DSE-inspired model for the pion GPD

José Rodríguez-Quintero

Univ. Huelva & CAFPE

3 December

In collaboration with
L. Chang, C. Mezrag, H. Moutarde, P. Tandy, C.D. Roberts, F. Sabatié
Describing a strongly coupled couple
GPDs in a nutshell:

- Short range $\rightarrow$ perturbation theory.
- Long range $\rightarrow$ nonperturbative objects: GPDs,
- which encodes the hadrons 3D partonic and spin structure.
- *Universality.*
GPDs in a nutshell:

- Short range $\rightarrow$ perturbation theory.
- Long range $\rightarrow$ nonperturbative objects: GPDs,
- which encodes the hadrons 3D partonic and spin structure.
- *Universality.*
- $H$ stands for the GPD,
- depending on the 3 variables: $x, \xi, t$. 

$$H(x, \xi, t)$$

$$x-\xi$$

$$x+\xi$$

$$(1-\xi)P^+ \quad t=\Delta \quad (1+\xi)P^-$$
Current GPDs models

- The most popular approach to model GPDs invokes the Double Distribution $F$ et $G$

$$H(x, \xi, t) = \int_{|\alpha| + |\beta| \leq 1} d\alpha \ d\beta (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi \alpha)$$
Current GPDs models

- The most popular approach to model GPDs invokes the Double Distribution $F$ et $G$

$$H(x, \xi, t) = \int_{|\alpha|+|\beta|\leq1} d\alpha \, d\beta (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi \alpha)$$

Current GPDs models

- The most popular approach to model GPDs invokes the Double Distribution $F$ et $G$

$$H(x, \xi, t) = \int_{|\alpha|+|\beta|\leq1} d\alpha \ d\beta (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi \alpha)$$


- Advantage: simple.
Current GPDs models

- The most popular approach to model GPDs invokes the Double Distribution $F$ et $G$

$$H(x, \xi, t) = \int_{|\alpha|+|\beta|\leq 1} d\alpha \ d\beta (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi \alpha)$$

- Avantadge : simple.
- Drawback : non flexible enough [C. Mezrag et al, PRD88(2013)014001]
Current GPDs models

- The most popular approach to model GPDs invokes the Double Distribution $F$ et $G$

$$H(x, \xi, t) = \int_{|\alpha|+|\beta|\leq1} d\alpha \ d\beta (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi \alpha)$$

- Avantadge : simple.
- Drawback : non flexible enough [C. Mezrag et al, PRD88(2013)014001]

Well-educated ansatz inspired by the asymptotic shape of Distribution Amplitudes.
Current GPDs models

- The most popular approach to model GPDs invokes the Double Distribution $F$ et $G$

$$H(x, \xi, t) = \int_{|\alpha|+|\beta|\leq 1} d\alpha \ d\beta (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi \alpha)$$

- Avantadge : simple.
- Drawback : non flexible enough [C. Mezrag et al, PRD88(2013)014001]

Well-educated ansatz inspired by the asymptotic shape of Distribution Amplitudes.

How can we compute GPDs on the ground of QCD?
DSE and BSE approach:

quark propagator:

\[
S(p; \mu) = -i \gamma \cdot p \sigma_V(p; \mu) + \sigma_S(p; \mu),
\]
\[
= \frac{1}{i \gamma \cdot p A(p; \mu) + B(p; \mu)}.
\]

\[
S^{-1}(p) = Z_2(i \gamma \cdot p + m) + Z_1 \int d^4 q g^2 D_{\mu\nu}(p - q) \frac{\lambda^a}{2} \gamma_{\mu} S(q) \frac{\lambda^a}{2} \Gamma_{\nu}(q, p),
\]
DSE and BSE approach:

quark propagator:

\[
S(p; \mu) = -i \gamma \cdot p \sigma_V(p; \mu) + \sigma_S(p; \mu),
\]
\[
= \frac{1}{i \gamma \cdot p A(p; \mu) + B(p; \mu)}.
\]

\[
S^{-1}(p) = Z_2(i \gamma \cdot p + m) + Z_1 \int d^4 q \, g^2 D_{\mu \nu}(p - q) \frac{\lambda^a}{2} \gamma_\mu S(q) \frac{\lambda^a}{2} \Gamma_\nu(q, p),
\]

Bethe-Salpeter amplitude:

\[
\Gamma_{\pi}(q, P; \mu) = \gamma_5 \left[ i E_{\pi}(q, P; \mu) + \gamma \cdot P F_{\pi}(q, P; \mu) + q \cdot P \gamma \cdot q G_{\pi}(q, P; \mu) + \sigma_{\mu \nu} q^\mu P^\nu H_{\pi}(q, P; \mu) \right].
\]
DSE and BSE approach:

quark propagator:

\[
S(p; \mu) = -i \gamma \cdot p \sigma_V(p; \mu) + \sigma_S(p; \mu),
= \frac{1}{i \gamma \cdot p A(p; \mu) + B(p; \mu)}.
\]

\[
S^{-1}(p) = Z_2(i \gamma \cdot p + m) + Z_1 \int d^4q \, g^2 D_{\mu\nu}(p - q) \frac{\lambda^a}{2} \gamma_\mu S(q) \frac{\lambda^a}{2} \Gamma_\nu(q, p),
\]

Bethe-Salpeter amplitude:

\[
\Gamma_\pi(q, P; \mu) = \gamma_5 \left[ i E_\pi(q, P; \mu) + \gamma \cdot P F_\pi(q, P; \mu) + q \cdot P \gamma \cdot q G_\pi(q, P; \mu) + \sigma_{\mu\nu} q^\mu P^\nu H_\pi(q, P; \mu) \right].
\]

Nakanishi representation in terms of complex conjugate poles
Scalar meson GPDs

Formal definition:

\[ H(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P + \frac{\Delta}{2} \bigg| \bar{q} \left( -\frac{z}{2} \right) \gamma^+ \left[ -\frac{z}{2}; \frac{z}{2} \right] q \left( \frac{z}{2} \right) \bigg| P - \frac{\Delta}{2} \right\rangle_{z^+ = 0, z_\perp = 0}, \]

( X. Ji, 1997; D. Müller, 1994; A. Radyushkin, 1997;)

GPD can be reconstructed starting from its Mellin moments!!!
Scalar meson GPDs

Formal definition:

\[
H(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P + \frac{\Delta}{2} \right| \bar{q} \left( -\frac{z}{2} \right) \gamma^+ \left[ -\frac{z}{2}, \frac{z}{2} \right] q \left( \frac{z}{2} \right) \left| P - \frac{\Delta}{2} \right\rangle z^+ = 0, z_\perp = 0,
\]

(X. Ji, 1997; D. Müller, 1994; A. Radyushkin, 1997;)

the Mellin moments of which can be formally expressed as (twist-2 operators)

\[
\mathcal{M}_m(\xi, t) = \int_{-1}^{1} dx \, x^m H(x, \xi, t)
\]

\[
= \frac{1}{2(P \cdot n)^{m+1}} \left\langle \pi, P + \frac{\Delta}{2} \right| \bar{\psi}(0) \gamma \cdot n (i \overset{\leftrightarrow}{D} \cdot n)^m \psi(0) \left| \pi, P - \frac{\Delta}{2} \right\rangle.
\]

(José Rodríguez-Quintero (Univ. Huelva & CAFPE))
Scalar meson GPDs

Formal definition:

\[ H(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P + \frac{\Delta}{2} \left| \bar{q} \left( -\frac{z}{2} \right) \gamma^+ \left[ -\frac{z}{2}; \frac{z}{2} \right] q \left( \frac{z}{2} \right) \right| P - \frac{\Delta}{2} \right\rangle \big|_{z^+=0, z_\perp=0}, \]

(X. Ji, 1997; D. Müller, 1994; A. Radyushkin, 1997;)

the Mellin moments of which can be formally expressed as (twist-2 operators)

\[ M_m(\xi, t) = \int_{-1}^{1} dx \ x^m H(x, \xi, t) \]

\[ = \frac{1}{2(P \cdot n)^{m+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{\psi}(0) \gamma \cdot n (i \overset{\leftrightarrow}{D} \cdot n)^m \psi(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle. \]

GPD can be reconstructed starting from its Mellin moments!!!
Simple analytical model for the Pion:

\[
\langle x^m \rangle = M_m(\xi, t) = \frac{1}{2(P \cdot n)^{m+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{\psi}(0) \gamma \cdot n (i \rightarrow D \cdot n)^m \psi(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle.
\]

Direct and crossed triangle diagrams:
Simple analytical model for the Pion:

Direct and crossed triangle diagrams with fully dressed vertices and propagators:

\[
2(P \cdot n)^{m+1} \langle x^m \rangle_u = \text{tr}_{C_{FD}} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_{\pi} \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \\
S(k - \frac{\Delta}{2}) i\Gamma_{e.m.} \cdot n S(k + \frac{\Delta}{2}) \\
\tau_- i\bar{\Gamma}_{\pi} \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P),
\]
Simple analytical model for the Pion:

\[
2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{\text{CFD}} \int \frac{d^4k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_\pi \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \]

\[
S(k - \frac{\Delta}{2}) \ i\gamma \cdot n \ S(k + \frac{\Delta}{2})
\]

\[
\tau_- i\vec{\Gamma}_\pi \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P),
\]

DSE and BS inspired simple ansätze:

\[
S(p) = [- i\gamma \cdot p + M] \Delta_M(p^2),
\]

\[
\Delta_M(s) = \frac{1}{s + M^2},
\]

\[
\Gamma_\pi(k, p) = i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} dz \rho_\nu(z) \left[ \Delta_M(k_{+z}^2) \right]^\nu;
\]

\[
\rho_\nu(z) = R_\nu(1 - z^2)^\nu,
\]

with \( k_{\pm z} = k \mp (1 - z)P/2 \), standing for the momentum fraction that the quark carries out.
Simple analytical model for the Pion:

Case $\xi = 0$ (dressing improved [H.L.L. Roberts et al. PRC83(2011)065206])

$$2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{C\!F\!D} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_{\pi} \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right)$$

$$S(k - \frac{\Delta}{2}) i P_T(-t = \Delta^2_{\perp}) \gamma \cdot n S(k + \frac{\Delta}{2})$$

$$\tau_+ i\bar{\Gamma}_{\pi} \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P),$$

DSE and BS inspired simple ansätze:

$$S(p) = \left[ -i \gamma \cdot p + M \right] \Delta_M(p^2),$$

$$\Delta_M(s) = \frac{1}{s + M^2},$$

$$\Gamma_{\pi}(k, p) = i \gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} dz \rho_{\nu}(z) \left[ \Delta_M(k_{+z}^2) \right]^{\nu};$$

$$\rho_{\nu}(z) = R_{\nu}(1 - z^2)^{\nu},$$

with $k_{\pm z} = k \mp (1 - z)P/2$, standing for the momentum fraction that the quark carries out.
Results for the Mellin moments

\[ \langle x^m \rangle_{I=0,1} = \lambda \int dx \, dy \, du \, dv \, dw \, dz \, dz' \left( \frac{M^2}{M'^2} \right)^{2\nu} \delta(1-x-y-u-v-w)x^{\nu-1}y^{\nu-1}\rho(z)\rho(z') \]
\[ \left[ (g-2\xi \, f)^m (g+1-2\xi \, f) \pm (g-2\xi \, f)^m (-g-1-2\xi \, f) \right] \]
\[ + \frac{1}{2} ((-2\xi \, f + g - 1)(g - 2\xi \, f)^m \pm (-2\xi \, f - g + 1)(-g - 2\xi \, f)^m) \]
\[ + \frac{m}{2} ((g - 2\xi \, f)^{m-1}((g - 2\xi \, f)^2 - \xi^2) \pm (-g - 2\xi \, f)^{m-1}((-g - 2\xi \, f)^2 - \xi^2)) \]
\[ + \frac{\Gamma(2\nu + 1)}{2M'^2\Gamma(2\nu)} (g - 2\xi \, f)^m \left( (g - 2\xi \, f)(tf^2 + P^2(g^2 - 2g) + \frac{t}{4} + M^2) \right. \]
\[ \left. + tf^2 + P^2g^2 - \frac{t}{4} + tf\xi + M^2 \right) \]
\[ \pm \frac{\Gamma(2\nu + 1)}{2M'^2\Gamma(2\nu)} (-g - 2\xi \, f)^m \left( (-g - 2\xi \, f)(tf^2 + P^2(g^2 - 2g) + \frac{t}{4} + M^2) \right. \]
\[ \left. - tf^2 - P^2g^2 + \frac{t}{4} + tf\xi - M^2 \right) \].

(68)
Results for the Mellin moments

\[ f(x, y, v, w, z, z') = \frac{1}{2} \left( -\frac{1+z'}{2}y + \frac{1+z}{2}x + v - w \right), \quad (55) \]

\[ g(x, y, u, z, z') = \left( \frac{1-z'}{2} \right) y + x \frac{1-z}{2} + u, \quad (56) \]

\[ M'(t, P^2, x, y, u, v, w, z, z')^2 = M^2 + \frac{t}{4} \left( -4f^2 + y \left( \frac{1+z'}{2} \right)^2 + x \left( \frac{1+z}{2} \right)^2 + v + w \right) + P^2 \left( -g^2 + \left( \frac{1-z'}{2} \right)^2 y + \left( \frac{1-z}{2} \right)^2 x + u \right). \quad (57) \]

\[ 2M'^2 \Gamma(2\nu) \left( 1 - \frac{t}{4} + tf\xi + M^2 \right) + \left( -g - 2\xi f \right)^m \left( -g - 2\xi f \right)(tf^2 + P^2(g^2 - 2g) + \frac{t}{4} + M^2) \]

\[ \frac{\Gamma(2\nu + 1)}{2M'^2 \Gamma2\nu} \left( -g - 2\xi f \right)^m \left( -g - 2\xi f \right)(tf^2 + P^2(g^2 - 2g) + \frac{t}{4} + M^2) \]

\[ \left( -g - 2\xi f \right)^m \left( -g - 2\xi f \right)(tf^2 + P^2(g^2 - 2g) + \frac{t}{4} + M^2) \quad (68) \]
Results for the Mellin moments

\[
\begin{align*}
    f(x, y, v, w, z, z') &= \frac{1}{2} \left( -\frac{1+z'}{2} y + \frac{1+z}{2} x + v - w \right), \\
    g(x, y, u, z, z') &= \left( \frac{1-z'}{2} \right) y + x \frac{1-z}{2} + u, \\
    M'(t, P^2, x, y, u, v, w, z, z')^2 &= M^2 + \frac{t}{4} \left( -4f^2 + y \left( \frac{1+z'}{2} \right)^2 + x \left( \frac{1+z}{2} \right)^2 + v + w \right) \\
    &+ P^2 \left( -g^2 + \left( \frac{1-z'}{2} \right)^2 y + \left( \frac{1-z}{2} \right)^2 x + u \right).
\end{align*}
\]

\[
\begin{align*}
    H^I=0(x, \xi, t) &= H^u_{\pi^\pm}(x, \xi, t) + H^d_{\pi^\pm}(x, \xi, t) \\
    &= H^u_{\pi^0}(x, \xi, t) + H^d_{\pi^0}(x, \xi, t), \\
    H^I=1(x, \xi, t) &= H^u_{\pi^+}(x, \xi, t) - H^d_{\pi^+}(x, \xi, t) \\
    &= -(H^u_{\pi^-}(x, \xi, t) - H^d_{\pi^-}(x, \xi, t)), \\
    0 &= H^u_{\pi^0}(x, \xi, t) - H^d_{\pi^0}(x, \xi, t).
\end{align*}
\]
Results for the Mellin moments

Mellin Moments at $\xi = 0$

- $m = 0$
- $m = 1$
- $m = 2$
- $m = 3$

José Rodríguez-Quintero (Univ. Huelva & CAFPE)
Properties of Mellin moments

Polynomiality:

$$\left\langle \pi, P + \frac{\Delta}{2} \left| \bar{\psi}(0) \gamma \cdot n(i \vec{D} \cdot n)^m \psi(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$

$$= n_\mu n_{\mu_1} \cdots n_{\mu_m} P^{\{\mu} \sum_{j=0}^{m} \binom{m}{j} F_{m,j}(t) P^{\mu_1} \cdots P^{\mu_j} \left( -\frac{\Delta}{2} \right)^{\mu_{j+1}} \cdots \left( -\frac{\Delta}{2} \right)^{\mu_m}$$

$$- n_\mu n_{\mu_1} \cdots n_{\mu_m} \frac{\Delta}{2} \sum_{j=0}^{m} \binom{m}{j} G_{m,j}(t) P^{\mu_1} \cdots P^{\mu_j} \left( -\frac{\Delta}{2} \right)^{\mu_{j+1}} \cdots \left( -\frac{\Delta}{2} \right)^{\mu_m}$$

$$\xi = -\frac{\Delta \cdot n}{2P \cdot n} \Rightarrow M_m(\xi, t) \text{ is a polynomial in } \xi \text{ of order } m + 1.$$
Properties of Mellin moments

Double distributions:

\[ F_{m,j}(t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, \beta^{m-j} \alpha^j \, F(\beta, \alpha, t) \]

\[ G_{m,j}(t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, \beta^{m-j} \alpha^j \, G(\beta, \alpha, t) \]
Properties of Mellin moments

\[ M_m(\xi, t) = n_\mu n_{\mu_1} \ldots n_{\mu_m} \sum_{j=0}^{m} \binom{m}{j} \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \beta^{m-j} \alpha^j \]

\[ F(\beta, \alpha, t) P^{\{\mu} P^{\mu_1} \ldots P^{\mu_j} \left(-\frac{\Delta}{2}\right)^{\mu_{j+1}} \ldots \left(-\frac{\Delta}{2}\right)^{\mu_m} \]

\[ -G(\beta, \alpha, t) \frac{\Delta}{2} P^{\mu_1} \ldots P^{\mu_j} \left(-\frac{\Delta}{2}\right)^{\mu_{j+1}} \ldots \left(-\frac{\Delta}{2}\right)^{\mu_m} \]
Properties of Mellin moments

\[ \mathcal{M}_m(\xi, t) = n_\mu n_{\mu_1} \ldots n_{\mu_m} \sum_{j=0}^{m} \binom{m}{j} \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \beta^{m-j} \alpha^j \]

\[ F(\beta, \alpha, t) P^{\mu_1 \ldots P^{\mu_j}} \left( -\frac{\Delta}{2} \right)^{\mu_{j+1}} \ldots \left( -\frac{\Delta}{2} \right)^{\mu_m} \]

\[ -G(\beta, \alpha, t) \Delta_2 \{ P^{\mu_1 \ldots P^{\mu_j}} \left( -\frac{\Delta}{2} \right)^{\mu_{j+1}} \ldots \left( -\frac{\Delta}{2} \right)^{\mu_m} \}

Time Rerversal Invariance

\[ \Delta \rightarrow -\Delta \]

\[ \mathcal{M}_m(\xi, t) \] is an even polynomial in \( \xi \) of order \( m + 1 \).

\[ F(\beta, \alpha) \] is even in \( \alpha \).

\[ G(\beta, \alpha) \] is odd in \( \alpha \).
From Mellin moments to Double Distributions (DD)

DD are directed linked to $H$:

$$H(x, \xi, t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \ (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \alpha \xi)$$

Double Distributions are the Radon transform of the GPD
From Mellin moments to Double Distributions (DD)

DD are directed linked to $H$:

$$H(x, \xi, t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \left(F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)\right) \delta(x - \beta - \alpha \xi)$$

Double Distributions are the Radon transform of the GPD

PDF case:

$$q(x) = H(x, 0, 0) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \ F(\beta, \alpha, t) \delta(x - \beta)$$

Form Factor case:

$$\mathcal{F}(t) = \int_{-1}^{1} dx \ H(x, \xi, t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \ F(\beta, \alpha, t)$$
Comparison with polynomial reconstruction

\[ H(x, 0, 0) = \int_0^1 d\beta \int_{1-\beta}^{1+\beta} d\alpha \delta(x - \beta) F(\beta, \alpha, t) \]

- Forward case: \( \xi = 0 \) and \( t = 0 \).
- Very good agreement.
- The polynomial reconstruction describes well the support properties of valence GPD: \( x \in [-\xi, 1] \)
Comparison with polynomial reconstruction

\[ H(x, 0, 0) = \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha \delta(x - \beta) F(\beta, \alpha, t) \]

- Forward case: \( \xi = 0 \) and \( t = 0 \).
- Very good agreement.
- The polynomial reconstruction describes well the support properties of valence GPD: \( x \in [-\xi, 1] \)

If the polynomial reconstruction works, why should we care about DDs?
Advantages of DDs

- Support is strictly respected.
Advantages of DDs

- Support is strictly respected.
- Reconstruction is exact even at $\xi \neq 0$ (no numerical noise).
Advantages of DDs

- Support is strictly respected.
- Reconstruction is exact even at $\xi \neq 0$ (no numerical noise).
- We can get analytic expressions. For the PDF($\nu = 1$):

$$q(x) = \frac{72}{25} (x^3(x(-2(x - 4)x - 15) + 30)\log(x)$$
$$+ (2x^2 + 3) (x - 1)^4 \log(1 - x)$$
$$+ x(x(x(2x - 5) - 15) - 3)(x - 1))$$
Analytical “resummation” \((t = 0)\)
Analytical “resummation” \((t = 0, \xi = 0)\)
The PDF appears not to be symmetric around $x = \frac{1}{2}$.
Limits of the triangle diagrams


\[ \psi(x, k^2_\perp) = \sqrt{\frac{15}{2\pi \sigma^2}} \frac{\sqrt{x(1-x)}}{1 + k^2_\perp/(4 \sigma^2 x(1-x))} \theta(x) \theta(1-x). \]

Non-skewed GPD overlap representation:

\[ H_\sigma(x, 0, -\Delta^2_\perp) = \int d^2 k_\perp \psi(x, k_\perp + (1-x)\Delta_\perp) \psi(x, k_\perp) \]

\[ = 30(1 - x)^2 x^2 \theta(x) \theta(1-x) C \left( \frac{\Delta^2_\perp}{4x^2 \sigma^2} (1 - x) \right); \]

\( C(z) \) decreasing monotonically away from its maximum value \( C(0) = 1 \), and encoding the \((x \leftrightarrow 1-x)\)-asymmetry.

The PDF appears not to be symmetric around \( x = \frac{1}{2} \).
Heuristic example: Light-cone wave function for a bound-state of two scalar particles

\[
\psi(x, k^2_\perp) = \sqrt{\frac{15}{2\pi \sigma^2}} \frac{\sqrt{x(1-x)}}{1 + k^2_\perp/(4 \sigma^2 x(1-x))} \theta(x) \theta(1-x).
\]

Non-skewed GPD overlap representation:
\[
H_\sigma(x, 0, -\Delta^2_\perp = 0) = \int d^2 k_\perp \psi^2(x, k_\perp) = 30(1 - x)^2 x^2 \theta(x) \theta(1 - x);
\]

The PDF-like function is symmetric under 
\[x \leftrightarrow 1 - x.\]
The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Part of the gluons contribution is neglected in the triangle diagram approach.
The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Part of the gluons contribution is neglected in the triangle diagram approach.
The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Part of the gluons contribution is neglected in the triangle diagram approach.

$$q_{BC}^{T}(x) = n_q \left[ x^3 (2x([x - 3]x + 5) - 15) \ln(x) - (2x^3 + 4x + 9) \right. $$
$$\left. \times (x - 1)^3 \ln(1 - x) - x(2x - 1)([x - 1]x - 9)(x - 1) \right]. \quad (13)$$
The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Part of the gluons contribution is neglected in the triangle diagram approach.

Adding this contribution allows us to recover a symmetric PDF [L. Chang et al., Phys.Lett.B737(2014)2329].
The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Part of the gluons contribution is neglected in the triangle diagram approach.

Adding this contribution allows us to recover a symmetric PDF [L. Chang et al., Phys.Lett.B737(2014)2329].

\[ q_{BC}(x) = n_q \left[ x^3 (2x(x-3)x+15) \ln(x) - (2x^3 + 4x + 9) \right. \]
\[ \times (x-1)^3 \ln(1-x) - x(2x-1)(x-9)(x-1) \]  

(13)
Pion form factor

QCD sum rule:

\[ F_\pi^q(t) = M_0(t) = \int_{-1}^{1} dx \ H^q(x, \xi, t) \]
Pion form factor

QCD sum rule:

\[ F_{\pi}^q(t) = M_0(t) = \int_{-1}^{1} dx \, H^q(x, \xi, t) \]
PDF Mellin moments

GPD forward limit:

\[ PDF(x) = H^q(x, 0, 0) \]
PDF Mellin moments

GPD forward limit:

$$PDF(x) = H^q(x, 0, 0)$$
The standard triangle diagram does not fully keep all the contributions from the gluons which bind the dressed-quark to the pion.

\[ 2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{CFD} \int \frac{d^4k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_\pi \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \\
S(k - \frac{\Delta}{2}) i\gamma \cdot n S(k + \frac{\Delta}{2}) \\
\tau_- i\bar{\Gamma}_\pi \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P), \]
The standard triangle diagram does not fully keep all the contributions from the gluons which bind the dressed-quark to the pion and a new contribution is needed.

\[
2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{CFD} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i \Gamma_\pi \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \\
S(k - \frac{\Delta}{2}) \tau_- \frac{\partial}{\partial k} \bar{\Gamma}_\pi \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P)
\]
Non-forward pion GPD

\[ 2(P \cdot n)^{m+1} \frac{1}{2} \langle x^m \rangle^u = \text{tr}_{CFD} \int \frac{d^4k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma \pi \left( \eta(k-P) + (1-\eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \]

\[ S(k - \frac{\Delta}{2}) \tau_- \frac{\partial}{\partial k} \tilde{\Gamma}_\pi \left( (1-\eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P) \]

\[ F^{BC}(\beta, \alpha, t), G^{BC}(\beta, \alpha, t) \]

\[ H^{BC}(x, \xi, t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \left( F^{BC}(\beta, \alpha, t) + \xi G^{BC}(\beta, \alpha, t) \right) \delta(x - \beta - \alpha\xi) \]

\[ H^{BC}(x, 0, 0) = \int_{-1+|x|}^{1-|x|} d\alpha \ F^{BC}(x, \alpha, 0) \]
Non-forward pion GPD

\[
2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{\text{CFD}} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_i \bar{\Gamma}_\pi \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \\
S(k - \frac{\Delta}{2}) \tau_- \frac{\partial}{\partial k} \bar{\Gamma}_\pi \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P)
\]

\[
H^{BC}(x, \xi, t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \left( F^{BC}(\beta, \alpha, t) + \xi G^{BC}(\beta, \alpha, t) \right) \delta(x - \beta - \alpha\xi)
\]

\[
H^{BC}(x, 0, 0) = \int_{-1+|x|}^{1-|x|} d\alpha \ F^{BC}(x, \alpha, 0) \equiv q_\pi^{BC}(x)
\]
Non-forward pion GPD

The full model:

\[ 2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{\text{CFD}} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_{\pi} \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \]

\[ S(k - \frac{\Delta}{2}) i\gamma \cdot n S(k + \frac{\Delta}{2}) \]

\[ \tau_- i\bar{\Gamma}_{\pi} \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P), \]

\[ 2(P \cdot n)^{m+1} \langle x^m \rangle^u = \text{tr}_{\text{CFD}} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_{\pi} \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \]

\[ S(k - \frac{\Delta}{2}) \tau_- \frac{\partial}{\partial k} \bar{\Gamma}_{\pi} \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P). \]
Non-forward pion GPD

\[
H(x, \xi, 0) = \int_{-1}^{1} d\beta \int_{-1 + |\beta|}^{1 - |\beta|} d\alpha \left( F(\beta, \alpha, 0) + \xi G(\beta, \alpha, 0) \right) \delta(x - \beta - \alpha \xi)
\]

GPD 3D-plot (t=0)
Non-forward pion GPD

\[ H(x, \xi, 0) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \ (F(\beta, \alpha, 0) + \xi G(\beta, \alpha, 0)) \delta(x - \beta - \alpha \xi) \]

GPD \ (t=0, \xi = 1)
Non-forward pion GPD

\[ H(x, \xi, 0) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \left( F(\beta, \alpha, 0) + \xi G(\beta, \alpha, 0) \right) \delta(x - \beta - \alpha \xi) \]

GPD \ (t=0, \xi = 1)

Problems at large $\xi$!!!

AVWT Identity $\Rightarrow$ Soft pion theorem [C. Mezrag et al., arXiv:1411:6634]
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

The pion GPD

$$H^q(x, 0, t) = \int_{-1+|x|}^{1-|x|} d\alpha \ (F^0(x, \alpha, t) + F^{BC}(x, \alpha, t))$$
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

The pion GPD

\[ H^q(x, 0, t) = \int_{-1+|x|}^{1-|x|} d\alpha \left( F^0(x, \alpha, t) + F^{BC}(x, \alpha, t) \right) \]

\[ H(x, 0, t) = H(x, 0, 0) N(t) C_\pi(x,t) F_\pi(t), \quad F(\beta, \alpha, t) = \frac{1}{\left(1 + \frac{t}{4M^2} (1 - \beta + \alpha)(1 - \beta + \alpha)\right)^2 \times (F_S(\beta, \alpha) + t \cdots)} \]

1 \[ = \ N(t) \int_{-1}^{1} dx \ H(x, 0, 0) C_\pi(x, t). \]
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

The pion GPD

$$H^q(x, 0, t) = \int_{-1+|x|}^{1-|x|} d\alpha \left( F^0(x, \alpha, t) + F^{BC}(x, \alpha, t) \right)$$

$$H(x, 0, t) = H(x, 0, 0) N(t) C_\pi(x, t) F_\pi(t), \quad F(\beta, \alpha, t) = \frac{1}{\left(1 + \frac{t}{4M^2}(1 - \beta)(1 - \beta)\right)^2} \times F_S(\beta, \alpha)$$

Simplified analytical model:

$$C(x, t) = \frac{1}{\left(1 + \frac{t}{4M^2}(1 - x)^2\right)^2}$$

Valuable to sketch the pion’s valence-quark GPD [C. Mezrag et al., arXiv:1411.6634]
Non-forward (non-skewed) pion GPD ($\xi = 0, \, t \neq 0$)

3D plot of GPD at $\zeta = 0.4$ GeV

$M = 0.4$ GeV

$H(x,0,t)$

$0.0$ $0.5$ $1.0$ $1.5$

$0.0$ $0.5$ $1.0$ $1.5$

$0.0$ $5$ $10$ $15$

$t/M^2$
Non-forward (non-skewed) pion GPD ($\xi = 0$, $t \neq 0$)

3D plot of GPD at $\zeta = 2$ GeV (DGLAP running; $x > \xi$)

$M = 0.4$ GeV
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

3D plot of GPD at $\zeta = 2$ GeV (DGLAP running; $x > \xi$)

$M = 0.4$ GeV

$H(x, 0, t; \zeta/\Lambda_{QCD} \to \infty) = \delta(x)F_\pi(t)$
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

$$q(x, |\vec{b}|) = \int \frac{d|\Delta_{\perp}|}{2\pi} |\Delta_{\perp}| J_0(|\vec{b}_{\perp}| |\Delta_{\perp}|) H(x, 0, -\Delta_{\perp}^2)$$

Impact parameter space GPD at $\zeta = 0.4$ GeV

$M = 0.4$ GeV
Non-forward (non-skewed) pion GPD ($\xi = 0$, $t \neq 0$)

$$q(x, |b|) = \int \frac{d|\vec{\Delta}_\perp|}{2\pi} |\vec{\Delta}_\perp| J_0(|b_\perp||\vec{\Delta}_\perp|)H(x, 0, -\Delta^2_\perp)$$

Impact parameter space GPD at $\zeta = 2$ GeV

$M = 0.4$ GeV
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

$$q(x, |\vec{b}|) = \int \frac{d|\vec{\Delta}_{\perp}|}{2\pi} |\vec{\Delta}_{\perp}| J_0(|\vec{b}_{\perp}| |\vec{\Delta}_{\perp}|) H(x, 0, -\Delta^2_{\perp})$$

Impact parameter space GPD at $\zeta = 2$ GeV

$M = 0.4$ GeV

The peak of probability, at $|\vec{b}_{\perp}| = 0$, drifts to $x = 0$, its height is diminished and the distribution in $|\vec{b}_{\perp}|$ broadens.
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

$$q(x, |\vec{b}|) = \int \frac{d|\vec{\Delta}_\perp|}{2\pi} |\vec{\Delta}_\perp| J_0(|\vec{b}_\perp||\vec{\Delta}_\perp|) H(x, 0, -\Delta^2_\perp)$$

$$\langle |\vec{b}_\perp|^2 \rangle = \int_{-1}^{1} dx \langle |\vec{b}_\perp(x; \zeta)|^2 \rangle = \int_{-1}^{1} dx \int_{0}^{\infty} d|\vec{b}_\perp| |\vec{b}_\perp|^3 \int_{0}^{\infty} d\Delta \Delta J_0(\vec{b}_\perp|\Delta) F_{\pi}(\Delta^2)$$

Impact parameter space GPD

$$\langle |\vec{b}_\perp|^2 \rangle = (0.52 \text{ fm})^2$$

$\zeta = 2 \text{ GeV}; \ \zeta = 0.4 \text{ GeV}; \ \zeta = 0.4 \text{ GeV} \ [c(x,t)=1]$. X
Non-forward (non-skewed) pion GPD ($\xi = 0, \ t \neq 0$)

\[ q(x, |\vec{b}|) = \int \frac{d|\vec{\Delta}|}{2\pi} |\vec{\Delta}| J_0(|\vec{b}| \cdot |\vec{\Delta}|) H(x, 0, -\Delta^2) \]

\[ \langle |\vec{b}_{\perp}|^2 \rangle = \int_{-1}^{1} dx \langle |\vec{b}_{\perp}(x; \zeta)|^2 \rangle = \int_{-1}^{1} dx \int_{0}^{\infty} d|\vec{b}_{\perp}| |\vec{b}_{\perp}|^3 \int_{0}^{\infty} d\Delta \Delta J_0(\vec{b}_{\perp} \cdot \Delta) F_\pi(\Delta^2) \]

Impact parameter space GPD

\[ r_\pi = \sqrt{3/2 \langle |\vec{b}_{\perp}|^2 \rangle} = 0.674 \text{ fm} \iff r_\pi = 0.672(8) \text{ fm} \ [PRD86(2012)010001] \]

\[ \zeta = 2 \text{ GeV}; \ \zeta = 0.4 \text{ GeV}; \ \zeta = 0.4 \text{GeV} \ [c(x,t)=1]. \]
We described a calculation of the pion’s valence dressed-quark GPD within the context of a RL-truncation of QCD Dyson-Schwinger equations and, in the appropriate limits, coupled the results with the pion’s PDF and form factor. Drawing analogy with the pion’s valence dressed-quark PDF, we argued that the impulse-approximation invoked to derive the triangle-diagram GPD computation from the well-known handbag diagram contribution to DVCS gives incomplete results. A correction valid in the neighbourhood of $\xi = 0$ and small $t$ is applied and we built a model for the non-skewed pion’s valence dressed-quark GPD as the Radon transform of a single amplitude (DD) which, as is consistent with significantly more known constraints than impulse-approximation’s, left us with a practicable improvement. We computed the impact-parameter-space GPD and provided with a sound picture of the dressed-quark structure of the pion. All the features resulting from running the results with leading-order DGLAP equations from a hadronic to a larger scale may be qualitatively understood. In the future, more realistic forms for the dressed propagators and vertices and better extensions to the entire kinematic domain of $\xi$ and $t$ may be potentially helpful to relate the phenomenology of hadron GPDs to the properties of QCD.
We described a calculation of the pion’s valence dressed-quark GPD within the context of a RL-truncation of QCD Dyson-Schwinger equations and, in the appropriate limits, coupled the results with the pion’s PDF and form factor.

Drawing analogy with the pion’s valence dressed-quark PDF, we argued that the impulse-approximation invoked to derive the triangle-diagram GPD computation from the well-known handbag diagram contribution to DVCS gives incomplete results.

A correction valid in the neighbourhood of $\xi = 0$ and small $t$ is applied and we built a model for the non-skewed pion’s valence dressed-quark GPD as the Radon transform of a single amplitude (DD) which, as is consistent with significantly more known constraints than impulse-approximation’s, left us with a practicable improvement.

We computed the impact-parameter-space GPD and provided with a sound picture of the dressed-quark structure of the pion. All the features resulting from running the results with leading-order DGLAP equations from a hadronic to a larger scale may be qualitatively understood.

In the future, more realistic forms for the dressed propagators and vertices and better extensions to the entire kinematic domain of $\xi$ and $t$ may be potentially helpful to relate the phenomenology of hadron GPDs to the properties of QCD.
Epilogue: conclusion and perspectives

- We described a calculation of the pion’s valence dressed-quark GPD within the context of a RL-truncation of QCD Dyson-Schwinger equations and, in the appropriate limits, coupled the results with the pion’s PDF and form factor.

- Drawing analogy with the pion’s valence dressed-quark PDF, we argued that the impulse-approximation invoked to derive the triangle-diagram GPD computation from the well-known handbag diagram contribution to DVCS gives incomplete results.

- A correction valid in the neighbourhood of $\xi = 0$ and small $t$ is applied and we built a model for the non-skewed pion’s valence dressed-quark GPD as the Radon transform of a single amplitude (DD) which, as is consistent with significantly more known constraints than impulse-approximation’s, left us with a practicable improvement.
We described a calculation of the pion’s valence dressed-quark GPD within the context of a RL-truncation of QCD Dyson-Schwinger equations and, in the appropriate limits, coupled the results with the pion’s PDF and form factor.

Drawing analogy with the pion’s valence dressed-quark PDF, we argued that the impulse-approximation invoked to derive the triangle-diagram GPD computation from the well-known handbag diagram contribution to DVCS gives incomplete results.

A correction valid in the neighbourhood of $\xi = 0$ and small $t$ is applied and we built a model for the non-skewed pion’s valence dressed-quark GPD as the Radon transform of a single amplitude (DD) which, as is consistent with significantly more known constraints than impulse-approximation’s, left us with a practicable improvement.

We computed the impact-parameter-space GPD and provided with a sound picture of the dressed-quark structure of the pion. All the features resulting from running the results with leading-order DGLAP equations from a hadronic to a larger scale may be qualitatively understood.

In the future, more realistic forms for the dressed propagators and vertices and better extensions to the entire kinematic domain of $\xi$ and $t$ may be potentially helpful to relate the phenomenology of hadron GPDs to the properties of QCD.
Epilogue: conclusion and perspectives

- We described a calculation of the pion’s valence dressed-quark GPD within the context of a RL-truncation of QCD Dyson-Schwinger equations and, in the appropriate limits, coupled the results with the pion’s PDF and form factor.
- Drawing analogy with the pion’s valence dressed-quark PDF, we argued that the impulse-approximation invoked to derive the triangle-diagram GPD computation from the well-known handbag diagram contribution to DVCS gives incomplete results.
- A correction valid in the neighbourhood of $\xi = 0$ and small $t$ is applied and we built a model for the non-skewed pion’s valence dressed-quark GPD as the Radon transform of a single amplitude (DD) which, as is consistent with significantly more known constraints than impulse-approximation’s, left us with a practicable improvement.
- We computed the impact-parameter-space GPD and provided with a sound picture of the dressed-quark structure of the pion. All the features resulting from running the results with leading-order DGLAP equations from a hadronic to a larger scale may be qualitatively understood.
- In the future, more realistic forms for the dressed propagators and vertices and better extensions to the entire kinematic domain of $\xi$ and $t$ may be potentially helpful to relate the phenomenology of hadron GPDs to the properties of QCD.
Thank you