Time-reversal, loop-antiloop symmetry and the Bessel equation*

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- the Bessel equation can be cast, by means of suitable transformations, into a system of two parametric oscillator equations, one for a damped oscillator, the other one for an amplified one.

- the group contraction mechanism

is involved in such a relation of the Bessel equation with the dissipation/amplification system,

it introduces the breakdown of the loop-antiloop symmetry around a preferred axis,

- this can be read off, in a given re-parametrization, as the breakdown of time-reversal symmetry,

- relation between infinite dimensional loop-algebras, such as the Virasoro-like algebra, and the Euclidean algebras $e(2)$ and $e(3)$.

motivations:

• Special functions!...

Wigner*: "the role which is common to all special functions is to be matrix elements of representations of the simplest Lie groups".

• growing interest in the couple of damped/amplified oscillators

- dissipation at classical and quantum level
- inflationary models of the Universe
- thermal field theories
- Chern-Simons gauge theory
- Bloch electrons in metals
- the dissipative quantum model of brain
- fractal self-similarity

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- **Preliminaries**

Bessel functions describe solutions with different Pontryagin number in the punctured plane $R^2/(0)$,

the elements of the homotopy group, Π_n, represented by differential operators acting on analytic functions:

$$\Pi_n \equiv \frac{\partial^n}{\partial z^n} \;, \qquad n \in N \;, \tag{1}$$

with $\Pi_n \cdot \Pi_m \;=\; \Pi_{n+m}$,

n is the loop number around the hole.

Two different kinds of behaviors:

$$\frac{\partial^n}{z\partial z^n}\,\varphi_m(z) \;=\; (-)^m\,\varphi_{m+n}(z)\;, \qquad \varphi_m(z) = \frac{J_m(z)}{z^m} \tag{2}$$

and

$$\frac{\partial^n}{z\partial z^n}\,\psi_m(z) \;=\; \psi_{m-n}(z)\;, \qquad \psi_m(z) = z^m J_m(z)\;, \tag{3}$$

on φ, Π_n acts in counter-clockwise way, on ψ in clockwise way.

$J_m(z)$ **is the planar Bessel function (Bessel function of integer order).**

Eqs.(2) and (3) are the differential formulae for the planar Bessel functions; analogous formulae are true for the functions

$$\varphi_m(z) = j_m(z) z^{-m}\;, \qquad \psi_m(z) = z^{(m+1)} j_m(z)\;, \tag{4}$$

where the j_m are the spherical Bessel functions.

We will relate below to these topological properties of the φ and ψ functions (cf. the "x_3- and x_4-reversal symmetry breakdown" discussed in the following).

To start, consider the spherical Bessel equation of order n (also called of fractional order):

$$\eta^2 \; J_{n;\eta\eta} + 2\eta \; J_{n;\eta} + [\eta^2 - n(n+1)] \; J_n \;\; = \;\; 0 \; . \tag{5}$$

n is an integer or zero number, $(n = 0, \pm 1, \pm 2, ...)$ and "$;\eta$". "$;\eta\eta$" denote first and second order derivatives, respectively.

the solutions of Eq. (5), so called spherical Bessel functions,
- constitute a complete set of (parametric) decaying functions,

- can be expressed in terms of the first and second kind Bessel functions and their linear combinations (the Hankel functions).

Eq. (5) is invariant under the transformation $n \rightarrow -(n+1)$.

J_n and $J_{-(n+1)}$ are both solutions of the same equation:
are degenerate solutions corresponding to the same eigenvalue $n(n+1)$ of the operator $\eta^2 \; \frac{d^2}{d\eta^2} + 2\eta \; \frac{d}{d\eta} + \eta^2$.

in Eq.(5) change of variables : $\eta \to \eta \equiv \epsilon x$ **with** $x \equiv e^{-t/\alpha}$
ϵ **and** α**: arbitrary parameters**
t **may be thought to denote, e.g., the time variable.**

Put $w_{n,l} \equiv J_n \ \cdot (x)^{-l}$**, Eq.(5) then becomes:**

$$\ddot{w}_{n,l} \ - \ \frac{2l+1}{\alpha} \ \dot{w}_{n,l} + \left[\frac{l(l+1)-n(n+1)}{\alpha^2} \ + \ (\frac{\epsilon}{\alpha})^2 \ e^{-2t/\alpha}\right] \ w_{n,l} \ = \ 0, \quad (6)$$

$\dot{w}$ **denotes derivative of** w **with respect to time** t**.**

the degeneracy between J_n **and** $J_{-(n+1)}$ **is removed by putting** $l(l+1) = n(n+1)$**,**

thus,a *partition* **is induced between the two solution sectors** $\{J_n\}$ **and** $\{J_{-(n+1)}\}$**: two** *different* **sets of equations are obtained, one for** $w_{n,l}$**, the other one for** $w_{-(n+1),l}$**, respectively.**

The set for $w_{n,l}$ is

$$\ddot{w}_{n,-(n+1)} + \frac{2n+1}{\alpha} \dot{w}_{n,-(n+1)} + [(\frac{\epsilon}{\alpha})^2 e^{-2t/\alpha}] w_{n,-(n+1)} = 0,$$

$$\ddot{w}_{n,n} - \frac{2n+1}{\alpha} \dot{w}_{n,n} + [(\frac{\epsilon}{\alpha})^2 e^{-2t/\alpha}] w_{n,n} = 0 ,$$

for $l = -(n+1)$ and for $l = n$, respectively.
Similarly, two equations for $w_{-(n+1),l}$ are obtained:

$$\ddot{w}_{-(n+1),-(n+1)} + \frac{2n+1}{\alpha} \dot{w}_{-(n+1),-(n+1)} + [(\frac{\epsilon}{\alpha})^2 e^{-2t/\alpha}] w_{-(n+1),-(n+1)} = 0,$$

$$\ddot{w}_{-(n+1),n} - \frac{2n+1}{\alpha} \dot{w}_{-(n+1),n} + [(\frac{\epsilon}{\alpha})^2 e^{-2t/\alpha}] w_{-(n+1),n} = 0,$$

for $l = -(n+1)$ and $l = n$, respectively.

Inspection of these equations shows that the symmetry under the transformation $n \to -(n+1)$ has been broken.

We choose the arbitrary parameters α and ϵ to be n-dependent: $\alpha \to \alpha_n$ and $\epsilon \to \epsilon_n$, which means that $\eta \to \eta_n \equiv \epsilon_n x_n$ with $x_n \equiv e^{-t/\alpha_n}$.

so that $\frac{2n+1}{\alpha_n} \equiv L$ and $\frac{\epsilon_n}{\alpha_n} \equiv \omega_0$ do not depend on n (and on time).

By setting $u_n \equiv w_{n,-(n+1)}$, and $v_n \equiv w_{n,n}$, the equations for $w_{n,l}$ are recognized to be nothing else than the equations for the damped/amplified parametric oscillators:

$$\begin{aligned} \ddot{u}_n + L\, \dot{u}_n + {\omega_n}^2(t) u_n &= 0, \\ \ddot{v}_n - L\, \dot{v}_n + {\omega_n}^2(t) v_n &= 0, \end{aligned} \qquad (7)$$

with frequency

$$\omega_n(t) = \omega_0 \, e^{-\frac{Lt}{2n+1}}. \qquad (8)$$

Eqs. (7) are sometimes called Hill-type equations.

Remarkably, the first of Eqs. (7), with $n = 1$ is commonly used in expanding geometry (inflationary) models of the Universe. In that case L denotes the Hubble constant.

• $\omega_n(t) \to \omega_0$, which is time-independent, for $n \to \infty$:
the frequency time-dependence is thus "graded" by the order n of the original Bessel equation.

• L and ω_0, which may be arbitrarily chosen, are characteristic parameters of the oscillator system.

Note that the choice of keeping L independent of n implies that $\alpha_{-(n+1)} = -\alpha_n$.

Then the transformation $n \to -(n+1)$ leads to solutions (corresponding to $J_{-(n+1)}$) which have frequencies exponentially increasing in time (cf. Eq. (8)).

These solutions can be respectively obtained from the ones of Eqs. (7) by time-reversal $t \to -t$ and exchanging u with v ("charge conjugation").

In the large n limit ($\omega_n \to \omega_0$) u_n and v_n are each the time-reversed of the other one and in that limit the two sectors $\{J_i\}$, $i = n, -(n+1)$ are mapped one into the other one.

• **We thus recognize the core of the relation between the spherical Bessel equation and the dissipation/amplification phenomenon:**

breakdown of the $n \to -(n+1)$ symmetry $\Leftrightarrow$ breakdown of time-reversal symmetry (the emergence of the *arrow of time*) in the manifold of the solutions (the spherical Bessel functions) $\{J_i\}$, $i = n, -(n+1)$.

Similar results can be obtained for the to the planar Bessel equation of order n:

$$\eta^2 J_{n;\eta\eta} + \eta \; J_{n;\eta} + [\eta^2 - n^2] \; J_n \;=\; 0, \tag{9}$$

$\pm$ values of n $\Leftrightarrow$ positive/negative rotations (loop/antiloop) around the x_3 axis,
i.e., they correspond to different orientations of the x_3 axis.
The related solutions are, therefore, different.

- **well known: the representations of the Euclidean groups $E(2)$ and $E(3)$ can be constructed in terms of the planar and spherical Bessel functions, respectively**

The root of the above symmetry breakdown features is in the structure of the $E(2)$ and $E(3)$ group (the Euclidean group in 3 and 4 dimensions, respectively):

- **The (simpler) case of $E(2)$ and of the planar Bessel equations.**

$E(2)$ is the group of the $T(\mathrm{v})R(\theta)$ transformations,
the group contraction of $SO(3)$,

$T(\mathrm{v})$: the translation in the plane by the vector v ($\equiv (a,b)$).
$R(\theta)$: the rotation of the plane around the origin by the angle θ.

The associated Lie algebra: two translation generators P_a, P_b and of the rotation generator M:

$$[P_a, P_b] = 0, \quad [P_a, M] = -P_b, \quad [P_b, M] = P_a. \tag{10}$$

The invariant operator of $E(2)$ is $P^2 = P_a^2 + P_b^2 = P_+ \ P_- = P_- \ P_+$, with $P_\pm \equiv P_a \pm iP_b$, which has non-positive eigenvalue, $-p^2$.

In order to study the P^2 eigenvalue equation, consider the 3D-Laplace equation $\nabla^2\psi = 0$,
which refers to an isotropic and homogeneous 3D-space.

Cylindrical coordinates (instead of the spherical or rectangular ones) $\rightarrow$ breakdown of the symmetry of 3D-spatial rotation group $SO(3)$: x_3 is differently treated with respect to the two remaining coordinates and this singles out a privileged axis for rotations.

Search for solutions of the 2D-Helmholtz equation of the type:

$$\psi(r,\theta,x_3) = \varphi(r,\theta)\cdot\sigma(x_3), \qquad \frac{\partial^2}{\partial x_3^2}\sigma(x_3) \equiv p^2\sigma(x_3). \tag{11}$$

for positive x_3, the solution: $\sigma = e^{-x_3 p}$
for negative x_3 the solution: $\sigma = e^{x_3 p}$

Let $\varphi(r,\theta) \equiv f(r)\cdot e^{in\theta}$, we obtain $f(r) = J_n(pr)$, being $J_n(pr)$ the solution of the planar Bessel equation of order n, with $\eta = pr$:

$$\eta^2 J_{n;\eta\eta} + \eta\ J_{n;\eta} + [\eta^2 - n^2]\ J_n \ = \ 0, \tag{12}$$

breakdown of the rotational symmetry of $SO(3)$ $\rightarrow$ contraction to $E(2)$ $\rightarrow$ difference (breakdown of loop-antiloop symmetry) in the double choice of the x_3 axis orientation: the *mirror* index $\pm n$ of the Bessel functions is associated to the *couple* of damped/amplified harmonic oscillators. It is a time-mirror index.

Note

the $SO(3)$ contraction to $E(2)$ manifests itself in local observations. However, in the local observation process the x_3 axis orientation is "locked"

$\rightarrow$ loss of symmetry under $n \rightarrow -n$ **=** breakdown of loop-antiloop symmetry.

Specifying the direction of the x_3 axis (choosing one of the two possible forms for σ) produces topologically inequivalent configurations.

• **The case of $E(3)$ and of the spherical Bessel equation: similar analysis and similar results.**

$E(3)$, which is the group contraction of SO(4), has six generators P_i (translations) and M_i (rotations), $i = 1, 2, 3$:

$$[P_i, P_j] = 0, \quad [M_i, M_j] = \epsilon_{ijk} M_k, \quad [P_i, M_j] = \epsilon_{ijk} P_k; \tag{13}$$

The $SO(3)$ subgroup generated by the $M_i' s$ is left unchanged in the contraction process. The algebra $e(3)$ has two invariants, $P^2 = \Sigma P_i^2$ and $\Sigma P_i \cdot M_i$.

Now, one may search for solutions of the type $\psi(x_1, x_2, x_3, x_4) = \varphi(r, \theta, \phi) \cdot \sigma(x_4)$, ($r, \theta, \phi$ spherical coordinates, $\sigma = e^{\pm x_4 p}$) x_4 may be considered to play the role of time t.

The resulting equation is solved by the function $\varphi = Y_{n,m}(\theta, \phi) \cdot J_n(pr)$ where $Y_{n,m}$ is the spherical harmonics and J_n is the solution of the spherical Bessel equation.

Again, the breakdown of the symmetry under the transformation $n \to -(n+1)$ is built in in the geometrical structure of the $E(3)$ group:

such a symmetry breakdown is nothing but the breakdown of the x_4 axis reversal symmetry (breakdown of the loop-antiloop symmetry), i.e. of time-reversal symmetry when x_4 plays the role of time variable.

Also in the present case, the $SO(4)$ contraction to $E(3)$ manifests itself in local observations and the x_4 axis orientation then gets "locked".

• **The loop-antiloop symmetry breakdown suggests to us to investigate the relation between the Euclidean groups and the loop algebras.**

For the case of the Virasoro algebra, which plays a central role in the conformal field theories, this goes as follows.

The Virasoro algebra $\mathcal{L}$ **of central charge** c **(**$[c,T]=0$ **for all the** T**'s):**

$$[T_n, T_m] = (n-m)T_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \ , \qquad m,n \ \in \mathbf{Z} \ . \tag{14}$$

The Z$_2$**-grading: divide the** T_n **into an even set** $L_0 \equiv \{A_n, c\}$ **and an odd set** $L_1 \equiv \{B_n\}$

$$A_n = \frac{1}{2}\left(T_{2n} + \frac{c}{8}\delta_{n,0}\right) \ , \qquad B_n = \frac{1}{2}T_{2n+1} \ , \tag{15}$$

so that $\mathcal{L} = L_0 \oplus L_1$ **and** $[L_0, L_0] \subseteq L_0 \ , \quad [L_0, L_1] \subseteq L_1 \ , \quad [L_1, L_1] \subseteq L_0 \ .$

Explicitly, the commutation relations of the graded generators:

$$[A_n, A_m] = (n-m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0} \ , \qquad (16)$$

$$[B_n, B_m] = (n-m)A_{n+m+1} + \frac{2c}{12}(n-\frac{1}{2})(n+\frac{1}{2})(n+\frac{3}{2})\delta_{n+m+1,0} \ , \qquad (17)$$

$$[A_n, B_m] = (n-m-\frac{1}{2})B_{n+m} \ . \qquad (18)$$

$\{A_n, c\}$ **is again a Virasoro algebra but with central charge** $2c$**.**

Consider then the Z_2**-graded contraction:**

$$[A_n, A_m] = (n-m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0} \ , \qquad (19)$$

$$[B_n, B_m] = 0 \ , \qquad (20)$$

$$[A_n, B_m] = (n-m-\frac{1}{2})B_{n+m} \ . \qquad (21)$$

Remark 1: in the centerless case ($c = 0$), the A_0 and $A_{\pm 1}$ generators close the algebra isomorphic to $so(3) \sim su(2)$

Remark 2: these generators and the operators $B_{-\frac{1}{2}}$, $B_{\frac{1}{2}}$ and $B_{-\frac{3}{2}}$ close the $e(3)$ isomorphic algebra.

This is shown by setting:

$$\begin{aligned} M_+ \equiv \ A_1 \ , \quad M_- \equiv \ A_{-1} \ , \quad M_3 \equiv \ iA_0 \ , \\ P_+ \equiv \ B_{\frac{1}{2}}, \quad P_- \equiv \ B_{-\frac{3}{2}} \ , \quad P_3 \equiv \ iB_{-\frac{1}{2}} \ , \end{aligned} \tag{22}$$

where the Ms and Ps satisfy the $e(3)$ commutation relations (13).

General extension of this result:
the algebra $\mathcal{E}_n \equiv \{A_0, A_{\pm n}\} \bigoplus \{B_{-\frac{1}{2}}, B_{\pm n-\frac{1}{2}}\}$ reproduces the $e(3)$ algebra for each integer value of n, provided the following positions are assumed:

$$\begin{aligned} M_+ \equiv \ \frac{1}{n}A_n \ , \quad M_- \equiv \frac{1}{n}A_{-n} \ , \quad M_3 \equiv \ \frac{i}{n}A_0 \ , \\ P_+ \equiv \ B_{n-\frac{1}{2}} \ , \quad P_- \equiv \ B_{-n-\frac{1}{2}} \ , \quad P_3 \equiv \ iB_{-\frac{1}{2}} \ . \end{aligned} \tag{23}$$

As final remark we notice that the $e(2)$-algebra can be obtained as a subalgebra of (23) by choosing $A_{\pm n} = 0$, for non-zero values of n.

The conclusion is that the extension of the Virasoro algebra by means of its Z_2-grading with the subsequent step of the Z_2-graded contraction appears as a n-graded hierarchy of Euclidean algebras.

This establish the relation between the couple of damped/amplified parametric oscillators graded by n and the loop algebras here considered.

Summing up,

the relation between the Bessel equation and the dissipation/amplificatio processes has been shown.

The breakdown of the $n \to -(n+1)$ ($n \to -n$) symmetry of the spherical (planar) Bessel equation may be represented as the breakdown of time-reversal symmetry in the manifold of the solutions (the Bessel functions).

In connection with the loop-antiloop symmetry of the Bessel equation, a n-graded hierarchy of Euclidean algebras appears as the extension of the Virasoro algebra.