Model Independent Features of Bose-Einstein Correlations

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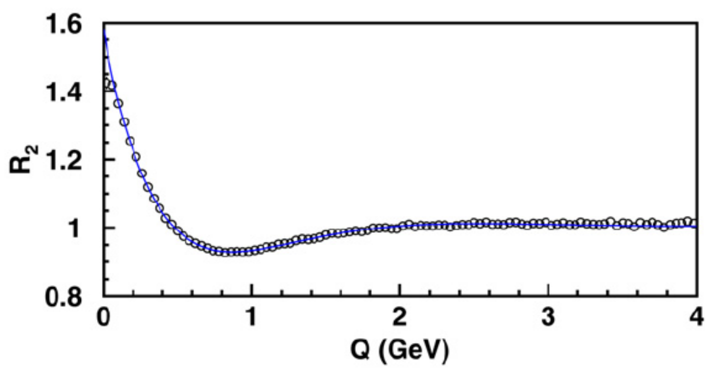

Fig. 1. The Bose–Einstein correlation function R_2 for events generated by PYTHIA. The curve corresponds to a fit of the one-sided Lévy parametrization, Eq. (13).

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Model-independent shape analysis:

- General introduction
- Edgeworth,
- Laguerre,
- Levy expansions
- Multivariate expansions

Summary

MODEL - INDEPENDENT SHAPE ANALYIS

experimental properties:

i) The correlation function tends to a constant for large values of the relative momentum Q.

ii) The correlation function has a non-trivial structure at a certain value of its argument.

The location of the non-trivial structure in the correlation function is assumed for simplicity to be close to $Q = 0$.

Model-independent but experimentally testable:

- w(t) measure in an abstract H-space
- approximate form of the correlations
- t: dimensionless scale variable

$$\int dt w(t) h_n(t) h_m(t) = \delta_{n,m},$$

$$f(t) = \sum_{n=0}^{\infty} f_n\, h_n(t),$$

$$f_n = \int dt w(t)\, f(t)\, h_n(t).$$

T. Csörgő and S: Hegyi, hep-ph/9912220, T. Csörgő, hep-ph/001233

e.g. $t = Q_I R_I$.

MODEL - INDEPENDENT SHAPE ANALYIS 2

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)},$$

$$C_2(t) = \mathcal{N}\left\{1 + \lambda_w\, w(t) \sum_{n=0}^{\infty} g_n h_n(t)\right\}$$

Model-independent AND experimentally testable:

- method for any approximate shape w(t)
- intercept parameter not the same as fit parameter $$\lambda_* = \lambda_w \sum_{n=0}^{\infty} g_n h_n(0)$$
- coefficients by numerical integration (fits to data) $$g_n = \int dt\, R_2(t) h_n(t)$$
- condition for applicability: experimentally testabe $$\int dt\, \left[R_2^2(t)/w(t)\right] < \infty$$

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EDGEWORTH EXPANSION: ~ GAUSSIAN

$$t = \sqrt{2} Q R_E,$$
$$w(t) = \exp(-t^2/2),$$
$$\int_{-\infty}^{\infty} dt \, \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m},$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[1 + \frac{\kappa_3}{3!} H_3(\sqrt{2} Q R_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2} Q R_E) + \ldots \right] \right\}.$$

$$H_n(t) = \exp(t^2/2) \left(-\frac{d}{dt} \right)^n \exp(-t^2/2).$$

$$H_1(t) = t,$$
$$H_2(t) = t^2 - 1,$$
$$H_3(t) = t^3 - 3t,$$
$$H_4(t) = t^4 - 6t^2 + 3, \ \ldots$$

3d generalization straightforward

- Applied by NA22, L3, STAR, PHENIX, ALICE, CMS (LHCb?)

LAGUERRE EXPANSIONS: ~ EXPONENTIAL

Model-independent but experimentally tested:

- w(t) exponential
- t: dimensionless
- Laguerre polynomials

$$t = QR_L,$$
$$w(t) = \exp(-t)$$
$$\int_0^\infty dt\, \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m},$$

$$C_2(Q) = \mathcal{N}\left\{1 + \lambda_L \exp(-QR_L)\left[1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + \ldots\right]\right\}$$

First successful tests

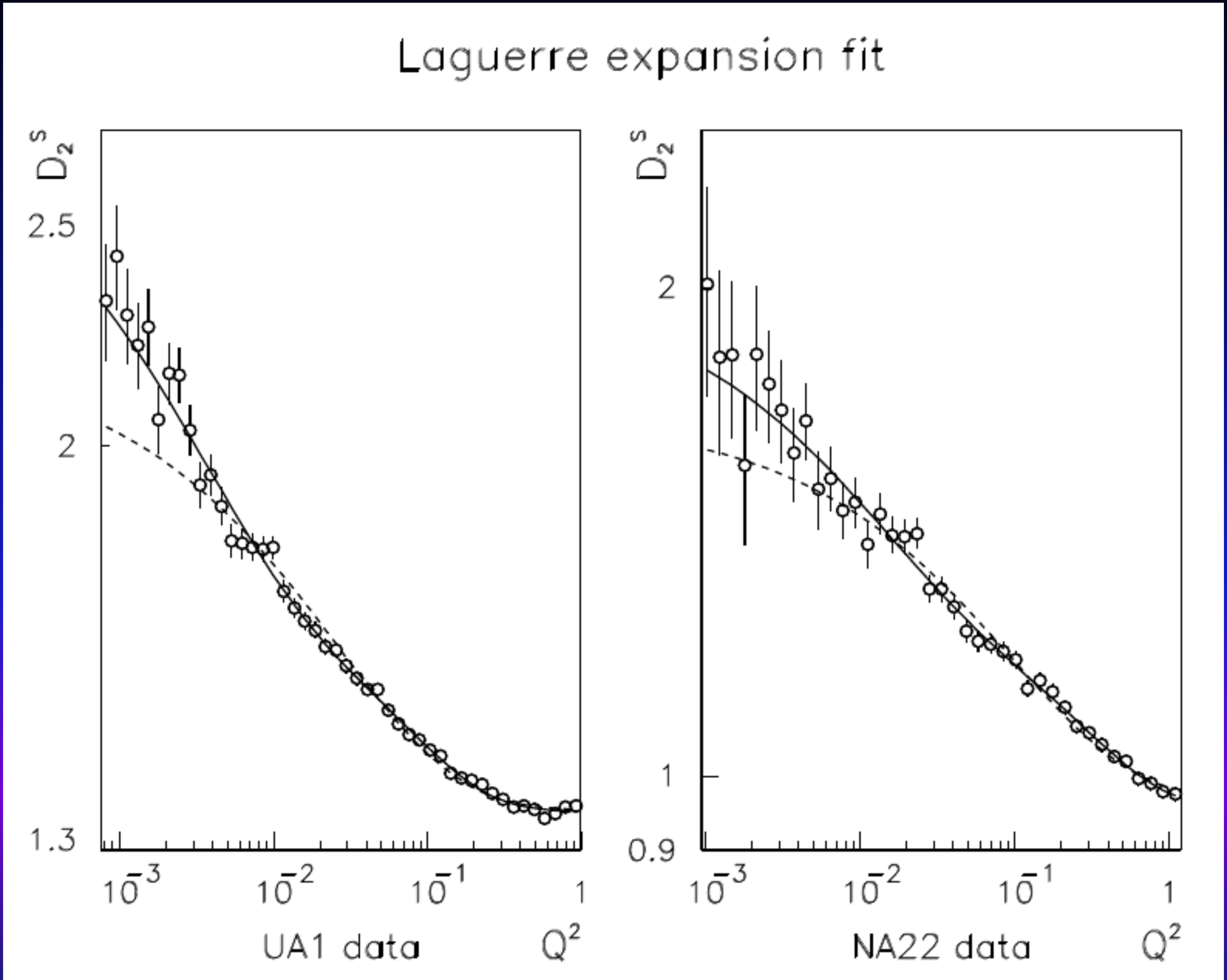
- NA22, UA1 data
- convergence criteria satisfied
- intercept parameter ~ 1

$$L_0(t) = 1,$$
$$L_1(t) = t - 1,$$

$$\int_0^\infty dt\, R_2^2(t) \exp(+t) < \infty,$$

$$\lambda_* = \lambda_L[1 - c_1 + c_2 - \ldots],$$
$$\delta^2\lambda_* = \delta^2\lambda_L\left[1 + c_1^2 + c_2^2 + \ldots\right] + \lambda_L^2\left[\delta^2 c_1 + \delta^2 c_2 + \ldots\right]$$

LAGUERRE EXPANSIONS: ~ superEXPONENTIAL



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LEVY EXPANSIONS: ~ LEVY

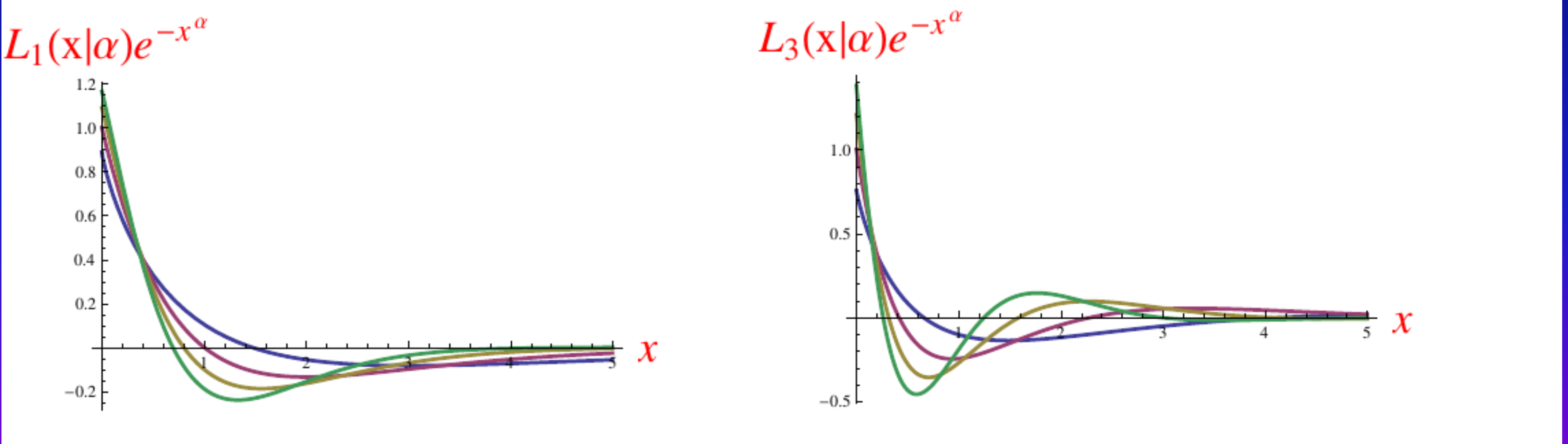
Model-independent but:

- Levy generalizes Gaussians and exponentials
- ubiquoutous in nature
- How far from a Levy?
- Need new set of polynomials orthonormal to a Levy weight

$$L_1(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx\;\, x^r f(x\,|\,\alpha) = \frac{1}{\alpha}\Gamma\left(\frac{r+1}{\alpha}\right)$$



Lévy polynomials of first and third order times the weight function e^{-x^α} for $\alpha = 0.8, 1.0, 1.2, 1.4$.

1st-order Lévy polynomial	$\gamma\left[1+\lambda e^{-R^\alpha Q^\alpha}[1+c_1 L_1(Q\|\alpha,R)]\right]$
3rd-order Lévy polynomial	$\gamma\left[1+\lambda e^{-R^\alpha Q^\alpha}[1+c_1 L_1(Q\|\alpha,R)+c_3 L_3(Q\|\alpha,R)]\right]$

M. de Kock, H. C. Eggers, T. Cs: arXiv:1206.1680v1 [nucl-th]

MULTIVARIATE EXPANSIONS

Trivial case: product of univariate functions

Non-trivial case:

example of angular correlations

- Lego plot (eta, phi)
- correlations on a unit sphere: t = (theta, phi)
- How far from a uniform distribution?
- w(t) = sin(theta)
- Spherical harmonics
- no applications yet …
- need:

  multivariate Levy

$$t = (\theta, \phi)$$

$$\int_0^{\pi} d\theta \sin(\theta) \int_0^{2\pi} d\phi \, \overline{Y}_k^n(\theta, \phi) Y_l^m(\theta, \phi) \propto \delta_{k,l}\delta_{m,n}$$

$$C_2(\theta, \phi) = 1 + \lambda_Y \sin(\theta)[1 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_l^m(\theta, \phi)].$$

MINIMAL MODEL ASSUMPTION: LEVY

experimental conditions:
(i) The correlation function tends to a constant for large values of the relative momentum Q.
(ii) The correlation function deviates from its asymptotic, large Q value in a certain domain of its argument.

(iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Model-independent but:

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)}$$

- Assumes that Coulomb can be corrected
- No assumptions about analyticity yet
- For simplicity, consider 1d case first
- For simplicity, consider factorizable x k
- Normalizations :
  - density
  - multiplicity
  - spectra

$$S(x, k) = f(x)\, g(k)$$

$$\int \mathrm{d}x\, f(x) = 1, \qquad \int \mathrm{d}k\, g(k) = \langle n \rangle,$$

$$N_1(k) = \int \mathrm{d}x\, S(x, k) = g(k).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MINIMAL MODEL ASSUMPTION: LEVY/GAUSS

$$\psi_{k_1,k_2}(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\exp(\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2) + \exp(\mathrm{i}k_1x_2 + \mathrm{i}k_2x_1)\right].$$

$$N_2(k_1,k_2) = \int \mathrm{d}x_1\mathrm{d}x_2\, S(x_1,k_1)\, S(x_2,k_2)\, |\psi_{k_1,k_2}(x_1,x_2)|^2,$$

Model-independent but:

- Assumes plane-wave propagation
- C2 measures a modulus squared Fourier-transform vs relative momentum
- Analyticity assumptions is an extra!
- Resulting correlations Gaussian
- Radius interpreted as variances

$$C_2(k_1,k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x\, \exp(\mathrm{i}q_{12}x)\, f(x),$$

$$q_{12} = k_1 - k_2.$$

$$\tilde{f}(q) \approx 1 + \mathrm{i}q\langle x\rangle - q^2\langle x^2\rangle/2 + \ldots,$$

$$C(q) = 1 + |\tilde{f}(q)|^2 \approx 2 - q^2(\langle x^2\rangle - \langle x\rangle^2) \approx 1 + \exp(-q^2R^2),$$

$$R = \sqrt{\langle x^2\rangle - \langle x\rangle^2}.$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MINIMAL MODEL ASSUMPTION: LEVY

Model-independent but:

- not assumes analyticity
- C2 measures a modulus squared Fourier-transform vs relative momentum
- Levy: Generalized Central Limit Theorems
- Stable for adding one more small random source
- Correlations non-Gaussian
- Radius not a variance
- 0 < alpha <= 2

$$C_2(k_1,k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x\, \exp(\mathrm{i}q_{12}x)\, f(x),$$

$$x = \sum_{i=1}^{n} x_i,$$

$$f(x) = \int \Pi_{i=1}^{n} \mathrm{d}x_i\, \Pi_{j=1}^{n} f_j(x_j)\, \delta(x - \sum_{k=1}^{n} x_k).$$

$$\tilde{f}_i(q) = \exp\left(iq\delta_i - |\gamma_i q|^\alpha\right), \qquad \prod_{i=1}^{n} \tilde{f}_i(q) = \exp\left(iq\delta - |\gamma q|^\alpha\right),$$

$$\gamma^\alpha = \sum_{i=1}^{n} \gamma_i^\alpha, \qquad \delta = \sum_{i=1}^{n} \delta_i.$$

$$\tilde{f}(q) = \prod_{i=1}^{n} \tilde{f}_i(q)$$

$$C(q;\alpha) = 1 + \lambda \exp\left(-|qR|^\alpha\right).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

UNIVARIATE LEVY EXAMPLES

Include some well known cases:

- alpha = 2

  Gaussian source, Gaussian C2

$$f(x) = \frac{1}{(2\pi R^2)^{1/2}} \exp\left[-\frac{(x-x_0)^2}{2R^2}\right]$$

$$C(q) = 1 + \exp\left(-q^2 R^2\right)$$

- alpha = 1

  Lorenzian source, exponential C2

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x-x_0)^2},$$

$$C(q) = 1 + \exp\left(-|q R|\right).$$

- asymmetric Levy:

  asymmetric support

  Streched exponential

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x-x_0)^{3/2}} \exp\left(-\frac{R}{8(x-x_0)}\right)$$

$$x_0 < x < \infty,$$

$$C(q) = 1 + \exp\left(-\sqrt{|q R|}\right).$$

- Radius not a variance,
- Variance infinite for Lorenz

  infinite for any alpha < 2

T. Cs, hep-ph/0001233, T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MULTIVARIATE LEVY DISTRIBUTIONS

$$C_2(k_1, k_2) = 1 + \lambda \exp\left[- \left(\sum_{i,j=1}^{3} R_{ij}^2 q_i q_j \right)^{\alpha/2} \right]$$

Model-independent but:

- A new parameter alpha generalizes Gauss
- Solved only for symmetric distributions
- Deep open problems in mathematical statistics

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

SUMMARY AND CONCLUSIONS

Several model-independent methods:

- Based on matching an abstract measure in H to the approximate shape of data
- Gaussian: Edgeworth expansions
- Exponential: Laguerre expansions
- Levy: Generalized Central Limit Theorems
- Levy expansions 0 < alpha <= 2
- Methods applicable e.g. in 2, and 3d (lego plot?)
- New directions: multivariate Levy expansions