t_n)]}{\omega_{\text{max}}(t-t_n)} \quad \text{(reconstruction formula)}. \]

- CA “time” \( n \sim \) discrete time \( t_n := n/l \rightarrow \) continuous time \( t \)
  bandwidth \( \omega_{\text{max}} := \pi/l \) (Nyquist rate)
applying Shannon’s reconstruction formula ...
the discrete “Schrödinger equation”, \( \dot{\psi}_n^\alpha = \hat{\tau}_n \hat{H}_{\alpha\beta} \psi_n^\beta \) ...
is mapped to continuous time Schrödinger equation:
\[
(\hat{D}_l - \hat{D}_{-l})\psi^\alpha(t) = 2 \sinh(l \partial_t) \psi^\alpha(t) = \frac{1}{i} H_{\alpha\beta} \psi^\beta(t),
\]
with \( \hat{D}_T f(t) := f(t + T) \) and naturally \( \hat{\tau}_n \equiv 1 \).

correction terms, \( |\partial^k \psi / \partial t^k| \ll l^{-k} = (\omega_{\text{max}} / \pi)^k \)
stationary states, \( 2 \sin(E_{\alpha} l) = \epsilon_{\alpha} \), for \( \hat{H} \to \text{diag}(\epsilon_0, \epsilon_1, \ldots) \)
\( \Rightarrow \) spectrum cut off by \( |E_{\alpha}| \leq \pi / 2l = \omega_{\text{max}} / 2 \)
Conservation Laws: discrete CA ↔ continuous QM

- CA & QM equations are both **linear!** – Then, CA → QM by
  \[\dot{\psi}_n := \psi_{n+1} - \psi_{n-1} \rightarrow \frac{1}{i} \sin(il\partial_t)\psi(t) , \text{ suggests ...}\]

- Theorem: For any \(\hat{G}\) with \([\hat{G}, \hat{H}] = 0\), it holds that
  \[\psi^{*\alpha} G_{\alpha\beta} \sin(il\partial_t)\psi^\beta + [\sin(il\partial_t)\psi^{*\alpha}] G_{\alpha\beta} \psi^\beta = 0 .\]

  In particular, wave function normalization conserved
  \[\psi^{*\alpha} \sin(il\partial_t)\psi^\alpha + [\sin(il\partial_t)\psi^{*\alpha}] \psi^\alpha = 0 . \text{ correlation fcts.}\]

- same commutator \([\hat{G}, \hat{H}] = 0 \Rightarrow \text{ CA & QM conserv. laws!}\]

- in all finite-\(l\) QM equations, continuum limit \(l \rightarrow 0\) works.
Some things to find out ...

- CA properties that become unitary symmetries in QM??

- CA observables? - Yes! Exist consistently with discrete analog of Poisson brackets and QM interpretation \([\text{EPJ WoC 78, 02005 (2014)}]\).

- CA Hamiltonians have interesting spectra? - Yes! Complete classification by mathematicians \([\text{J.McKee & C.Smyth, 2007]}\).

- QM approximation scheme with bandwidth limited wave fct.s? - Feasible \([\text{D.Gigli}]\).

- What is relativistic/QFT version of CA \(\leftrightarrow\) QM map??
Consistent CA observables?

Discrete Poisson brackets ...

- recall: only variational derivatives for discrete variables
  \[ \delta g(f) := \frac{[g(f + \delta f) - g(f - \delta f)]}{2}, \quad f, \delta f \in \mathbb{Z}. \]

- → define: \[ \{A, B\} := \delta_x^\alpha A \delta_p^\alpha B - \delta_x^\alpha B \delta_p^\alpha A . \]

- for constant, linear, or quadratic polynomials \( A, B \), variational derivatives independent of \( \delta_x, \delta_p \) and bracket corresponds to ordinary Poisson bracket, in all respects.

⇒ \( \) CA observables can be chosen as real quadratic forms in
  \[ \psi_n^\alpha := x_n^\alpha + ip_n^\alpha ; \text{a closed algebra endowed with } \{ , \} . \]

  E.g., \[ \dot{\psi}^\alpha = \{\psi^\alpha, \mathcal{H}\} , \text{with } \mathcal{H} := \psi^*\alpha H_{\alpha\beta}\psi^\beta / 2 . \]
On the relation between Hamiltonian CA and QM ...

- MAP: CA $\leftrightarrow$ QM based on
  - integer valued action principle with arbitrary variations
  - sampling theory ($\Rightarrow$ bandwidth limited wave fcts.)
  $\Rightarrow$ replaces Jordan’s separability assumption ($\Rightarrow$ linearity)

- Schrödinger equation with correction terms, $\sim (l\partial_t)^k$,
  - incorporating discreteness scale $l$ . . . $\Rightarrow$ GUP [D.Gigli]

- $[\hat{G}, \hat{H}] = 0$ $\Rightarrow$ corresp. conservation laws for CA & QM