Variational Study of SU(3) Gauge Theory by Stationary Variance

The Stationary Variance as a tool for QCD

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Outline

Brief review of the method of Stationary Variance:

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- General Perturbation Theory (PT)

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- Dynamical Mass and Constraints of Gauge Invariance
- Outlook: QCD by Stationary Variance

Perturbation Theory

PHYSICAL LIMITS OF PERTURBATION THEORY

For a given Hamiltonian H we take any solvable Hamiltonian H_0 that satisfies:

$$H \approx H_0.$$

What does it mean ?

Perturbation Theory

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$$H \approx H_0.$$

What does it mean ?
It means that we *define* the interaction V as

$$V = H - H_0$$

and the matrix elements of V must be "small".
While mathematical bounds can be found for the asymptotic convergence, **the choice of H_0 usually stems from physics!**

Perturbation Theory
OPTIMIZATION

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- UV freedom $\rightarrow H_0$ becomes exact

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Perturbation Theory works in the UV but breaks down in the IR
PT breaks down because of the bad choice of H_0:
optimization is mandatory, and must be based on physics

First Order Optimized Perturbation Theory

THE VARIATIONAL METHOD IN QUANTUM MECHANICS

$H_0(\lambda)$ solvable $\rightarrow V(\lambda) = H - H_0(\lambda)$

$$H_0(\lambda)|\Psi(\lambda)\rangle = E_0(\lambda)|\Psi(\lambda)\rangle$$
$$E^{(1)}(\lambda) = E_0(\lambda) + \langle\Psi(\lambda)|V(\lambda)|\Psi(\lambda)\rangle$$

we find the well known result that

$$E^{(1)}(\lambda) = \langle\Psi(\lambda)|H|\Psi(\lambda)\rangle.$$

The variational method yields the best $H_0(\lambda) \approx H$

First Order Optimized Perturbation Theory

FIELD THEORY: THE GAUSSIAN EFFECTIVE POTENTIAL (GEP)

In the Lagrangian formalism

$$S = S_0 + (S - S_0) \;\rightarrow\; S_I = S - S_0$$

But in field theory:
S_0 solvable $\rightarrow$ Gaussian Functional

$$S_0[g;\Psi] = \int \Psi_a(x) g^{-1}{}_{ab}(x,y) \Psi_b(y) \mathrm{d}x\mathrm{d}y.$$

Here $g_{ab}(x,y)$ is a trial correlator and is equivalent to **an infinite set of free parameters.**

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Here $g_{ab}(x,y)$ is a trial correlator and is equivalent to **an infinite set of free parameters.**
The first-order effective potential is $V^{(1)} = \langle 0|H|0\rangle$

$$\frac{\delta V^{(1)}}{\delta g} = 0 \Longrightarrow \textbf{GEP} = \textbf{Optimized First Order P.T.}$$

First Order Optimized Perturbation Theory

THE EFFECTIVE POTENTIAL BY STANDARD P.T.

$$e^{-\Gamma[\Psi]} = \int_{1PI} \mathcal{D}_{\Psi'}\, e^{-S_0[g;\Psi+\Psi']}\, e^{-S_I[g;\Psi+\Psi']} \qquad \text{(effective action)}$$

$$\langle X \rangle = \frac{\int_{1PI} \mathcal{D}_\Psi X\, e^{-S_0[\Psi]}}{\int \mathcal{D}_\Psi e^{-S_0[\Psi]}} \to \qquad V = V_0 + \frac{1}{\mathcal{V}} \log\langle e^{-S_I}\rangle \quad \text{(eff. potential)}$$

$$\text{where :} \qquad \log\langle e^{-S_I}\rangle = \mathcal{V}\sum_{n=1}^{\infty} V_n = -\langle S_I\rangle - \frac{1}{2!}\langle [S_I - \langle S_I\rangle]^2\rangle - \frac{1}{3!}\langle [S_I - \langle S_I\rangle]^3\rangle + \dots$$

First and second order terms are:

$$V_1 = -\frac{1}{\mathcal{V}}\langle S_I\rangle, \qquad V_2 = -\frac{1}{2!\mathcal{V}}\langle [S_I - \langle S_I\rangle]^2\rangle$$

Second Order Optimized Perturbation Theory

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Optimize by "Stationary Variance":
$\sigma^2 \sim V_2 = -\frac{1}{2!\mathcal{V}} \langle [S_I - \langle S_I \rangle]^2 \rangle$
(S_0 adds only disconnected graphs)

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- $V_2 \approx$ Error Estimate (of the asympthotic expansion)
- $\sigma^2 = -V_2 > 0$ and stationary when the trial propagator approaches the exact propagator

SU(3) Yang-Mills by Stationary Variance

LAGRANGIAN AND NOTATION

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{fix}$$

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}\left(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right); \quad \mathcal{L}_{fix} = -\frac{1}{\xi}\mathrm{Tr}\left[(\partial_\mu \hat{A}^\mu)(\partial_\nu \hat{A}^\nu)\right]$$

$$\text{Quantum effective action} \to e^{i\Gamma[A']} = \int_{1PI} \mathcal{D}_A e^{iS[A'+A]} J_{FP}[A'+A]$$

$$\text{Faddev} - \text{Popov det.} \to J_{FP}[A] = \int \mathcal{D}_{\omega,\omega^\star} e^{iS_{gh}[A,\omega,\omega^\star]}$$

$$A' = 0 \ \to e^{i\Gamma} = \int_{1PI} \mathcal{D}_{A,\omega,\omega^\star} e^{iS_0[A,\omega,\omega^\star]} e^{iS_I[A,\omega,\omega^\star]}$$

$$S_{tot} = S_0 + S_I = \int \mathcal{L}_{YM} \mathrm{d}^4x + \int \mathcal{L}_{fix} \mathrm{d}^4x + S_{gh}$$

but what is S_0 ?

SU(3) Yang-Mills by Stationary Variance

TRIAL FUNCTIONS

$$S_0 = \frac{1}{2}\int A^{a\mu}\, {D^{-1}}^{ab}_{\mu\nu}\, A^{b\nu} + \int \omega_a^{\star}\, {G^{-1}}_{ab}\, \omega_b$$

$$S_I = S_{tot} - S_0 = S_2 + \int \mathrm{d}^4x \left[\mathcal{L}_{gh} + \mathcal{L}_3 + \mathcal{L}_4\right]$$

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$$S_2 = \frac{1}{2}\int A^{a\mu}\left[{D_0}^{-1\,ab}_{\ \ \mu\nu} - {D^{-1}}^{ab}_{\mu\nu}\right] A^{b\nu} + \int \omega^\star_a \left[{G_0}^{-1}{}_{ab} - {G^{-1}}_{ab}\right] \omega_b$$

$$D_0{}^{ab}_{\mu\nu}(p) = -\tfrac{\delta_{ab}}{p^2}\left[\eta_{\mu\nu} + (\xi - 1)\tfrac{p_\mu p_\nu}{p^2}\right]; \quad G_{0ab}(p) = \tfrac{\delta_{ab}}{p^2}$$

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$$\mathcal{L}_3 = -g\, f_{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu}$$

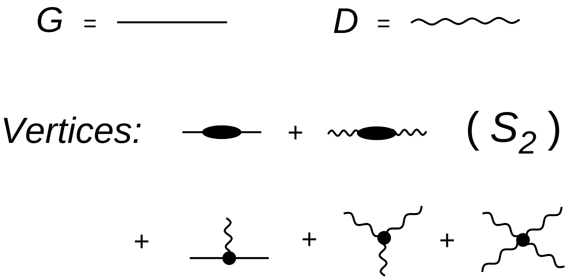
$$\mathcal{L}_4 = -\frac{1}{4}g^2 f_{abc}f_{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}$$

$$\mathcal{L}_{gh} = -g\, f_{abc}(\partial_\mu \omega^\star_a)\omega_b A^{c\mu}$$

SU(3) Yang-Mills by Stationary Variance

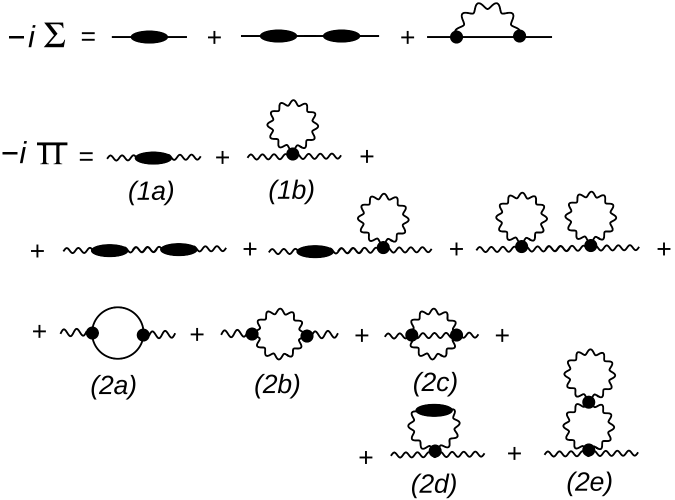
VERTEX GRAPHS

The unknown trial propagators D, G are the free-particle lines:



SU(3) Yang-Mills by Stationary Variance

SELF-ENERGY GRAPHS



SU(3) Yang-Mills by Stationary Variance

STATIONARY EQUATIONS

By the general connection betweem self-enery and functional derivatives (F. Siringo, Phys. Rev. D **88**, 056020 (2013), arXiv:1308.1836)

$$\frac{\delta V_n}{\delta D^{ab}_{\mu\nu}(p)} = \frac{i}{2}\left(\Pi^{\nu\mu,ba}_n(p) - \Pi^{\nu\mu,ba}_{n-1}(p)\right)$$

$$\frac{\delta V_n}{\delta G_{ab}(p)} = -i\left(\Sigma^{ba}_n(p) - \Sigma^{ba}_{n-1}(p)\right)$$

the stationary equations are

$$\Pi^{\nu\mu,ab}_2(p) = \Pi^{\nu\mu,ab}_1(p)$$
$$\Sigma^{ba}_2(p) = \Sigma^{ba}_1(p)$$

SU(3) Yang-Mills by Stationary Variance

LAZY GAUGE

Lorentz invariance $\rightarrow D^{ab}_{\mu\nu}(p) = \delta_{ab}\left[\eta_{\mu\nu}A(p) + p_\mu p_\nu B(p)\right]$

SU(3) Yang-Mills by Stationary Variance

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Lazy gauge $\rightarrow \; t_{\mu\nu} = \eta_{\mu\nu}$

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In Feynman gauge that would simplify things:

$$D^{ab}_{\mu\nu}(p) = \delta_{ab}\eta_{\mu\nu}D(p) = \delta_{ab}\eta_{\mu\nu}\frac{f(p)}{-p^2} = f(p){D_0}^{ab}_{\mu\nu}$$

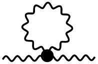
SU(3) Yang-Mills by Stationary Variance

FIRST ORDER

$$-i\Sigma_1^{ab}(p) = i\delta_{ab}\left[p^2 - G^{-1}(p)\right]$$

$$-i\Pi_{1a}^{\mu\nu,ab} = i\delta_{ab}\eta^{\mu\nu}\left[-p^2 - D^{-1}\right]$$

$$-i\Pi_{1b}^{\mu\nu,ab} = i\delta_{ab}\eta^{\mu\nu}(3Ng^2)I_0^{(1)}$$



$$M^2 = 3Ng^2 I_0^{(1)} = 3Ng^2 \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} D(k_E)$$

$$\Pi_1 = D^{-1} - \Delta^{-1} \qquad \Delta(p) = \frac{1}{-p^2 + M^2}$$

$$G(p) = \frac{1}{p^2}$$

$$D(p) = \Delta(p)$$

J.M. Cornwall, Phys. Rev. D **26**, 1453 (1982)

$$\Pi_2 = \Pi_2^\star + (\Pi_1)^2 D$$
$$\Sigma_2 = \Sigma_2^\star + (\Sigma_1)^2 G$$

Insertion of first order functions yields:

$$G(p) = \frac{1}{p^2} - \frac{\Sigma_2^\star(p)}{p^4}$$
$$D(p) = \Delta(p) - [\Delta(p)]^2 \Pi_2^\star(p)$$

(we only need 1PI graphs). In terms of dressing functions:

$$\chi(p_E) = \left[1 + \frac{1}{p_E^2}\Sigma_2^\star(p_E)\right]$$
$$f(p_E) = \frac{p_E^2}{p_E^2 + M^2}\left[1 - \frac{\Pi_2^\star(p_E)}{p_E^2 + M^2}\right]$$

SU(3) Yang-Mills by Stationary Variance

REGULARIZATION

Cutoff $p_E^2 < \Lambda^2$: $\quad \Lambda \to \Lambda' \Longrightarrow g(\Lambda) \to g(\Lambda')$

$$f_R(p/\mu,\ \mu) = \frac{f_B(p/\Lambda, g)}{Z(g, \mu)}$$

$$Z(g, \mu) = f_B(\mu/\Lambda, g(\Lambda))$$

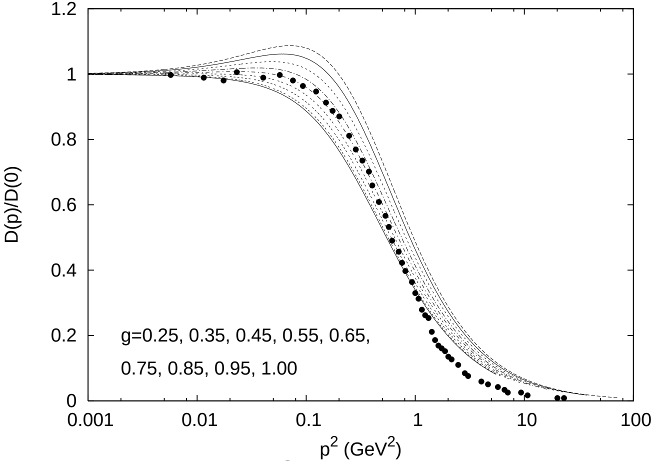
$$f_R(p/\mu, \mu) = \frac{f_B(p/\Lambda, g(\Lambda))}{f_B(\mu/\Lambda, g(\Lambda))} = \frac{f_B(p/\Lambda', g(\Lambda'))}{f_B(\mu/\Lambda', g(\Lambda'))}$$

f_R is independent of Λ and g and

$$f_B(p/\Lambda, g(\Lambda)) = K(g, g')\, f_B(p/\Lambda', g(\Lambda'))$$

SU(3) Yang-Mills by Stationary Variance

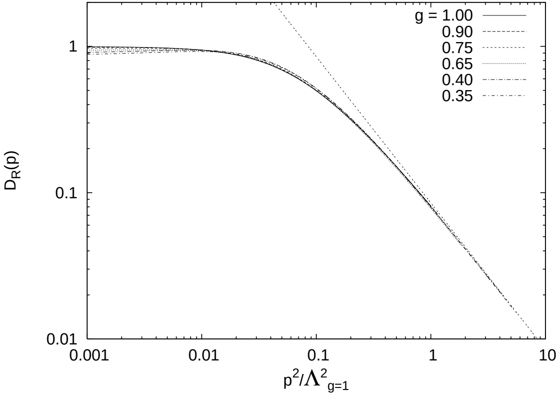
GLUON PROPAGATOR



- Scale fixed by $M = 0.5$ GeV.
- dots are lattice data (Landau gauge, $g = 1.02$, L=96) Bogolubsky et al., Phys. Lett. B 676, 69 (2009).

SU(3) Yang-Mills by Stationary Variance

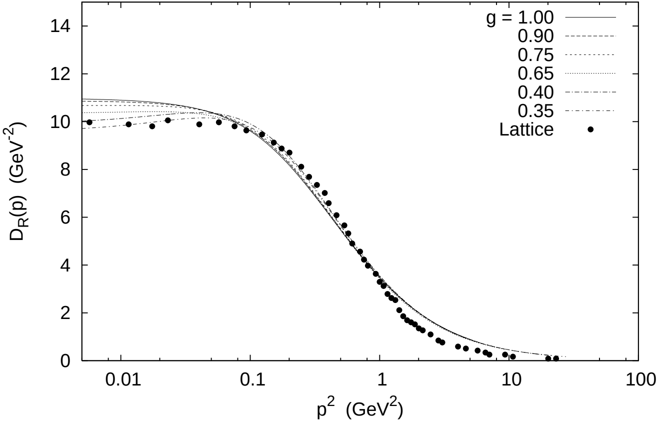
SCALING OF GLUON PROPAGATOR



- $g(1) = 1$, for $g = 1$ the curve is not rescaled.
- Asymptotic behavior $D(p) \approx z/p^2$, $z = 0.085$.

SU(3) Yang-Mills by Stationary Variance

RENORMALIZED GLUON PROPAGATOR



- Scale fixed by lattice data of Bogolubsky et al., Phys. Lett. B 676, 69 (2009). (Landau gauge, $g = 1.02$, L=96)

SU(3) Yang-Mills by Stationary Variance

EXPANSION AROUND THE TRIAL PROPAGATOR

The trial gluon propagator D is just the zeroth-order approximation of the optimized perturbation expansion:

$$D_{(n)}^{-1} = D^{-1} - \Pi^{\star}_{(n)}$$

Let us explore higher orders:

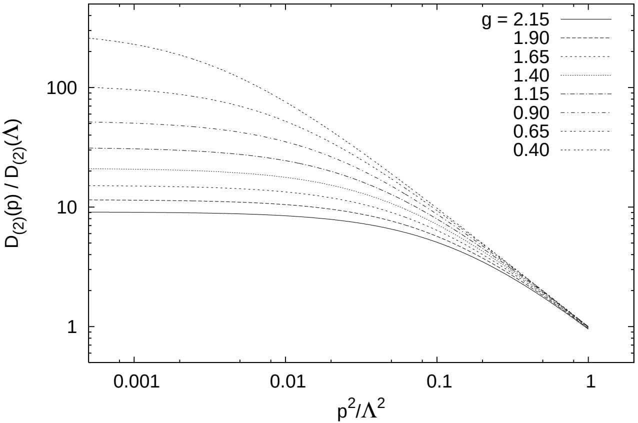
$$D_{(0)}{}_{\mu\nu}(p) = \eta_{\mu\nu} D(p)$$

$$D_{(1)}^{-1}{}_{\mu\nu}(p) = \eta_{\mu\nu}(p^2 + M^2)$$

$$D_{(2)}^{-1}{}_{\mu\nu}(p) = \eta_{\mu\nu}(p^2 + M^2) - \Pi^{\star}_{2}{}_{\mu\nu}(p)$$

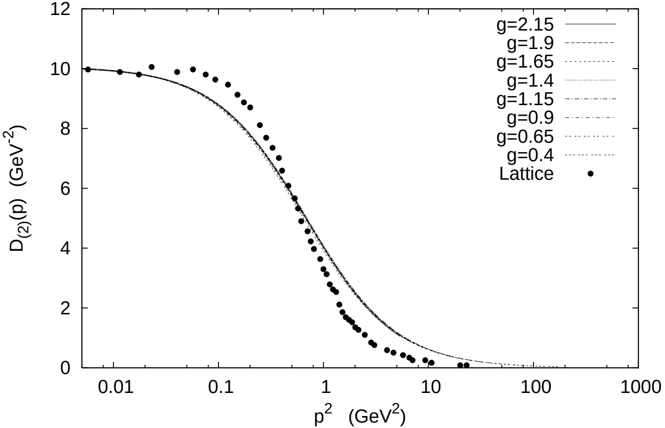
SU(3) Yang-Mills by Stationary Variance

BARE SECOND-ORDER PROPAGATOR



SU(3) Yang-Mills by Stationary Variance

RENORMALIZED SECOND-ORDER PROPAGATOR



- Scale fixed by rough fit of lattice data
- Dots are lattice data (Landau gauge, $g = 1.02$, L=96) Bogolubsky et al., Phys. Lett. B 676, 69 (2009).

SU(3) Yang-Mills by Stationary Variance

DYNAMICAL MASS AND GAUGE INVARIANCE

By Lorentz invariance:

$$\Pi_{\mu\nu}(p) = \left[\eta_{\mu\nu}\Pi'(p) + \frac{p_\mu p_\nu}{p^2}\Pi''(p)\right]$$

$$D_{(2)}{}_{\mu\nu}(p) = \eta_{\mu\nu}D_{(2)}(p) + p_\mu p_\nu D''_{(2)}(p)$$

where the physical part reads:

$$\left[D_{(2)}(p)\right]^{-1} = p^2 + M^2 - \Pi_2^{\star\prime}(p).$$

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We actually find

$$\Pi^\star_{2\,\mu\nu}(p) \approx -(\delta m^2)\eta_{\mu\nu} - \pi(p)\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \text{ with } \pi(0) = 0$$

as expected by gauge invariance (since $\Pi_{1\,\mu\nu} \sim \eta_{\mu\nu}$)

SU(3) Yang-Mills by Stationary Variance

DYNAMICAL MASS AND GAUGE INVARIANCE

The constraint $\Longrightarrow \Pi^{\star}_{2\,\mu\nu}(p) \approx -(\delta m^2)\eta_{\mu\nu} - \pi(p)\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$,

with $\pi(0) = 0$ and $\Pi^{\star}_{2\,\mu\nu}(p) = \left[\eta_{\mu\nu}\Pi_2{}'(p) + \frac{p_\mu p_\nu}{p^2}\Pi_2{}''(p)\right]$

requires that:

The constraint $\Longrightarrow \Pi^{\star}_{2\,\mu\nu}(p) \approx -(\delta m^2)\eta_{\mu\nu} - \pi(p)\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$,

with $\pi(0) = 0$ and $\Pi^{\star}_{2\,\mu\nu}(p) = \left[\eta_{\mu\nu}\Pi_2{}'(p) + \frac{p_\mu p_\nu}{p^2}\Pi_2{}''(p)\right]$

requires that:

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SU(3) Yang-Mills by Stationary Variance

DYNAMICAL MASS AND GAUGE INVARIANCE

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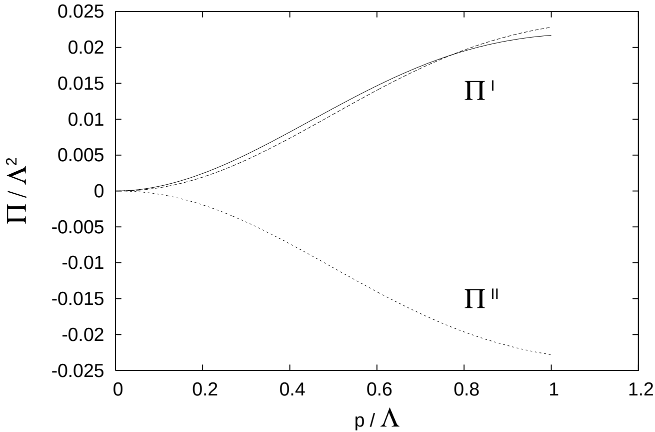
We actually find $\Longrightarrow$ $m^2(p) \sim (p^2)^{-1.5}$

for any coupling $g = 0.9 - 1.4$

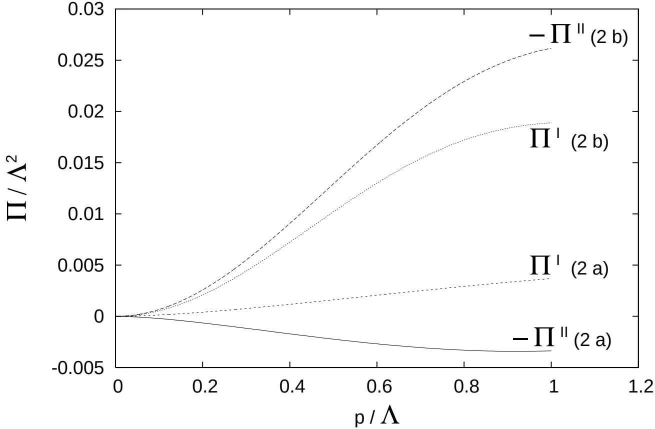
SU(3) Yang-Mills by Stationary Variance

POLARIZATION FUNCTIONS



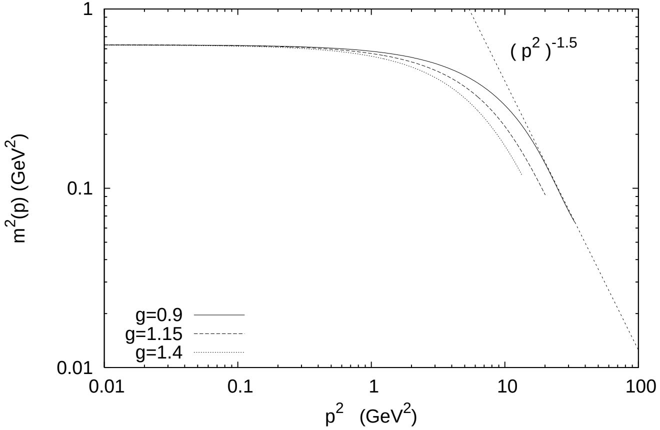
SU(3) Yang-Mills by Stationary Variance

POLARIZATION FUNCTIONS (single terms)



SU(3) Yang-Mills by Stationary Variance

DYNAMICAL MASS



CONCLUDING REMARKS

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THANK YOU

Variational Study of SU(3) Gauge Theory by Stationary Variance

The Stationary Variance as a tool for QCD

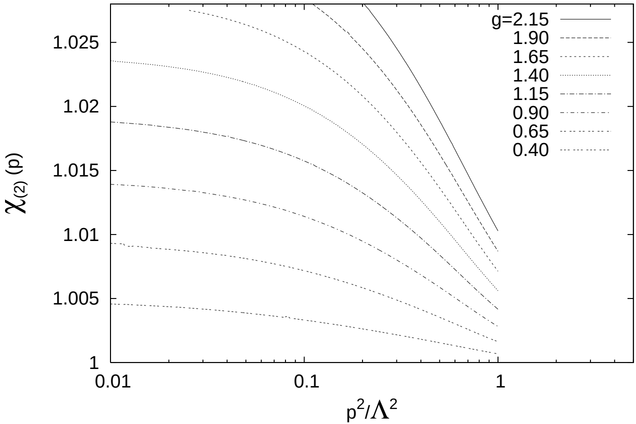
Fabio Siringo

Department of Physics and Astronomy
University of Catania, Italy

DISCRETE 2014 - London, 2-6 December 2014

SU(3) Yang-Mills by Stationary Variance

GHOST DRESSING FUNCTION



Stationary Equations

TWO-POINT NON-LOCAL VERTEX

$$S[\Psi] = \int \Psi_a(x){\Delta^{-1}}_{ab}(x,y)\Psi_b(y)\mathrm{d}x\mathrm{d}y + \int \mathcal{L}_{int}(x)\mathrm{d}x$$

$\Delta_{ab}(x,y)$ = standard free-particle propagators

$$\text{Def.} \rightarrow S_0[g;\Psi] = \int \Psi_a(x){g^{-1}}_{ab}(x,y)\Psi_b(y)\mathrm{d}x\mathrm{d}y.$$

$$S_I[\Psi] = \int \Psi_a(x)v_2^{ab}(x,y)\Psi_b(y)\mathrm{d}x\mathrm{d}y + \int \mathcal{L}_{int}(x)\mathrm{d}x$$

where the new non-local vertex v_2 is

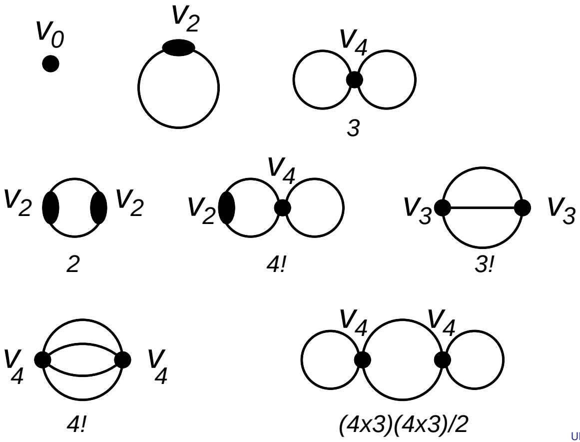
$$v_2^{ab}(x,y) = \left[{\Delta^{-1}}_{ab}(x,y) - {g^{-1}}_{ab}(x,y)\right] =$$

Self Energy $\rightarrow$ $\Sigma_{v_2}^{ab} = {g^{-1}}_{ab}(x,y) - {\Delta^{-1}}_{ab}(x,y)$

Stationary Equations

VACUUM GRAPHS (scalar theory)

$$\mathcal{L}_I = v_0 + v_1\phi(x) + v_3\phi^3(x) + v_4\phi^4(x) + \int \phi(x)v_2(x,y)\phi(y)\mathrm{d}^4y$$



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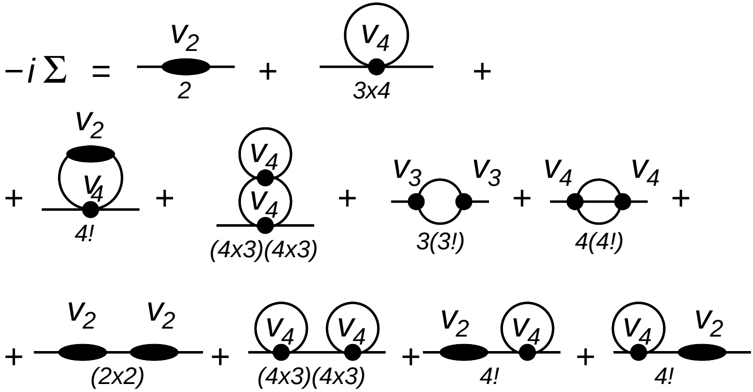
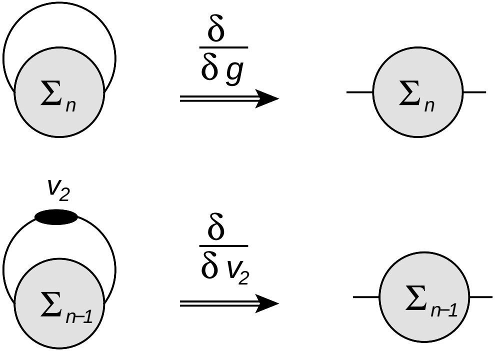


Stationary Equations

GENERAL CONNECTION



$$\frac{\delta V_n}{\delta g_{ab}(k)} = \pm\frac{i}{2}\left(\Sigma_n^{ba}(k) - \Sigma_{n-1}^{ba}(k)\right) ; \qquad \frac{\delta V^{(n)}}{\delta g(k)} = \pm\frac{i}{2}\Sigma_n(k)$$

F. Siringo, Phys. Rev. D **88**, 056020 (2013), arXiv:1308.1836

UNIVERSITY of CATANIA, ITALY

Stationary Equations

GAUSSIAN EFFECTIVE POTENTIAL (GEP)

The first order optimization equation

$$\frac{i}{2}\Sigma_1 = \frac{\delta V^{(1)}}{\delta g} = 0$$

is equivalent to the self-consistency requirement $\Sigma_1 = 0$:

$$\Sigma_1^{ab} = \Sigma_{v2}^{ab} + \Sigma_{int}^{ab} = g_{ab}^{-1} - \Delta_{ab}^{-1} - \delta M_{ab}^2 = 0$$

yielding the gap equations of the GEP

$$g_{ab}^{-1} = \Delta_{ab}^{-1} + \delta M_{ab}^2$$
$$\delta M_{ab}^2 = -\Sigma_{int}^{ab}[g]$$

The Method of Stationary Variance

A COMPARISON OF STRATEGIES

- Stationary Variance $\rightarrow$ Stationary V_2

  $0 = \frac{\delta V_2}{\delta g_{ab}(k)} = \pm \frac{i}{2} \left(\Sigma_2(k) - \Sigma_1(k) \right) \Longrightarrow \Sigma_2^{ba}(k) = \Sigma_1^{ba}(k)$

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- Minimal Sensitivity $\rightarrow$ Stationary $V^{(2)} = V_0 + V_1 + V_2$
  $0 = \frac{\delta V^{(2)}}{\delta g(k)} = \pm \frac{i}{2} \Sigma_2(k) \Longrightarrow \Sigma_2(k) = 0$
  but $V_2 < 0$ and $V^{(2)}$ can be unbounded $\rightarrow$ no solutions

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- GEP: optimal $g(x,y)$ fixed at higher orders
  $0 = \frac{\delta V^{(1)}}{\delta g(k)} = \pm \frac{i}{2} \Sigma_1(k) \Longrightarrow \Sigma_1(k) = 0$
  (useless when $\langle \mathcal{L}_{int} \rangle = 0 \quad \to \quad g(x,y) = \Delta(x,y)$)

All of them can be extended to higher orders
Convergence $\Longrightarrow$ differences should decrease order by order

SU(3) Yang-Mills by Stationary Variance

FEYNMAN GAUGE

The choice $\xi = 1$ would simplify things if

$$D^{ab}_{\mu\nu}(p) = \delta_{ab}\eta_{\mu\nu}D(p) = \delta_{ab}\eta_{\mu\nu}\frac{f(p)}{-p^2} = f(p){D_0}^{ab}_{\mu\nu}$$

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but at each order of P.T., in Feynman gauge

$$D^{(n)}{}_{\mu\nu}(p) = D^{(n)}(p)P_{\mu\nu} + \frac{p_\mu p_\nu}{p^4}, \text{ where } P_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$$

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Here we have the freedom of taking $D^{ab}_{\mu\nu}(p)$ as general as we can (no mass would arise for the gluon at each order of P.T.)

Lorentz invariance $\rightarrow D^{ab}_{\mu\nu}(p) = \delta_{ab}\left[\eta_{\mu\nu}A(p) + p_\mu p_\nu B(p)\right]$

SU(3) Yang-Mills by Stationary Variance

LAZY GAUGE

Lorentz invariance $\rightarrow D^{ab}_{\mu\nu}(p) = \delta_{ab}\left[\eta_{\mu\nu}A(p) + p_\mu p_\nu B(p)\right]$

A less general choice $\rightarrow D^{ab}_{\mu\nu}(p) = \delta_{ab}t_{\mu\nu}(p)D(p)$

$$\frac{\delta}{\delta D(p)} = \sum_{ab,\mu\nu}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\delta D^{ab}_{\mu\nu}(k)}{\delta D(p)}\frac{\delta}{\delta D^{ab}_{\mu\nu}(k)} = \sum_{ab,\mu\nu}\delta_{ab}t_{\mu\nu}(p)\frac{\delta}{\delta D^{ab}_{\mu\nu}(p)}$$

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Defining $\Pi_n(p) = \frac{1}{4(N^2-1)} \sum_{ab,\mu\nu} \delta_{ab} t_{\mu\nu}(p) \Pi^{\mu\nu,ab}_n(p)$
the stationary equation reads

$$\frac{\delta V_n}{\delta D^{ab}_{\mu\nu}(p)} = \frac{i}{2}\left(\Pi^{\nu\mu,ba}_n(p) - \Pi^{\nu\mu,ba}_{n-1}(p)\right) \Longrightarrow \Pi_2(p) = \Pi_1(p)$$

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Lazy gauge $\to \; t_{\mu\nu} = \eta_{\mu\nu}$

The Method of Stationary Variance

EXACT PROPAGATOR

$$S[\Psi] \approx \int \Psi_a(x) {G^{-1}}_{ab}(x,y) \Psi_b(y) \mathrm{d}x\mathrm{d}y$$

$$S_I = \int \Psi_a(x) \left[{G^{-1}}_{ab}(x,y) - {g^{-1}}_{ab}(x,y) \right] \Psi_b(y) \mathrm{d}x\mathrm{d}y$$

$\Sigma_1 = g^{-1} - G^{-1}$ $\rightarrow$ higher-order graphs are reducible
$\Sigma_2 = \Sigma_1 \, g \, \Sigma_1$ $\rightarrow$ then Σ_1 factorizes

$$\Sigma_1^{ab}(k) = 0 \qquad \text{(fixed first order GEP)}$$

$$\Sigma_1^{ac}(k) \left[\delta_{cb} - g_{cd}(k) \Sigma_1^{db}(k) \right] = 0 \qquad \text{(minimal variance)}$$

$$\Sigma_1^{ac}(k) \left[g_{cd}(k) \Sigma_1^{db}(k) \right] = 0 \qquad \text{(minimal sensitivity)}$$

They are all satisfyed by $\Sigma_1 = 0 \Longrightarrow g_{ab}(k) = G_{ab}(k)$ (exact)