Dynamical generation of flavor mixing*

Massimo Blasone
Dipartimento di Fisica, Università di Salerno & INFN Salerno, Italy

- Flavor mixing in Quantum Field Theory;
- Flavor states for mixed neutrinos;
- Dynamical generation of flavor mixing;
- Entanglement in flavor neutrino states.

  M.B., M.V.Gargiulo and G.Vitiello, work in progress;
Motivations

• CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

• Theoretical interest: origin of mixing in the Standard Model;

• Bargmann superselection rule*: coherent superposition of states with different masses is not allowed in non-relativistic QM;

• Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles†; oscillation formulas‡;

Neutrino oscillations in QM *

\[ |\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle \]

\[ |\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle \]

– Time evolution:

\[ |\nu_e(t)\rangle = \cos \theta e^{-iE_1 t} |\nu_1\rangle + \sin \theta e^{-iE_2 t} |\nu_2\rangle \]

– Flavor oscillations:

\[ P_{\nu_e \rightarrow \nu_e}(t) = |\langle \nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta \sin^2 \left( \frac{\Delta E}{2} t \right) = 1 - P_{\nu_e \rightarrow \nu_\mu}(t) \]

– Flavor conservation:

\[ |\langle \nu_e|\nu_e(t)\rangle|^2 + |\langle \nu_\mu|\nu_e(t)\rangle|^2 = 1 \]

Quantum field theory of fermion mixing

– Mixing relations for two Dirac fields

\[
\nu_e(x) = \cos \theta \, \nu_1(x) + \sin \theta \, \nu_2(x)
\]

\[
\nu_\mu(x) = -\sin \theta \, \nu_1(x) + \cos \theta \, \nu_2(x)
\]

\(\nu_1, \nu_2\) are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

\[
\mathcal{L} = \bar{\nu}_1 (i \not\! \partial - m_1) \nu_1 + \bar{\nu}_2 (i \not\! \partial - m_2) \nu_2
\]

and

\[
\mathcal{L} = \bar{\nu}_e (i \not\! \partial - m_e) \nu_e + \bar{\nu}_\mu (i \not\! \partial - m_\mu) \nu_\mu - m_{e\mu} \left( \bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e \right)
\]

with \(m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta, m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta.\)
Currents and charges for mixed fermions *

− Lagrangian in the mass basis:

\[ \mathcal{L} = \bar{\nu}_m (i \not{\partial} - M_d) \nu_m \]

where \( \nu^T_m = (\nu_1, \nu_2) \) and \( M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \).

\[ \mathcal{L} \] invariant under global \( U(1) \) with conserved charge \( Q = \) total charge.

− Consider now the \( SU(2) \) transformation:

\[ \nu'_m = e^{i\alpha_j \tau_j} \nu_m \quad ; \quad j = 1, 2, 3. \]

with \( \tau_j = \sigma_j/2 \) and \( \sigma_j \) being the Pauli matrices.

The associated currents are:

\[ \delta \mathcal{L} = i \alpha_j \bar{\nu}_m [\tau_j, M_d] \nu_m = -\alpha_j \partial_\mu J^{\mu}_{m,j} \]

\[ J^{\mu}_{m,j} = \bar{\nu}_m \gamma^\mu \tau_j \nu_m \]

– The charges \( Q_{m,j}(t) \equiv \int d^3x J^0_{m,j}(x) \), satisfy the \( su(2) \) algebra:

\[ [Q_{m,j}(t), Q_{m,k}(t)] = i \epsilon_{jkl} Q_{m,l}(t). \]

– The Casimir operator is proportional to the total charge: \( C_m = \frac{1}{2} Q \).

• \( Q_{m,3} \) is conserved \( \Rightarrow \) charge conserved separately for \( \nu_1 \) and \( \nu_2 \):

\[ Q_1 = \frac{1}{2} Q + Q_{m,3} = \int d^3x \, \nu_1^\dagger(x) \nu_1(x) \]

\[ Q_2 = \frac{1}{2} Q - Q_{m,3} = \int d^3x \, \nu_2^\dagger(x) \nu_2(x). \]

These are the flavor charges in the absence of mixing.
The currents in the flavor basis

– Lagrangian in the flavor basis:

\[ \mathcal{L} = \bar{\nu}_f (i \not\partial - M) \nu_f \]

where \( \nu^T_f = (\nu_e, \nu_\mu) \) and \( M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix} \).

– Consider the \( SU(2) \) transformation:

\[ \nu'_f = e^{i \alpha_j \tau_j} \nu_f \quad ; \quad j = 1, 2, 3. \]

with \( \tau_j = \sigma_j/2 \) and \( \sigma_j \) being the Pauli matrices.

– The charges \( Q_{f,j} \equiv \int d^3x J^0_{f,j} \) satisfy the \( su(2) \) algebra:

\[ [Q_{f,j}(t), Q_{f,k}(t)] = i \epsilon_{jkl} Q_{f,l}(t). \]

– The Casimir operator is proportional to the total charge \( C_f = C_m = \frac{1}{2} Q \).
• $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between $\nu_e$ and $\nu_\mu$.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3x \, \nu_e^\dagger(x) \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3x \, \nu_\mu^\dagger(x) \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

We have:

$$Q_e(t) = \cos^2 \theta \, Q_1 + \sin^2 \theta \, Q_2 + \sin \theta \cos \theta \int d^3x \, [\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1]$$

$$Q_\mu(t) = \sin^2 \theta \, Q_1 + \cos^2 \theta \, Q_2 - \sin \theta \cos \theta \int d^3x \, [\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1]$$
In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

\[ Q_e(t) \equiv \int d^3x \nu_e^\dagger(x) \nu_e(x) \quad ; \quad Q_\mu(t) \equiv \int d^3x \nu_\mu^\dagger(x) \nu_\mu(x) \]

– They are not conserved charges \( \Rightarrow \) flavor oscillations.

– They are still (approximately) conserved in the vertex \( \Rightarrow \) define flavor neutrinos as their eigenstates

● Problem: find the eigenstates of the above charges.
Neutrino mixing in QFT

- Mixing relations for two Dirac fields

\[
\nu_e(x) = \cos \theta \, \nu_1(x) + \sin \theta \, \nu_2(x)
\]
\[
\nu_\mu(x) = -\sin \theta \, \nu_1(x) + \cos \theta \, \nu_2(x)
\]

can be written as*

\[
\nu_\alpha^e(x) = G^{-1}_\theta(t) \, \nu_\alpha^1(x) \, G_\theta(t)
\]
\[
\nu_\alpha^\mu(x) = G^{-1}_\theta(t) \, \nu_\alpha^2(x) \, G_\theta(t)
\]

- Mixing generator:

\[
G_\theta(t) = \exp \left[ \theta \int d^3x \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right] = \exp[2i\theta \, Q_{m,2}(t)]
\]

For \( \nu_e \), we get \( \frac{d^2}{d\theta^2} \nu_\alpha^e = -\nu_\alpha^e \) with initial conditions \( \nu_\alpha^e|_{\theta=0} = \nu_1^\alpha \), \( \frac{d}{d\theta} \nu_\alpha^e|_{\theta=0} = \nu_2^\alpha \).

– $\nu_i$ are free Dirac field operators:

$$
\nu_i(x) = \sum_{k,r} \frac{e^{ik \cdot x}}{\sqrt{V}} [u_{k,i}^r(t) \alpha_{k,i}^r + v_{-k,i}^r(t) \beta_{-k,i}^{r\dagger}], \quad i = 1, 2.
$$

– Anticommutation relations:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(x-y)\delta_{\alpha\beta}\delta_{ij}; \quad \{\alpha_{k,i}^r, \alpha_{q,j}^{s\dagger}\} = \{\beta_{k,i}^r, \beta_{q,j}^{s\dagger}\} = \delta^3(k-q)\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u_{k,i}^r(t) = e^{-i\omega_{k,i} t} u_{k,i}^r; \quad v_{k,i}^r(t) = e^{i\omega_{k,i} t} v_{k,i}^r; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u_{k,i}^{r\dagger} u_{k,i}^s = v_{k,i}^{r\dagger} v_{k,i}^s = \delta_{rs}, \quad u_{k,i}^{r\dagger} v_{-k,i}^s = 0, \quad \sum_r (u_{k,i}^{r\alpha\dagger} u_{k,i}^{r\beta} + v_{-k,i}^{r\alpha\dagger} v_{-k,i}^{r\beta}) = \delta_{\alpha\beta}.$$

– Fock space for $\nu_1, \nu_2$:

$$\mathcal{H}_{1,2} = \{\alpha_{1,2}^{\dagger}, \beta_{1,2}^{\dagger}, |0\rangle_{1,2}\}.$$
• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G^{-1}_\theta(t) |0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V \to \infty} 1,2 \langle 0 | 0(t) \rangle_{e,\mu} = \lim_{V \to \infty} e V \int \frac{d^3k}{(2\pi)^3} \ln (1 - \sin^2 \theta |V_k|^2)^2 = 0$$

with

$$|V_k|^2 \equiv \sum_{r,s} |v^r_{-k,1} u^s_{k,2}|^2 \neq 0 \quad \text{for} \quad m_1 \neq m_2$$
Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:
  - finite \# of degrees of freedom.
  - unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:
  - infinite \# of degrees of freedom.
  - \( \infty \) many unitarily inequivalent representations of the field algebra \( \Leftrightarrow \) many vacua.
  - The mapping between interacting and free fields is “weak”, i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

• The “flavor vacuum” $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state*:

$$
|0\rangle_{e,\mu} = \prod_{k,r} \left[ (1 - \sin^2 \theta |V_k|^2) - \epsilon^r \sin \theta \cos \theta |V_k| (\alpha_{k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} + \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger}) 
+ \epsilon^r \sin^2 \theta |V_k||U_k| (\alpha_{k,1}^{r\dagger} \beta_{-k,1}^{r\dagger} - \alpha_{k,2}^{r\dagger} \beta_{-k,2}^{r\dagger}) + \sin^2 \theta |V_k|^2 \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger} \right] |0\rangle_{1,2}
$$

− Condensation density:

$$
e,\mu \langle 0(t) | \alpha_{k,i}^{r\dagger} \alpha_{k,i}^{r\dagger} | 0(t) \rangle_{e,\mu} = e,\mu \langle 0(t) | \beta_{k,i}^{r\dagger} \beta_{k,i}^{r\dagger} | 0(t) \rangle_{e,\mu} = \sin^2 \theta |V_k|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

Condensation density for mixed fermions

- $V_k = 0$ when $m_1 = m_2$ and/or $\theta = 0$.

- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \rightarrow \frac{1}{2}$ for $\frac{(m_2-m_1)^2}{m_1 m_2} \rightarrow \infty$.

- $|V_k|^2 \approx \frac{(m_2-m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$. 
• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

\[
\alpha_{k,e}^r(t) = \cos \theta \alpha_{k,1}^r + \sin \theta \left( U_k^*(t) \alpha_{k,2}^r + \epsilon^r V_k(t) \beta_{-k,2}^{r\dagger} \right) \\
\alpha_{k,\mu}^r(t) = \cos \theta \alpha_{k,2}^r - \sin \theta \left( U_k^*(t) \alpha_{k,1}^r - \epsilon^r V_k(t) \beta_{-k,1}^{r\dagger} \right) \\
\beta_{-k,e}^r(t) = \cos \theta \beta_{-k,1}^r + \sin \theta \left( U_k^*(t) \beta_{-k,2}^r - \epsilon^r V_k(t) \alpha_{k,1}^{r\dagger} \right) \\
\beta_{-k,\mu}^r(t) = \cos \theta \beta_{-k,2}^r - \sin \theta \left( U_k^*(t) \beta_{-k,1}^r + \epsilon^r V_k(t) \alpha_{k,1}^{r\dagger} \right)
\]

• Mixing transformation = Rotation + Bogoliubov transformation.

- Bogoliubov coefficients:

\[
U_k(t) = u_{k,2}^{r\dagger} u_{k,1}^r e^{i(\omega_{k,2} - \omega_{k,1})t} ; \quad V_k(t) = \epsilon^r u_{k,1}^{r\dagger} v_{-k,2}^r e^{i(\omega_{k,2} + \omega_{k,1})t}
\]

\[
|U_k|^2 + |V_k|^2 = 1
\]
• The flavor charge operators are diagonal in the flavor ladder operators:

\[ Q_{\nu\sigma}(t) \equiv \int d^3x : \nu^\dagger_\sigma(x) \nu_\sigma(x) : \]

\[ = \sum_r \int d^3k \left( \alpha^r_{k,\sigma}(t) \alpha^r_{k,\sigma}(t) - \beta^r_{-k,\sigma}(t) \beta^r_{-k,\sigma}(t) \right), \quad \sigma = e, \mu. \]

Here \( \vdots \vdots \) denotes normal ordering with respect to the flavor vacuum:

\[ \vdots A \vdots \equiv A - e,\mu \langle 0 | A | 0 \rangle_{e,\mu} \]

• Define flavor neutrino states with definite momentum and helicity:

\[ |\nu^r_{k,\sigma}\rangle \equiv \alpha^r_{k,\sigma}(0) |0\rangle_{e,\mu} \]

– Such states are eigenstates of the flavor charges (at \( t=0 \)):

\[ Q_{\nu\sigma} : |\nu^r_{k,\sigma}\rangle = |\nu^r_{k,\sigma}\rangle \]
We have, for an electron neutrino state:

\[ Q_{k,\nu_e}(t) \equiv \langle \nu_{k,e}^r \| Q_\nu(t) \| \nu_{k,e}^r \rangle \]

\[ = \left| \left\{ \alpha_{k,\sigma}^r(t), \alpha_{k,e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-k,\sigma}^{r\dagger}(t), \alpha_{k,e}^{r\dagger}(0) \right\} \right|^2 \]

Neutrino oscillation formula (exact result)*:

\[ Q_{k,\nu_e}(t) = 1 - |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \]

\[ Q_{k,\nu_\mu}(t) = |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \]

- For \( k \gg \sqrt{m_1 m_2} \), \( |U_k|^2 \to 1 \) and \( |V_k|^2 \to 0 \).

Lepton charge violation for Pontecorvo states

- Pontecorvo states:

\[
\begin{align*}
|\nu^r_{k,e}\rangle_P &= \cos \theta |\nu^r_{k,1}\rangle + \sin \theta |\nu^r_{k,2}\rangle \\
|\nu^r_{k,\mu}\rangle_P &= -\sin \theta |\nu^r_{k,1}\rangle + \cos \theta |\nu^r_{k,2}\rangle ,
\end{align*}
\]

are not eigenstates of the flavor charges.

⇒ violation of lepton charge conservation in the production/detection vertices, at tree level:

\[
P\langle \nu^r_{k,e} | : Q_e(0) : |\nu^r_{k,e}\rangle_P = \cos^4 \theta + \sin^4 \theta + 2|U_k| \sin^2 \theta \cos^2 \theta < 1 ,
\]

for any \(\theta \neq 0, k \neq 0\) and for \(m_1 \neq m_2\).

Other results

- Rigorous mathematical treatment for any number of flavors *

- Three flavor fermion mixing: CP violation†;

- QFT spacetime dependent neutrino oscillation formula‡;

- Boson mixing§;

- Majorana neutrinos¶;

● Flavor vacuum and cosmological constant*

● Flavor vacuum induced by condensation of D-particles.†

● Geometric phase for mixed particles‡.

* M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);
The issue of Lorentz invariance

– Canonical energy-momentum tensor for flavor fields:

\[
\begin{align*}
T_{\rho\sigma} &= \bar{\nu}e i \gamma_\rho \partial_\sigma \nu_e - \eta_{\rho\sigma} \bar{\nu}e (i \gamma^\lambda \partial_\lambda - m_e) \nu_e \\
&+ \bar{\nu}_\mu i \gamma_\rho \partial_\sigma \nu_\mu - \eta_{\rho\sigma} \bar{\nu}_\mu (i \gamma^\lambda \partial_\lambda - m_\mu) \nu_\mu \\
&+ \eta_{\rho\sigma} m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e)
\end{align*}
\]

– Define momentum and Hamiltonian operators:

\[
\begin{align*}
P^i &= \int d^3x \, T^{0i}; \\
H &= \int d^3x \, T^{00}.
\end{align*}
\]

One finds:

\[
P^i |\nu_{k,\sigma}\rangle = k^i |\nu_{k,\sigma}\rangle,
\]

but

\[
H |\nu_{k,\sigma}\rangle \neq \Omega_{k,\sigma} |\nu_{k,\sigma}\rangle.
\]

● This happens because: \([H, Q_\sigma] \neq 0.\)
Possible scenarios

- $\nu_e$ and $\nu_\mu$ are not fundamental; the fundamental objects are $\nu_1$ and $\nu_2^*$;

- $\nu_e$ and $\nu_\mu$ are fundamental but Poincaré invariance is broken (es. nonlinearly realized\(^\dagger\) as in DSR\(^\ddagger\)) $\Rightarrow$ modified dispersion relations;

- $\nu_e$ and $\nu_\mu$ are fundamental and Poincaré invariance is recovered in the vertices.

\(^\dagger\)M. B., J. Magueijo, P. Pires-Pacheco, EPL (2005);
\(^\ddagger\)J. Magueijo, L. Smolin, Phys. Rev. D (2003);
Flavor mixing as a non-abelian gauge theory

Let us return to the Lagrangian:

\[ \mathcal{L} = \bar{\nu}_e (i \not\partial - m_e) \nu_e + \bar{\nu}_\mu (i \not\partial - m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e). \]

The field equations:

\[ i\partial_0 \nu_e = (-i\alpha \cdot \nabla + \beta m_e) \nu_e + \beta m_{e\mu} \nu_\mu \]
\[ i\partial_0 \nu_\mu = (-i\alpha \cdot \nabla + \beta m_\mu) \nu_\mu + \beta m_{e\mu} \nu_e. \]

can be written compactly:

\[ iD_0 \nu_f = (-i\alpha \cdot \nabla + \beta M_d) \nu_f, \]
with \( \nu_f = (\nu_e, \nu_\mu)^T \), \( M_d = \text{diag}(m_e, m_\mu) \).

Non-abelian covariant derivative:

\[ D_0 := \partial_0 + i m_{e\mu} \beta \sigma_1, \]
with \( m_{e\mu} = \frac{1}{2} \tan 2\theta \delta m \) and \( \delta m := m_\mu - m_e. \)

Gauge connection:

\[ A_\mu := \frac{1}{2} A^a_\mu \sigma_a = n_\mu \delta m \frac{\sigma_1}{2} \in su(2), \]
with \( n^{\mu} := (1, 0, 0, 0)^T \), so that:

\[ D_\mu = \partial_\mu + i g \beta A_\mu. \]

We define \( g \equiv \tan 2\theta \) as the coupling constant for the mixing interaction.

The equations of motion and the Lagrangian read:

\[ (i \gamma^\mu D_\mu - M_d) \nu_f = 0, \]
\[ \mathcal{L} = \bar{\nu}_f (i \gamma^\mu D_\mu - M_d) \nu_f. \]
• Define a new energy-momentum tensor:

\[ \tilde{T}_{\rho\sigma} = \bar{\nu}_f i \gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma} \bar{\nu}_f (i \gamma^\lambda D_\lambda - M_d) \nu_f. \]

• Momentum and Hamiltonian operators:

\[
\begin{align*}
\tilde{P}^i &= \int d^3x \tilde{T}^{0i} \\
&= i \int d^3x \nu^\dagger_e \partial^i \nu_e + i \int d^3x \nu^\dagger_\mu \partial^i \nu_\mu \\
&\equiv \tilde{P}_e^i(t) + \tilde{P}_\mu^i(t), \quad i = 1, 2, 3; \\
\end{align*}
\]

\[
\begin{align*}
\tilde{H}(t) &= \int d^3x \tilde{T}^{00} \\
&= \int d^3x \nu^\dagger_e (-i \alpha \cdot \nabla + \beta m_e) \nu_e + \int d^3x \nu^\dagger_\mu (-i \alpha \cdot \nabla + \beta m_\mu) \nu_\mu \\
&\equiv \tilde{H}_e(t) + \tilde{H}_\mu(t).
\end{align*}
\]

This Hamiltonian does not generate time evolution.
Flavor fields in a different mass basis

- Flavor fields can be expanded also as*

\[ \nu_\sigma(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_r \left[ u^r_{k,\sigma}(t) \bar{\alpha}^r_{k,\sigma}(t) + v^r_{-k,\sigma}(t) \bar{\beta}^r_{-k,\sigma}(t) \right] e^{i k \cdot x}, \quad \sigma = e, \mu, \]

with \( u^r_{k,\sigma}(t) = u^r_{k,\sigma} e^{-i \omega_{k,\sigma} t} \) and \( v^r_{-k,\sigma}(t) = v^r_{-k,\sigma} e^{i \omega_{k,\sigma} t} \).

The spinor basis is defined by:

\[
(\alpha \cdot k + m_\sigma \beta) u^r_{k,\sigma} = \omega_{k,\sigma} u^r_{k,\sigma}
\]
\[
(\alpha \cdot k + m_\sigma \beta) v^r_{-k,\sigma} = -\omega_{k,\sigma} v^r_{-k,\sigma},
\]

where \( \omega_{k,\sigma} = \sqrt{k^2 + m_\sigma^2} \).

*K. Fujii, C. Habe, T. Yabuki  Phys. Rev. D (1999);
Operators in different bases are connected by a Bogoliubov transformation:

\[
\begin{pmatrix}
\tilde{\alpha}^{r}_{k,\sigma}(t) \\
\tilde{\beta}^{r\dagger}_{-k,\sigma}(t)
\end{pmatrix}
= J^{-1}(t) \begin{pmatrix}
\alpha^{r}_{k,\sigma}(t) \\
\beta^{r\dagger}_{-k,\sigma}(t)
\end{pmatrix} J_{\mu}(t),
\]

with generator:

\[
J(t) = \prod_{k,r} \exp \left\{ i \sum_{(\sigma,j)} \xi^{k}_{\sigma,j} \left[ \alpha^{r\dagger}_{k,\sigma}(t) \beta^{r\dagger}_{-k,\sigma}(t) + \beta^{r}_{-k,\sigma}(t) \alpha^{r}_{k,\sigma}(t) \right] \right\},
\]

where \((\sigma,j) = (e,1), (\mu,2)\), \(\xi^{k}_{\sigma,j} = (\chi_{\sigma} - \chi_{j})/2\) and \(\chi_{\sigma} = \arctan(\mu_{\sigma}/|k|)\), \(\chi_{j} = \arctan(m_{j}/|k|)\).

New flavor vacuum:

\[
|\tilde{0}(t)\rangle_{e\mu} = J^{-1}(t)|0(t)\rangle_{e\mu}.
\]
– Momentum and Hamiltonian operators are both diagonalized:

$$\tilde{P}_\sigma(t) = \sum_r \int d^3 k \; k \left( \tilde{\alpha}_{k,\sigma}^\dagger(t) \tilde{\alpha}_{k,\sigma}(t) + \tilde{\beta}_{k,\sigma}^\dagger(t) \tilde{\beta}_{k,\sigma}(t) \right),$$

$$\tilde{H}_\sigma(t) = \sum_r \int d^3 k \; \omega_{k,\sigma} \left( \tilde{\alpha}_{k,\sigma}^\dagger(t) \tilde{\alpha}_{k,\sigma}(t) - \tilde{\beta}_{k,\sigma}^\dagger(t) \tilde{\beta}_{k,\sigma}(t) \right).$$

– Flavor charges remain diagonal ([Q_\sigma(t), J(t)] = 0):

$$\tilde{Q}_\sigma(t) = \sum_r \int d^3 k \; \left( \tilde{\alpha}^\dagger_{k\sigma}(t) \tilde{\alpha}_{k\sigma}(t) - \tilde{\beta}^\dagger_{-k\sigma}(t) \tilde{\beta}_{-k\sigma}(t) \right).$$

• The new flavor states

$$|\tilde{\nu}_{k,\sigma}^r(t)\rangle = \tilde{\alpha}^\dagger_{k,\sigma}(t)|0(t)\rangle e^\mu.$$

are locally eigenstates of a four momentum operator:

$$\begin{pmatrix} \tilde{H}_\sigma(t) \\ \tilde{P}_\sigma(t) \end{pmatrix} |\tilde{\nu}_{k,\sigma}^r(t)\rangle = \begin{pmatrix} \omega_{k,\sigma} \\ k \end{pmatrix} |\tilde{\nu}_{k,\sigma}^r(t)\rangle,$$
Define the Lorentz generators:

\[
\tilde{M}^{\lambda\rho}(t) = \int d^3x \left( \tilde{T}^{0\rho} x^\lambda - \tilde{T}^{0\lambda} x^\rho \right) + \frac{1}{2} \int d^3x \nu_f^\dagger \sigma^\lambda \rho \nu_f = \tilde{M}^{\lambda\rho}_e(t) + \tilde{M}^{\lambda\rho}_\mu(t),
\]

where \[\sigma^\mu\nu = -\frac{i}{2} [\gamma^\mu, \gamma^\nu].\]

Algebra of equal-time commutators of the generators \((\sigma, \sigma' = e, \mu)\).

\[
[\tilde{P}_\sigma^\mu, \tilde{P}_\sigma'^\nu] = 0 \quad ; \quad [\tilde{M}_\sigma^{\mu\nu}, \tilde{P}_\sigma'^\lambda] = i\delta_{\sigma\sigma'} \left( \eta^{\mu\lambda} \tilde{P}_\sigma^\nu - \eta^{\nu\lambda} \tilde{P}_\sigma^\mu \right) ;
\]

\[
[\tilde{M}_\sigma^{\mu\nu}, \tilde{M}_\sigma'^{\lambda\rho}] = i\delta_{\sigma\sigma'} \left( \eta^{\mu\lambda} \tilde{M}_\sigma^{\nu\rho} - \eta^{\nu\lambda} \tilde{M}_\sigma^{\mu\rho} - \eta^{\mu\rho} \tilde{M}_\sigma^{\nu\lambda} + \eta^{\nu\rho} \tilde{M}_\sigma^{\mu\lambda} \right).
\]

- The Poincaré structure is preserved in the interaction vertices.
Physical picture (optical analogy)

- Flavor neutrinos are (locally) on-shell particles, with masses:
  \[ m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \quad m_\mu = m_2 \cos^2 \theta + m_1 \sin^2 \theta. \]

- Oscillations arise because of interaction with the external gauge field.

- Lorentz symmetry breaking is due to the external field.

- The vacuum acts as a sort of refractive medium ("neutrino aether") with respect to neutrinos.

- Optical analogy: flavor neutrinos as polarizations of the light, oscillations induced by birefringence\(^*\).

The tail of the tritium $\beta$ spectrum for:
- a massless neutrino (dotted line);
- fundamental flavor states (continuous line);
- superimposed prediction for 2 mass states (short-dashed line):
We used $m_e = 1.75$ KeV, $m_1 = 1$ KeV, $m_2 = 4$ KeV, $\theta = \pi/6$. 
Thermodynamic analogy

Identify

\[ F \equiv \tilde{H}, \quad T = g = \tan 2\theta \]

and write

\[ H - F = TS, \]

\[ S = \int d^3x \bar{\nu}_f A_0 \nu_f = \frac{1}{2} \delta m \int d^3x (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e). \]

- \( F \) is the energy that can be extracted from neutrinos through scattering experiments.

- Each of the two neutrinos can be considered as an open (dissipative) system.
Plot of expectation values on $|\nu_e(0)\rangle$ of $F_e(t)$, $F_\mu(t)$ and $2TS_e(t)$, as functions of dimensionless time $T = (\omega_2 - \omega_1)t$ and $\theta = \pi/6$. Scale on vertical axis is normalized to $\omega_\mu$. 
Dynamical generation of flavor mixing*

- The non trivial nature of flavor vacuum should result from the SSB process and the Higgs mechanism in the Standard Model;

- We consider dynamical symmetry breaking in a toy model with two flavors and quartic interaction term, as a generalization of Nambu and Jona-Lasinio model†;

- The approach of Umezawa, Takahashi and Kamefuchi for describing mass generation using inequivalent representations‡ is suitable for our purposes.

†Y. Nambu and G. Jona-Lasinio, Phys. Rev. (1961);
Dynamical mass generation and inequivalent representations

Consider a free Dirac field (at finite volume $V$):

$$\psi = \frac{1}{\sqrt{V}} \sum_{k,r} \left[ u_k^r a_k^r e^{-ik \cdot x} + v_k^r b_k^{r\dagger} e^{ik \cdot x} \right], \quad a_k^r |0\rangle = b_k^{r\dagger} |0\rangle = 0$$

The same field operator can be expanded as

$$\psi = \frac{1}{\sqrt{V}} \sum_{k,r} \left[ u_k^r(\vartheta, \phi) \alpha_k^r e^{ik \cdot x} + v_k^r(\vartheta, \phi) \beta_k^{r\dagger} e^{-ik \cdot x} \right],$$

with

$$\alpha_k^r = \cos \vartheta_k^r a_k^r + \sin \vartheta_k^r e^{i\varphi_k^r} b_{-k}^{r\dagger}$$
$$\beta_{-k}^r = \cos \vartheta_k^r b_{-k}^r - \sin \vartheta_k^r e^{i\varphi_k^r} a_k^{r\dagger}$$

and

$$u_k^r(\vartheta, \phi) = u_k^r \cos \vartheta_k + v_{-k}^r e^{-i\varphi_k^r} \sin \vartheta_k,$$
$$v_k^r(\vartheta, \phi) = v_k^r \cos \vartheta_k - u_{-k}^r e^{i\varphi_k^r} \sin \vartheta_k.$$
The above transformation is a Bogoliubov transformation, inducing inequivalent representations for different values of the parameters \((\vartheta, \varphi)\):

\[
|0(\vartheta, \varphi)\rangle = \prod_{k,r} \left[ \cos \vartheta_k - e^{i\varphi_k} \sin \vartheta_k \ a_k^{r\dagger} \ b_{-k}^{r\dagger} \right] |0\rangle
\]

with \(\alpha_k^r |0(\vartheta, \varphi)\rangle = \beta_k^r |0(\vartheta, \varphi)\rangle = 0\).
V-limit for operator products

In the infinite volume limit, one has the following relations:

\[ V\text{-lim} \left[ \int d^3 x \, \bar{\psi}_\alpha(x) \psi_\beta(x) \right] = \int d^3 x : \bar{\psi}_\alpha(x) \psi_\beta(x) : + \int d^3 x \, iS^-_{\alpha\beta}(\vartheta, \varphi), \]

\[ V\text{-lim} \left[ \int d^3 x \, \bar{\psi}_\alpha(x) \psi_\beta(x) \bar{\psi}_\gamma(x) \psi_\delta(x) \right] = \]

\[ = iS^-_{\alpha\beta}(\vartheta, \varphi) \int d^3 x : \bar{\psi}_\gamma(x) \psi_\delta(x) : + iS^+_{\gamma\delta}(\vartheta, \varphi) \int d^3 x : \bar{\psi}_\alpha(x) \psi_\beta(x) : + \]

\[ + iS^-_{\alpha\delta}(\vartheta, \varphi) \int d^3 x : \bar{\psi}_\gamma(x) \psi_\beta(x) : + iS^+_{\gamma\beta}(\vartheta, \varphi) \int d^3 x : \bar{\psi}_\alpha(x) \psi_\delta(x) : + \]

\[ + \int d^3 x \sum \text{contractions} S^+(\vartheta, \varphi)S^+(\vartheta, \varphi). \]

\( S^\pm_{\alpha\beta}(\vartheta, \varphi) \) are free two-point Wightman functions evaluated in \( |0(\vartheta, \varphi)\rangle \):

\[ iS^+_{\alpha\beta}(\vartheta, \varphi) = \langle 0(\vartheta, \varphi)|\bar{\psi}_\alpha(x) \psi_\beta(x)|0(\vartheta, \varphi)\rangle, \]

\[ iS^-_{\alpha\beta}(\vartheta, \varphi) = \langle 0(\vartheta, \varphi)|\bar{\psi}_\alpha(x) \psi_\beta(x)|0(\vartheta, \varphi)\rangle \]
We consider the following Hamiltonian:

\[ H = H_0 + H_{\text{int}}, \]

\[ H_0 = \int d^3x \bar{\psi} (-i \gamma \cdot \nabla + m) \psi, \]

\[ H_{\text{int}} = \lambda \int d^3x \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2 \right]. \]

In the lowest order in the Yang-Feldman eq. the \( V \)-limit of \( H \) gives:

\[ V\lim [H] = \bar{H}_0 + c - \text{number}. \]

with

\[ \bar{H}_0 = H_0 + \delta H_0 \]

\[ \delta H_0 = \int d^3x (f \bar{\psi}\psi + ig\bar{\psi}\gamma_5\psi) \]

where \( f, g \) depend on the set of parameters \( (\vartheta, \varphi) \):

\[ f = \lambda C_s, \quad g = \lambda C_p. \]
\[ C_p \equiv i \lim_{V \to \infty} \langle 0(\vartheta, \varphi) | \bar{\psi}(x) \gamma_5 \psi(x) | 0(\vartheta, \varphi) \rangle \]
\[
= \frac{2}{(2\pi)^3} \int d^3k \sin 2\vartheta_k \sin \varphi_k 
\]

\[ C_s \equiv \lim_{V \to \infty} \langle 0(\vartheta, \varphi) | \bar{\psi}(x) \psi(x) | 0(\vartheta, \varphi) \rangle \]
\[
= -\frac{2}{(2\pi)^3} \int d^3k \left[ \frac{m}{\omega_k} \cos 2\vartheta_k - \frac{k}{\omega_k} \sin 2\vartheta_k \cos \varphi_k \right]. 
\]
We then require that $\bar{H}_0$ has the form of a free Hamiltonian:

$$\bar{H}_0 = \sum_r \int d^3k \, E_k \left( \alpha_r^{r\dagger} \alpha_r + \beta_r^{r\dagger} \beta_r \right) + W_0.$$ 

with

$$E_k = \sqrt{k^2 + M^2} \quad ; \quad W_0 = -2 \int d^3k \, E_k.$$ 

by fixing the Bogoliubov transformation parameters. One obtains:

$$\cos 2\vartheta_r^k = \frac{1}{E_k} \left[ \omega_k + f \frac{m}{\omega_k} \right]$$

$$\cos \varphi(k, r) = -f \frac{k}{\omega_k} \frac{1}{\sqrt{g^2 + f^2 (k^2/\omega_k^2)}}$$

$$M^2 = (m + f)^2 + g^2.$$
Two possibilities:

\[ C_p = 0, \quad M = m - \frac{2\lambda}{(2\pi)^3} M \int \frac{d^3k}{E_k}, \]

\[ m = 0, \quad 1 + \frac{2\lambda}{(2\pi)^3} \int \frac{d^3k}{E_k} = 0. \]

The second case is only allowed for \( \lambda < 0. \)
Dynamical generation of flavor mixing

We consider the following Hamiltonian:

\[ H = H_0 + H_{\text{int}} \]

\[ H_0 = \int d^3x \bar{\Psi} \left( -i \gamma^i \partial_i + M_0 \right) \Psi \]

with \( \Psi^T = (\psi_I, \psi_{II}) \) and \( M_0 = \text{diag}(m_I, m_{II}) \).

The interaction Hamiltonian \( H_{\text{int}} \) can be assumed in the generic form

\[ H_{\text{int}} = (\bar{\psi} \Gamma \psi)(\bar{\psi} \Gamma' \psi), \]

where \( \Gamma \) and \( \Gamma' \) are some doublet spinor matrices.

In this case the \( V \)-limit renormalization term \( \delta H_0 \) has the following structure

\[ \delta H_0 = \delta H_0^I + \delta H_0^{II} + \delta H_{\text{mix}} \]

\[ = f_I \bar{\psi}_I \psi_I + f_{II} \bar{\psi}_{II} \psi_{II} + h \left( \bar{\psi}_I \psi_{II} + \bar{\psi}_{II} \psi_I \right). \]
Generalized Bogoliubov transformation

We consider the $4 \times 4$ canonical transformation

\[
\begin{pmatrix}
\alpha_A \\
\alpha_B \\
\beta_A^+ \\
\beta_B^+
\end{pmatrix}
=\begin{pmatrix}
c_\theta \rho_{AI} & s_\theta \rho_{AII} & c_\theta \lambda_{AI} & s_\theta \lambda_{AII} \\
-s_\theta \rho_{BI} & c_\theta \rho_{BII} & -s_\theta \lambda_{BI} & c_\theta \lambda_{BII} \\
c_\theta \lambda_{AI} & s_\theta \lambda_{AII} & c_\theta \rho_{AI} & s_\theta \rho_{AII} \\
-s_\theta \lambda_{BI} & c_\theta \lambda_{BII} & -s_\theta \rho_{BI} & c_\theta \rho_{BII}
\end{pmatrix}
\begin{pmatrix}
a_I \\
a_{II} \\
b_I^+ \\
b_{II}^+
\end{pmatrix}
\]

where $c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$ and

\[
\rho_{ab} \equiv \cos \frac{\chi_a - \chi_b}{2}, \quad \lambda_{ab} \equiv \sin \frac{\chi_a - \chi_b}{2}, \quad \chi_a \equiv \cot^{-1}\left[\frac{k}{m_a}\right], \quad a, b = I, II, A, B.
\]

Thus we have three parameters $(\theta, m_A, m_B)$ to fix in terms of $(f_I, f_{II}, h)$ in order to diagonalize the Hamiltonian.
Partial diagonalization

A possible representation is obtained by a partial diagonalization of $\bar{H}_0$, leaving untouched $\delta \mathcal{H}_{\text{mix}}$:

$$\bar{H}_0 = \sum_{\sigma = e, \mu} \bar{\psi}_\sigma (-i\gamma \cdot \nabla + m_\sigma) \psi_\sigma + h(\bar{\psi}_e \psi_\mu + \bar{\psi}_\mu \psi_e).$$

Such a representation is obtained by setting

$$\theta \rightarrow 0, \quad m_A \rightarrow m_e \equiv m_I + f_I, \quad m_B \rightarrow m_\mu \equiv m_{II} + f_{II}.$$ 

The vacuum is denoted as

$$|0(\theta = 0, m_e, m_\mu)\rangle \equiv |0\rangle_{e\mu},$$

In this representation we have

$$e,\mu \langle 0|\bar{H}_0|0\rangle_{e,\mu} = -2 \int d^3k \left( \sqrt{k^2 + m_e^2} + \sqrt{k^2 + m_\mu^2} \right),$$

since $e,\mu \langle 0|\delta \mathcal{H}_{\text{mix}}|0\rangle_{e,\mu} = 0.$
Complete diagonalization

Another possibility is to require that the Hamiltonian $\tilde{H}_0$ becomes fully diagonal in two fermion fields, $\psi_1$ and $\psi_2$, with masses $m_1$ and $m_2$:

$$\tilde{H}_0 = \sum_{j=1,2} \bar{\psi}_j (-i \gamma \cdot \nabla + m_j) \psi_j.$$ 

The condition for the complete diagonalization is found to be:

$$\theta \rightarrow \bar{\theta} \equiv \frac{1}{2} \tan^{-1} \left[ \frac{2h}{m_\mu - m_e} \right],$$

$$m_A \rightarrow m_1 \equiv \frac{1}{2} \left( m_e + m_\mu - \sqrt{(m_\mu - m_e)^2 + 4h^2} \right),$$

$$m_B \rightarrow m_2 \equiv \frac{1}{2} \left( m_e + m_\mu + \sqrt{(m_\mu - m_e)^2 + 4h^2} \right),$$

where we introduced the notation $m_e = m_I + f_I$, $m_\mu = m_{II} + f_{II}$. 
We set

\[ |0(\bar{\theta}, m_1, m_2)\rangle \equiv |0\rangle_{1,2}, \]

The vev of the Hamiltonian in this representation has the form:

\[ _{1,2}\langle 0|H_0|0\rangle_{1,2} = -2 \int d^3k \left( \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} \right). \]
Notice that the difference in energy among $|0\rangle_{12}$ and $|0\rangle_{e\mu}$ is given by

$$\Delta E \equiv e\mu\langle 0|\bar{H}_0|0\rangle_{e\mu} - 12\langle 0|\bar{H}_0|0\rangle_{12}$$

$$\Delta E = -2 \int d^3p \left[ \sqrt{p^2 + m_e^2} + \sqrt{p^2 + m_\mu^2} - \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2} \right].$$

For large $p$, i.e. $p \gg m_{1,2}$, we have $\sqrt{p^2 + m^2} \simeq p + \frac{m^2}{2p}$ and

$$\Delta E \simeq \int d^3p \frac{(m_2 - m_1)^2}{2p} \sin^2 2\theta.$$

We now compare this result with the flavor vacuum condensation density, namely

$$e\mu\langle 0|H|0\rangle_{e\mu} = \int d^3p \left( \omega_2 + \omega_1 \right) |V_p|^2 \sin^2 \theta.$$

For large $p$, this has the same form of $\Delta E$ since $|V_p|^2 \simeq \frac{(m_2 - m_1)^2}{4p^2}$ for $p \gg \sqrt{m_1m_2}$.
Decomposition of mixing generator *

Mixing generator function of $m_1$, $m_2$, and $\theta$. Try to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp \left\{ \theta \sum_{k,r} \left[ (\alpha_{k,1}^{r\dagger} \alpha_{k,2}^r + \beta_{-k,1}^{r\dagger} \beta_{-k,2}^r) e^{i\psi_k} - (\alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + \beta_{-k,2}^{r\dagger} \beta_{-k,1}^r) e^{-i\psi_k} \right] \right\},$$

$$B_i(\Theta_i) \equiv \exp \left\{ \sum_{k,r} \Theta_{k,i} \ e^r [\alpha_{k,i}^r \beta_{-k,i}^{r\dagger} e^{-i\phi_{k,i}} - \beta_{-k,i}^{r\dagger} \alpha_{k,i}^r e^{i\phi_{k,i}}] \right\}, \quad i = 1, 2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1) B_2(\Theta_2)$$

* M.B., M.V. Gargiulo and G. Vitiello, work in progress
We find:

\[ G_\theta = B(\Theta_1, \Theta_2) R(\theta) B^{-1}(\Theta_1, \Theta_2) \]

which is realized when the \( \Theta_{k,i} \) are chosen as:

\[
U_k = e^{-i\psi_k} \cos(\Theta_{k,1} - \Theta_{k,2}) \\
V_k = e^{\frac{(\phi_{k,1} + \phi_{k,2})}{2}} \sin(\Theta_{k,1} - \Theta_{k,2})
\]

The \( B_i(\Theta_{k,i}), i = 1, 2 \) are ordinary Bogoliubov transformations which introduce a mass shift, and \( R(\theta) \) is a rotation. Their action on the vacuum is given by:

\[
|\tilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} = \prod_{k,r} \left[ \cos \Theta_{k,i} + \epsilon^r \sin \Theta_{k,i} \alpha_{k,i}^{r\dagger} \beta_{-k,i}^{r\dagger} \right] |0\rangle_{1,2}
\]

\[
R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}
\]
Moreover,

$$|0\rangle_{e,\mu} = |0\rangle_{1,2} + [B(\Theta_1, \Theta_2), R^{-1}(\theta)] |\tilde{0}\rangle_{1,2}$$

which shows that the condensate structure of the flavor vacuum arises as a consequence of the non vanishing commutator $[B, R^{-1}]$.

<table>
<thead>
<tr>
<th>$⟨H_{k,1} + H_{k,2}⟩$</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-(\omega_{k,1} + \omega_{k,2})$</td>
<td>$</td>
</tr>
<tr>
<td>$-(k + k)$</td>
<td>$B^{-1}(\Theta_1, \Theta_2)</td>
</tr>
<tr>
<td>$-k(2\cos^2 \theta + (\frac{\omega_{k,1}}{\omega_{k,2}} + \frac{\omega_{k,2}}{\omega_{k,1}})\sin^2 \theta)$</td>
<td>$R^{-1}(\theta)B^{-1}(\Theta_1, \Theta_2)</td>
</tr>
<tr>
<td>$-(\omega_{k,1} + \omega_{k,2})(1 - 2\sin^2 \theta \sin^2(\Theta_{k,1} - \Theta_{k,2}))$</td>
<td>$B(\Theta_1, \Theta_2)R^{-1}(\theta)B^{-1}(\Theta_1, \Theta_2)</td>
</tr>
</tbody>
</table>
Plot of vacuum energies for $H_1$ and $H_2$ for the different vacua given in Table for $\theta = \pi/6$, $m_1 = 20$, $m_2 = 150$, $k = 80$. 
Conclusions

• Mixing transformations are not trivial in Q.F.T. \(\Leftrightarrow\) they are associated to inequivalent representations.

• The vacuum for mixed fields has the structure of a \(SU(N)\) generalized coherent state (condensate of particle-antiparticle pairs).

• Dynamical generation of mixing: not just a rotation!

• Neutrino oscillations as a resource for quantum information;

• Extension to QFT: Entanglement vs. inequivalent representations;
Entanglement in neutrino oscillations.

- Flavor mixing and entanglement;

- Entanglement in neutrino oscillations:
  - Flavor entanglement;
  - Decoherence;

- Neutrino oscillations as a resource for quantum information.

- Flavor entanglement in Quantum Field Theory.
Entanglement in particle mixing

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle$$

• Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

where $|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass $m_i$.

⇒ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \cos \theta |10\rangle + \sin \theta |01\rangle.$$
Single-particle entanglement

– A state like $|\psi\rangle_{A,B} = |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc.).

*J.van Enk, Phys. Rev. A (2005), (2006);
Protocols for extraction of single-particle entanglement

One photon is split, creating an entangled one-photon state.

Each photon mode interacts with a two-level atom. Resonance is tuned to give a $\pi$ pulse, if a photon is present. The excitation is transferred to the atomic pair.

One excitation is distributed between two atoms. A Bell state of excited-ground states is created.

One atom is split between two traps, creating an entangled one-atom state.

Each atomic trap interacts with an attenuated atomic beam. Resonance is tuned to create a molecule if one atom is found in the trap. The traps are left empty, and the atom is transferred to the beams.

The (dark grey) trapped atom is distributed between two (light grey) atomic beams. A Bell state of molecule–atom states is created.
Multipartite entanglement in neutrino mixing

- Neutrino mixing (three flavors):

\[ |\nu_f\rangle = U(\tilde{\theta}, \delta) |\nu_m\rangle \]

with \( |\nu_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T \) and \( |\nu_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T \).

- Mixing matrix (MNSP)

\[ U(\tilde{\theta}, \delta) = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}s_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}s_{13}
\end{pmatrix}, \]

where \((\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta)\), \(c_{ij} \equiv \cos \theta_{ij}\) and \(s_{ij} \equiv \sin \theta_{ij}\).

- Correspondence with three-qubit states:

\[ |\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle, \]
\[ |\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle \]

Multipartite entanglement

– Characterization of entanglement for multipartite systems is a non-trivial task. Several approaches have been developed: global entanglement, tangle, geometric measures\(^\ast\), etc...

In the 3-qubit case, the two fundamental classes\(^\dagger\) of states are those of the \(GHZ\) state and of the \(W\) state:

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),
\]

\[
|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle).
\]

\(^\ast\) T.C. Wei and P.M. Goldbart Phys. Rev. A (2003);

(Flavor) Entanglement in neutrino oscillations*

- Two-flavor neutrino states

\[ |\nu^{(f)}\rangle = U(\tilde{\theta}, \delta) |\nu^{(m)}\rangle \]

where \( |\nu^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T \) and \( |\nu^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T \) and \( U(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).

- Flavor states at time \( t \):

\[ |\nu^{(f)}(t)\rangle = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1} |\nu^{(f)}\rangle \equiv \tilde{U}(t)|\nu^{(f)}\rangle, \]

with \( U_0(t) = \begin{pmatrix} e^{-iE_1t} & 0 \\ 0 & e^{-iE_2t} \end{pmatrix} \).

- Transition probability for $\nu_\alpha \rightarrow \nu_\beta$

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\tilde{U}_{\alpha \beta}(t)|^2.$$  

- We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu \equiv |01\rangle_f,$$

- Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle = \tilde{U}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \tilde{U}_{\alpha \mu}(t)|0\rangle_e|1\rangle_\mu, \quad \alpha = e, \mu.$$
Entanglement measure

– Let $\rho = |\psi\rangle \langle \psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$

and

$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\}, \quad 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$

– Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$\rho_{A_n} \equiv \rho_{i_1,i_2,\ldots,i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1,j_2,\ldots,j_{N-n}}[\rho]$
• Linear entropy associated to such a bipartition:

\[ S_L^{(A_n;B_{N-n})}(\rho) = \frac{d}{d-1} \left( 1 - Tr_{A_n}[\rho_{A_n}^2] \right), \]

\( d \) is the Hilbert-space dimension: \( d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}. \)

• Average linear entropy (global entanglement):

\[ \langle S_L^{(n:N-n)}(\rho) \rangle = \left( \begin{array}{c} N \\ n \end{array} \right)^{-1} \sum_{A_n} S_L^{(A_n;B_{N-n})}(\rho), \]

sum over all the possible bi-partitions of the system in two subsystems, respectively with \( n \) and \( N - n \) elements (\( 1 \leq n < N \)).
Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle \langle \nu_e(t)|$, and trace over mode $\mu \Rightarrow \rho^{(e)}_e$.

- The associated linear entropy is:

$$S_L^{(e;\mu)}(\rho^{(e)}) = 4 |\tilde{U}_{e\mu}(t)|^2 |\tilde{U}_{ee}(t)|^2 = 4 P_{\nu_e\rightarrow\nu_e(t)} P_{\nu_e\rightarrow\nu_\mu(t)}$$

- The linear entropy for the state $\rho^{(\alpha)}$ is:

$$S_L^{(e;\mu)} = S_L^{(\mu;e)} = \langle S_L^{(1:1)} \rangle = 4 |\tilde{U}_{\alpha\mu}(t)|^2 |\tilde{U}_{\alpha e}(t)|^2$$

$$= 4|\tilde{U}_{\alpha e}(t)|^2 (1 - |\tilde{U}_{\alpha e}(t)|^2)$$

$$= 4|\tilde{U}_{\alpha\mu}(t)|^2 (1 - |\tilde{U}_{\alpha\mu}(t)|^2).$$

- Linear entropy given by product of transition probabilities!
Linear entropy $S_{Le}^{(e; \mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2 \theta = 0.314$. Transition probabilities $P_{\nu_e \rightarrow \nu_e}$ (dashed) and $P_{\nu_e \rightarrow \nu_\mu}$ (dot-dashed) are reported for comparison.
Neutrino oscillations as a resource for quantum information

- Single-particle entanglement encoded in flavor states $|\nu^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

- Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with spatially separated modes.

Charged-current interaction between a neutrino $\nu_\alpha$ with flavor $\alpha$ and a nucleon $N$ gives a lepton $\alpha^-$ and a baryon $X$:

$$\nu_\alpha + N \rightarrow \alpha^- + X.$$
Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \rightarrow \alpha^- + p$ with $\alpha = e, \mu$.

A spatially nonuniform magnetic field $B(r)$ constrains the momentum of the outgoing lepton within a solid angle $\Omega_i$, and ensures spatial separation between lepton paths.

The reaction produces a superposition of electronic and muonic spatially separated states.
Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle = \Lambda_e |1_e\rangle |0_\mu\rangle + \Lambda_\mu |0_e\rangle |1_\mu\rangle,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, represents the lepton qubit.

The coefficients $\Lambda_\alpha$ are proportional to $\widetilde{U}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

Analogy with single-photon system: quantum uncertainty on the so-called “which path” of the photon at the output of an unbalanced beam splitter $\Leftrightarrow$ uncertainty on the “which flavor” of the produced lepton.

The coefficients $\Lambda_\alpha$ plays the role of the transmissivity and of the reflectivity of the beam splitter.
Entanglement for neutrino oscillations in QFT

- Extension of the above analysis to QFT
- Non-trivial nature of mixing transformations in QFT
- Dynamical symmetry approach to entanglement
- Entropic measures in QFT

*M.B., F.Dell’Anno, F.Illuminati and S.De Siena, EPL (2014).*
Entanglement in relativistic systems

• Necessity for a treatment of entanglement in the context of Quantum Field Theory.*

• Lorentz invariance of entanglement: two particles entangled in spin and momentum†.

Dynamical symmetry approach to entanglement

• Entanglement can be characterized by total variance of the operators generating the dynamical algebra*.

– Consider the observables $X_i$ elements of the basis of a Lie algebra $\mathcal{L}$ such that the Lie group $G = \exp(i\mathcal{L})$ defines the dynamic symmetry of the system.

• Entanglement of a state $|\psi\rangle$ is given by the total amount of uncertainty:

$$\Delta(\psi) = \sum_i \left( |\langle \psi | X_i^2 |\psi\rangle \rangle - |\langle \psi | X_i |\psi\rangle^2 \right)$$

Entanglement for flavor neutrino states in QFT I

– Entanglement for flavor neutrino states in QFT can be expressed by means of the variances of the neutrino charges: $Q_i, Q_\sigma(t)$

– Variance of $Q_i \Rightarrow$ static entanglement:

$$\Delta Q_i(\nu_e) = \langle \nu_{k,e}^r | Q_i^2(t) | \nu_{k,e}^r \rangle - \langle \nu_{k,e}^r | Q_i | \nu_{k,e}^r \rangle^2$$

$$= \cos^2 \theta \sin^2 \theta$$

– Variance of $Q_\sigma \Rightarrow$ flavor entanglement:

$$\Delta Q_\sigma(\nu_e)(t) = \langle \nu_{k,e}^r | Q_\sigma^2(t) | \nu_{k,e}^r \rangle - \langle \nu_{k,e}^r | Q_\sigma(t) | \nu_{k,e}^r \rangle^2$$

$$= Q_{\nu_e \rightarrow \nu_e}^k(t) Q_{\nu_e \rightarrow \nu_\mu}^k(t)$$

in formal agreement with results obtained in QM.
Rewrite QFT flavor neutrino state as:

$$|\nu_e(t)\rangle = [U_{ee}(t)\alpha_e^\dagger + U_{e\mu}(t)\alpha_\mu^\dagger + U_{e\mu}^e(t)\alpha_e^\dagger\alpha_\mu^\dagger\beta_e^\dagger + U_{ee}^\mu(t)\alpha_e^\dagger\alpha_\mu^\dagger\beta_\mu^\dagger] |0\rangle_{e,\mu}$$

with

$$U_{ee}(t) = e^{-i\omega_1 t} \left[ \cos^2 \theta + \sin^2 \theta (e^{-i(\omega_2-\omega_1)t}|U|^2 + e^{-i(\omega_2+\omega_1)t}|V|^2) \right]$$

$$U_{e\mu}(t) = e^{-i\omega_1 t} U \cos \theta \sin \theta (e^{-i(\omega_2-\omega_1)t} - 1)$$

$$U_{e\mu}^e(t) = e^{-i\omega_1 t} V \cos \theta \sin \theta (1 - e^{-i(\omega_2+\omega_1)t})$$

$$U_{ee}^\mu(t) = e^{-i\omega_1 t} UV \sin^2 \theta (e^{-i(\omega_2+\omega_1)t} - e^{-i(\omega_2-\omega_1)t})$$

$$|U_{ee}(t)|^2 + |U_{e\mu}(t)|^2 + |U_{e\mu}^e(t)|^2 + |U_{ee}^\mu(t)|^2 = 1.$$
Linear entropies for $|\nu_e\rangle$ associated to unbalanced bipartitions $S_L^{(\alpha;\beta,\gamma,\delta)}$:

\[
S_L^{(\nu_e;\nu_\mu,\overline{\nu}_e,\overline{\nu}_\mu)} = 4|U_{e\mu}|^2(1 - |U_{e\mu}|^2), \\
S_L^{(\nu_\mu;\nu_e,\overline{\nu}_e,\overline{\nu}_\mu)} = 4|U_{ee}|^2(1 - |U_{ee}|^2), \\
S_L^{(\overline{\nu}_e;\nu_e,\nu_\mu,\overline{\nu}_\mu)} = 4|U_{e\mu}^{\overline{e}\overline{e}}|^2(1 - |U_{e\mu}^{\overline{e}\overline{e}}|^2), \\
S_L^{(\overline{\nu}_\mu;\nu_e,\nu_\mu,\overline{\nu}_e)} = 4|U_{e\mu}^{\mu\overline{e}}|^2(1 - |U_{e\mu}^{\mu\overline{e}}|^2).
\]

In the quantum mechanical limit, first two expressions reduce to the Pontecorvo analogs, while the other two go to zero.
The linear entropies associated with balanced bipartitions $S_{L,e}^{(a,b;c,d)}$ are:

\[
S_{L,e}^{(\nu_e,\nu_\mu;\bar{\nu}_e,\bar{\nu}_\mu)} = \frac{4}{3} \left[ 1 - (|U_{ee}|^2 + |U_{e\mu}|^2)^2 - (|U_{e\bar{e}}|^2 + |U_{e\bar{\mu}}|^2)^2 \right],
\]

\[
S_{L,e}^{(\nu_e,\bar{\nu}_e;\nu_\mu,\bar{\nu}_\mu)} = \frac{4}{3} \left[ 1 - (|U_{ee}|^2 + |U_{e\bar{e}}|^2)^2 - (|U_{e\mu}|^2 + |U_{e\bar{\mu}}|^2)^2 \right],
\]

\[
S_{L,e}^{(\nu_\mu,\bar{\nu}_\mu;\nu_e,\bar{\nu}_e)} = \frac{4}{3} \left[ 1 - (|U_{ee}|^2 + |U_{e\bar{e}}|^2)^2 - (|U_{e\mu}|^2 + |U_{e\bar{\mu}}|^2)^2 \right].
\]
The linear entropies \( S_L^{(a;b,c,d)} \), coincide, apart from a constant factor, with the variances associated with the particle number:

\[
\langle (\Delta N_a)^2 \rangle \equiv \langle N_a \rangle (1 - \langle N_a \rangle) = \frac{1}{4} S_L^{(a;b,c,d)}.
\]

On the other hand, the linear entropies \( S_L^{(a,b;c,d)} \) are proportional to variances of linear combination of number operators

\[
\langle [\Delta (N_e + N_\mu)]^2 \rangle \equiv \langle [\Delta Q_e]^2 \rangle = \frac{3}{4} S_{L,e}^{(\nu_e,\nu_\mu;\bar{\nu}_e,\bar{\nu}_\mu)},
\]

\[
\langle [\Delta (N_e - N_{\bar{e}})]^2 \rangle \equiv \langle [\Delta Q_{\bar{e}}]^2 \rangle = \frac{3}{4} S_{L,e}^{(\nu_e,\bar{\nu}_e;\nu_\mu,\bar{\nu}_\mu)},
\]

\[
\langle [\Delta (N_e - N_{\bar{\mu}})]^2 \rangle = \frac{3}{4} S_{L,e}^{(\nu_e,\bar{\nu}_\mu;\nu_\mu,\bar{\nu}_e)}.
\]
Linear entropies: \( S_{L}^{(\nu;\nu,\bar{\nu},\bar{\nu})} \) (double-dot-dashed line), \( S_{L}^{(\nu,\nu,\bar{\nu},\bar{\nu})} \) (dot-dashed line), \( S_{L}^{(\bar{\nu},\nu,\nu,\bar{\nu})} \) (dotted line), \( S_{L}^{(\bar{\nu},\nu,\nu,\bar{\nu})} \) (dashed line), and the average linear entropy \( \langle S_{L}^{(1:3)} \rangle \) (full line) as functions of the scaled time \( \tau = (\omega_2 - \omega_1)t \).
The linear entropies $S_{L,e}^{(\nu_e,\nu_\mu;\bar{\nu}_e,\bar{\nu}_\mu)}$ (double-dot-dashed line), $S_{L,e}^{(\nu_e,\bar{\nu}_e;\nu_\mu,\bar{\nu}_\mu)}$ (dot-dashed line), $S_{L,e}^{(\nu_e,\nu_\mu;\nu_\mu,\bar{\nu}_e)}$ (dotted line), and the average linear entropy $\langle S_{L,e}^{(2:2)} \rangle$ (full line) as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$. The mixing angle $\theta$ is fixed at the experimental value $\sin^2 \theta = 0.314$; the parameters $x$ and $p$ are fixed as $x = 10$ and $p = 5$. 