Time-reversal, loop-antiloop symmetry and the Bessel equation*

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Fractals, coherent states and self-similarity induced noncommutative geometry, Phys. Lett. A 376, 2527 (2012);
• the Bessel equation can be cast, by means of suitable transformations, into a system of two parametric oscillator equations, one for a damped oscillator, the other one for an amplified one.

• the group contraction mechanism is involved in such a relation of the Bessel equation with the dissipation/amplification system,

it introduces the breakdown of the loop-antiloop symmetry around a preferred axis,

• this can be read off, in a given re-parametrization, as the breakdown of time-reversal symmetry,

• relation between infinite dimensional loop-algebras, such as the Virasoro-like algebra, and the Euclidean algebras $e(2)$ and $e(3)$.
motivations:

- Special functions!...

  Wigner*: ”the role which is common to all special functions is to be matrix elements of representations of the simplest Lie groups”.

- growing interest in the couple of damped/amplified oscillators
  - dissipation at classical and quantum level
  - inflationary models of the Universe
  - thermal field theories
  - Chern-Simons gauge theory
  - Bloch electrons in metals
  - the dissipative quantum model of brain
  - fractal self-similarity

*J.D. Talman, Special Functions. A group theoretical approach, (Benjamin, New York, 1968)
Preliminaries

Bessel functions describe solutions with different Pontryagin number in the punctured plane $\mathbb{R}^2/(0)$,

the elements of the homotopy group, $\Pi_n$, represented by differential operators acting on analytic functions:

$$\Pi_n \equiv \frac{\partial^n}{\partial z^n}, \quad n \in \mathbb{N},$$

with $\Pi_n \cdot \Pi_m = \Pi_{n+m}$,

$n$ is the loop number around the hole.
Two different kinds of behaviors:

\[
\frac{\partial^n}{z \partial z^n} \varphi_m(z) = (-)^m \varphi_{m+n}(z) , \quad \varphi_m(z) = \frac{J_m(z)}{z^m}
\]  \hspace{1cm} (2)

and

\[
\frac{\partial^n}{z \partial z^n} \psi_m(z) = \psi_{m-n}(z) , \quad \psi_m(z) = z^m J_m(z)
\]  \hspace{1cm} (3)

on \( \varphi \), \( \Pi_n \) acts in counter-clockwise way, on \( \psi \) in clockwise way.

\( J_m(z) \) is the planar Bessel function (Bessel function of integer order).

Eqs.(2) and (3) are the differential formulae for the planar Bessel functions; analogous formulae are true for the functions

\[
\varphi_m(z) = j_m(z) z^{-m} , \quad \psi_m(z) = z^{(m+1)} j_m(z)
\]  \hspace{1cm} (4)

where the \( j_m \) are the spherical Bessel functions.
We will relate below to these topological properties of the $\varphi$ and $\psi$ functions (cf. the “$x_3$- and $x_4$-reversal symmetry breakdown” discussed in the following).
To start, consider the spherical Bessel equation of order \( n \) (also called of fractional order):

\[
\eta^2 J_{n;\eta\eta} + 2\eta J_{n;\eta} + [\eta^2 - n(n + 1)] J_n = 0. 
\]  
(5)

\( n \) is an integer or zero number, \((n = 0, \pm 1, \pm 2, \ldots)\) and “; \( \eta \)”. “; \( \eta\eta \)” denote first and second order derivatives, respectively.

the solutions of Eq. (5), so called spherical Bessel functions,  
- constitute a complete set of (parametric) decaying functions,  
- can be expressed in terms of the first and second kind Bessel functions and their linear combinations (the Hankel functions).

Eq. (5) is invariant under the transformation \( n \to -(n + 1) \).

\( J_n \) and \( J_{-(n+1)} \) are both solutions of the same equation:  
are degenerate solutions corresponding to the same eigenvalue \( n(n+1) \) of the operator \( \eta^2 \frac{d^2}{d\eta^2} + 2\eta \frac{d}{d\eta} + \eta^2 \).
in Eq. (5) change of variables: \( \eta \rightarrow \eta \equiv \epsilon x \) with \( x \equiv e^{-t/\alpha} \)

\( \epsilon \) and \( \alpha \): arbitrary parameters

\( t \) may be thought to denote, e.g., the time variable.

Put \( w_{n,l} \equiv J_n \cdot (x)^{-l} \), Eq. (5) then becomes:

\[
\begin{align*}
\ddot{w}_{n,l} - \frac{2l + 1}{\alpha} \dot{w}_{n,l} + \left[ \frac{l(l + 1) - n(n + 1)}{\alpha^2} + \left( \frac{\epsilon}{\alpha} \right)^2 e^{-2t/\alpha} \right] w_{n,l} &= 0, \quad (6)
\end{align*}
\]

\( \dot{w} \) denotes derivative of \( w \) with respect to time \( t \).

the degeneracy between \( J_n \) and \( J_-(n+1) \) is removed by putting \( l(l+1) = n(n + 1) \),

thus, a partition is induced between the two solution sectors \( \{J_n\} \) and \( \{J_-(n+1)\} \): two different sets of equations are obtained, one for \( w_{n,l} \), the other one for \( w_{-(n+1),l} \), respectively.
The set for $w_{n,l}$ is

$$\ddot{w}_{n,-(n+1)} + \frac{2n+1}{\alpha} \dot{w}_{n,-(n+1)} + [\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}] w_{n,-(n+1)} = 0,$$

$$\ddot{w}_{n,n} - \frac{2n+1}{\alpha} \dot{w}_{n,n} + [\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}] w_{n,n} = 0,$$

for $l = -(n+1)$ and for $l = n$, respectively.

Similarly, two equations for $w_{-(n+1),l}$ are obtained:

$$\ddot{w}_{-(n+1),-(n+1)} + \frac{2n+1}{\alpha} \dot{w}_{-(n+1),-(n+1)} + [\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}] w_{-(n+1),-(n+1)} = 0,$$

$$\ddot{w}_{-(n+1),n} - \frac{2n+1}{\alpha} \dot{w}_{-(n+1),n} + [\left(\frac{\epsilon}{\alpha}\right)^2 e^{-2t/\alpha}] w_{-(n+1),n} = 0,$$

for $l = -(n+1)$ and $l = n$, respectively.

Inspection of these equations shows that the symmetry under the transformation $n \rightarrow -(n+1)$ has been broken.
We choose the arbitrary parameters $\alpha$ and $\epsilon$ to be $n$-dependent: $\alpha \to \alpha_n$ and $\epsilon \to \epsilon_n$, which means that $\eta \to \eta_n \equiv \epsilon_n x_n$ with $x_n \equiv e^{-t/\alpha_n}$.

so that $\frac{2n+1}{\alpha_n} \equiv L$ and $\frac{\epsilon_n}{\alpha_n} \equiv \omega_0$ do not depend on $n$ (and on time).

By setting $u_n \equiv w_{n,-(n+1)}$, and $v_n \equiv w_{n,n}$, the equations for $w_{n,l}$ are recognized to be nothing else than the equations for the damped/amplified parametric oscillators:

\[
\begin{align*}
\ddot{u}_n + L \dot{u}_n + \omega_n^2(t) u_n &= 0, \\
\ddot{v}_n - L \dot{v}_n + \omega_n^2(t) v_n &= 0,
\end{align*}
\]

with frequency

\[
\omega_n(t) = \omega_0 e^{-\frac{Lt}{2n+1}}.
\]

Eqs. (7) are sometimes called Hill-type equations.

Remarkably, the first of Eqs. (7), with $n = 1$ is commonly used in expanding geometry (inflationary) models of the Universe. In that case $L$ denotes the Hubble constant.
• $\omega_n(t) \rightarrow \omega_0$, which is time-independent, for $n \rightarrow \infty$: the frequency time-dependence is thus "graded" by the order $n$ of the original Bessel equation.

• $L$ and $\omega_0$, which may be arbitrarily chosen, are characteristic parameters of the oscillator system.

Note that the choice of keeping $L$ independent of $n$ implies that $\alpha_{-(n+1)} = -\alpha_n$.

Then the transformation $n \rightarrow -(n+1)$ leads to solutions (corresponding to $J_{-(n+1)}$) which have frequencies exponentially increasing in time (cf. Eq. (8)).

These solutions can be respectively obtained from the ones of Eqs. (7) by time-reversal $t \rightarrow -t$ and exchanging $u$ with $v$ ("charge conjugation").
In the large $n$ limit ($\omega_n \to \omega_0$) $u_n$ and $v_n$ are each the time-reversed of the other one and in that limit the two sectors $\{J_i\}$, $i = n, -(n + 1)$ are mapped one into the other one.

- We thus recognize the core of the relation between the spherical Bessel equation and the dissipation/amplification phenomenon:

breakdown of the $n \to -(n+1)$ symmetry $\Leftrightarrow$ breakdown of time-reversal symmetry (the emergence of the arrow of time) in the manifold of the solutions (the spherical Bessel functions) $\{J_i\}$, $i = n, -(n + 1)$.

Similar results can be obtained for the to the planar Bessel equation of order $n$:

$$\eta^2 J_{n;\eta} + \eta J_{n;\eta} + [\eta^2 - n^2] \ J_n = 0,$$

(9)

$\pm$ values of $n$ $\Leftrightarrow$ positive/negative rotations (loop/antiloop) around the $x_3$ axis,
i.e., they correspond to different orientations of the $x_3$ axis.
The related solutions are, therefore, different.
• well known: the representations of the Euclidean groups $E(2)$ and $E(3)$ can be constructed in terms of the planar and spherical Bessel functions, respectively

The root of the above symmetry breakdown features is in the structure of the $E(2)$ and $E(3)$ group (the Euclidean group in 3 and 4 dimensions, respectively):

• The (simpler) case of $E(2)$ and of the planar Bessel equations.

$E(2)$ is the group of the $T(v)R(\theta)$ transformations, the group contraction of $SO(3),$

$T(v)$: the translation in the plane by the vector $v \equiv (a, b)).$
$R(\theta)$: the rotation of the plane around the origin by the angle $\theta.$
The associated Lie algebra: two translation generators $P_a, P_b$ and of the rotation generator $M$:

\[
[P_a, P_b] = 0, \quad [P_a, M] = -P_b, \quad [P_b, M] = P_a. \tag{10}
\]

The invariant operator of $E(2)$ is $P^2 = P_a^2 + P_b^2 = P_+ P_- = p^2$, with $P_\pm \equiv P_a \pm iP_b$, which has non-positive eigenvalue, $-p^2$.

In order to study the $P^2$ eigenvalue equation, consider the 3D-Laplace equation $\nabla^2 \psi = 0$, which refers to an isotropic and homogeneous 3D-space.

Cylindrical coordinates (instead of the spherical or rectangular ones) → breakdown of the symmetry of 3D-spatial rotation group $SO(3)$: $x_3$ is differently treated with respect to the two remaining coordinates and this singles out a privileged axis for rotations.
Search for solutions of the 2D-Helmholtz equation of the type:

\[ \psi(r, \theta, x_3) = \varphi(r, \theta) \cdot \sigma(x_3), \quad \frac{\partial^2}{\partial x_3^2} \sigma(x_3) \equiv p^2 \sigma(x_3). \]  \hspace{1cm} (11)

for positive \( x_3 \), the solution: \( \sigma = e^{-x_3 p} \)
for negative \( x_3 \) the solution: \( \sigma = e^{x_3 p} \)

Let \( \varphi(r, \theta) \equiv f(r) \cdot e^{i n \theta} \), we obtain \( f(r) = J_n(pr) \), being \( J_n(pr) \) the solution of the planar Bessel equation of order \( n \), with \( \eta = pr \):

\[ \eta^2 J_n;\eta \eta + \eta J_n;\eta + [\eta^2 - n^2] J_n = 0, \]  \hspace{1cm} (12)

breakdown of the rotational symmetry of \( SO(3) \rightarrow \) contraction to \( E(2) \rightarrow \) difference (breakdown of loop-antiloop symmetry) in the double choice of the \( x_3 \) axis orientation: the mirror index \( \pm n \) of the Bessel functions is associated to the \textit{couple} of damped/amplified harmonic oscillators. It is a time-mirror index.
Note

the $SO(3)$ contraction to $E(2)$ manifests itself in local observations. However, in the local observation process the $x_3$ axis orientation is "locked"

$\rightarrow$ loss of symmetry under $n \rightarrow -n = \text{breakdown of loop-antiloop symmetry.}$

Specifying the direction of the $x_3$ axis (choosing one of the two possible forms for $\sigma$) produces topologically inequivalent configurations.
• The case of $E(3)$ and of the spherical Bessel equation: similar analysis and similar results.

$E(3)$, which is the group contraction of $SO(4)$, has six generators $P_i$ (translations) and $M_i$ (rotations), $i = 1, 2, 3$:

$$[P_i, P_j] = 0, \quad [M_i, M_j] = \epsilon_{ijk} M_k, \quad [P_i, M_j] = \epsilon_{ijk} P_k; \quad (13)$$

The $SO(3)$ subgroup generated by the $M_i$'s is left unchanged in the contraction process. The algebra $e(3)$ has two invariants, $P^2 = \sum P_i^2$ and $\sum P_i \cdot M_i$.

Now, one may search for solutions of the type
$$\psi(x_1, x_2, x_3, x_4) = \varphi(r, \theta, \phi) \cdot \sigma(x_4), \quad (r, \theta, \phi \text{ spherical coordinates}, \quad \sigma = e^{\pm x_4 p})$$
x_4 may be considered to play the role of time $t$.

The resulting equation is solved by the function $\varphi = Y_{n,m}(\theta, \phi) \cdot J_n(pr)$ where $Y_{n,m}$ is the spherical harmonics and $J_n$ is the solution of the spherical Bessel equation.
Again, the breakdown of the symmetry under the transformation $n \rightarrow -(n + 1)$ is built in in the geometrical structure of the $E(3)$ group:

such a symmetry breakdown is nothing but the breakdown of the $x_4$ axis reversal symmetry (breakdown of the loop-antiloop symmetry), i.e. of time-reversal symmetry when $x_4$ plays the role of time variable.

Also in the present case, the $SO(4)$ contraction to $E(3)$ manifests itself in local observations and the $x_4$ axis orientation then gets "locked".
The loop-antiloop symmetry breakdown suggests to us to investigate the relation between the Euclidean groups and the loop algebras.

For the case of the Virasoro algebra, which plays a central role in the conformal field theories, this goes as follows.

The Virasoro algebra $L$ of central charge $c$ ($[c,T] = 0$ for all the $T$’s):

$$[T_n,T_m] = (n - m)T_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} , \quad m,n \in \mathbb{Z}.$$  \hspace{1cm} (14)

The $\mathbb{Z}_2$-grading: divide the $T_n$ into an even set $L_0 \equiv \{A_n,c\}$ and an odd set $L_1 \equiv \{B_n\}$

$$A_n = \frac{1}{2} \left( T_{2n} + \frac{c}{8}\delta_{n,0} \right) , \quad B_n = \frac{1}{2} T_{2n+1} ,$$ \hspace{1cm} (15)

so that $L = L_0 \oplus L_1$ and $[L_0,L_0] \subseteq L_0 , \quad [L_0,L_1] \subseteq L_1 , \quad [L_1,L_1] \subseteq L_0$. 
Explicitly, the commutation relations of the graded generators:

\[ [A_n, A_m] = (n - m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0} , \tag{16} \]

\[ [B_n, B_m] = (n - m)A_{n+m+1} + \frac{2c}{12}(n - \frac{1}{2})(n + \frac{1}{2})(n + \frac{3}{2})\delta_{n+m+1,0} , \tag{17} \]

\[ [A_n, B_m] = (n - m - \frac{1}{2})B_{n+m} . \tag{18} \]

\{A_n, c\} is again a Virasoro algebra but with central charge \(2c\).

Consider then the \(Z_2\)-graded contraction:

\[ [A_n, A_m] = (n - m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0} , \tag{19} \]

\[ [B_n, B_m] = 0 , \tag{20} \]

\[ [A_n, B_m] = (n - m - \frac{1}{2})B_{n+m} . \tag{21} \]
Remark 1: in the centerless case \((c = 0)\), the \(A_0\) and \(A_{\pm 1}\) generators close the algebra isomorphic to \(so(3) \sim su(2)\)

Remark 2: these generators and the operators \(B_{-\frac{1}{2}}, B_{\frac{1}{2}}\) and \(B_{-\frac{3}{2}}\) close the \(e(3)\) isomorphic algebra.

This is shown by setting:

\[
M_+ \equiv A_1, \quad M_- \equiv A_{-1}, \quad M_3 \equiv iA_0 , \\
P_+ \equiv B_{\frac{1}{2}}, \quad P_- \equiv B_{-\frac{3}{2}}, \quad P_3 \equiv iB_{-\frac{1}{2}},
\]

where the \(M\)s and \(P\)s satisfy the \(e(3)\) commutation relations \((13)\).

General extension of this result:
the algebra \(\mathcal{E}_n \equiv \{A_0, A_{\pm n}\} \oplus \{B_{-\frac{1}{2}}, B_{\pm n-\frac{1}{2}}\}\) reproduces the \(e(3)\) algebra for each integer value of \(n\), provided the following positions are assumed:

\[
M_+ \equiv \frac{1}{n} A_n, \quad M_- \equiv \frac{1}{n} A_{-n}, \quad M_3 \equiv \frac{i}{n} A_0 , \\
P_+ \equiv B_{n-\frac{1}{2}}, \quad P_- \equiv B_{-n-\frac{1}{2}}, \quad P_3 \equiv iB_{-\frac{1}{2}}. 
\]
As final remark we notice that the $e(2)$-algebra can be obtained as a subalgebra of (23) by choosing $A_{\pm n} = 0$, for non-zero values of $n$.

The conclusion is that the extension of the Virasoro algebra by means of its $\mathbb{Z}_2$-grading with the subsequent step of the $\mathbb{Z}_2$-graded contraction appears as a $n$-graded hierarchy of Euclidean algebras.

This establish the relation between the couple of damped/amplified parametric oscillators graded by $n$ and the loop algebras here considered.
Summing up,

the relation between the Bessel equation and the dissipation/amplification processes has been shown.

The breakdown of the $n \to -(n+1)$ ($n \to -n$) symmetry of the spherical (planar) Bessel equation may be represented as the breakdown of time-reversal symmetry in the manifold of the solutions (the Bessel functions).

In connection with the loop-antiloop symmetry of the Bessel equation, a $n$-graded hierarchy of Euclidean algebras appears as the extension of the Virasoro algebra.