Soft gluons and the ordering problem

René Ángeles
Jeff Forshaw
Mike Seymour

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Introduction / outline

1.- On the breaking of factorisation

- Is it possible to factorise all ***collinear*** singularities into process independent & universal functions?

Current understanding: *"No. Only for processes involving 0 (e.g. e+e-) or 1 (e.g. DIS) incoming partons or for sufficiently inclusive observables".*

- Which interactions break factorisation?

2.- Concrete example: Gaps between Jets and the ordering problem

- The breakdown of factorisation was anticipated using an algorithm, based on kT ordering, for computing the leading soft corrections to a hard process.

- The ordering problem: Is kT the correct ordering variable?.

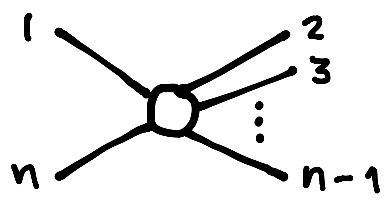
3.- kT ordering is correct at the first two non-trivial orders.

On the breakdown of collinear factorisation

Forshaw, Seymour & Siodmok hep/1206.6363
Catani, de Florian, Rodrigo hep/1112.4405

Notation I: NLO soft corrections to a hard process

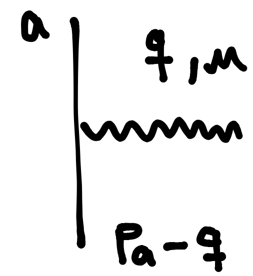
Hard process amplitude:
a vector in colour and spin spaces

$|n\rangle \equiv$ 

$2p_a \cdot p_b \sim Q^2$ for all $a, b \in \{1, ..n\}$

Soft corrections $q \ll Q$

Feynman rules simplify to

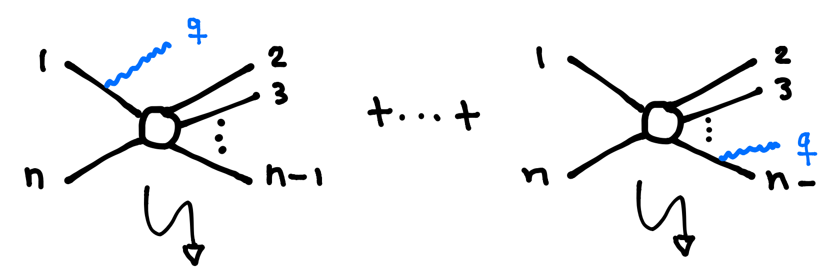


$$\simeq g \frac{T_a \, p_a^{\mu}}{p_a \cdot q + \delta}$$

$$\delta = \begin{cases} 0 \text{ if } q \text{ Real} \\ i\epsilon \text{ if } q \text{ Virtual \& } a \text{ Outgoing} \\ -i\epsilon \text{ if } q \text{ Virtual \& } a \text{ Incoming} \end{cases}$$

$\mathbf{T}_a$: Colour generator of parton a

One real emission



Hard process plus soft emission amplitude

$$J(q)\,|n\rangle \equiv \left[g \, \frac{T_1 \; p_1 \cdot \varepsilon}{p_1 \cdot q} + \cdots + g \, \frac{T_{n-1} \; p_{n-1} \cdot \varepsilon}{p_{n-1} \cdot q} \right] |n\rangle$$

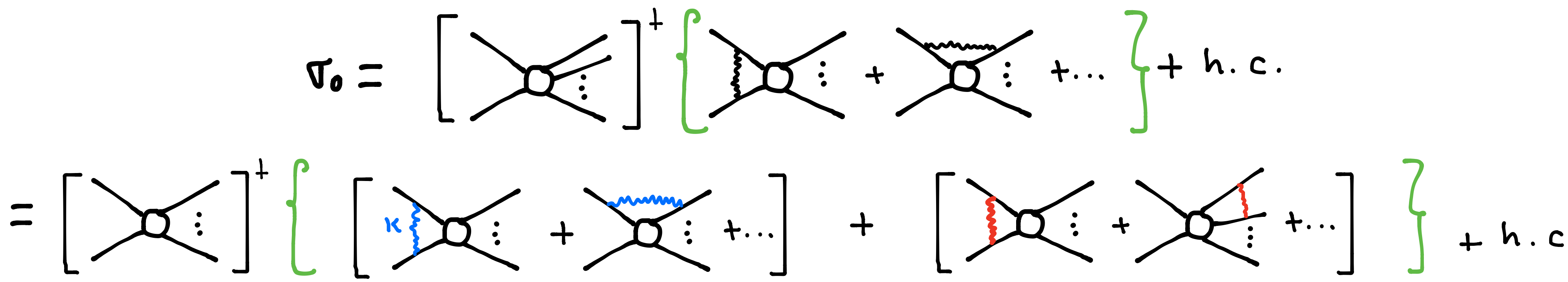
Observable ϕ

$$\sigma_1 = \int^{Q} d[q] \langle n | \, J(q) \cdot J(q) \, | n \rangle \, \phi(q)$$

Notation II: One loop

Virtual emissions have two types of contributions:

$\sigma_0 =$



$$= \frac{1}{2} \langle n | \left\{ -\int^{Q} d[k]\, J(k)\cdot J(k) + \sum_{a,b \in \text{in or out}} C^{ab}(0,Q) \right\} | n \rangle + h.c.$$

Eikonal gluons

Coulomb (Glauber) gluons (no real counterpart)

$$E^{n} \equiv \sum_{a,b}^{n} \int^{Q} d[k]\, T_a\cdot T_b \frac{p_a\cdot p_b}{p_a\cdot k\; p_b\cdot k}$$

$$C^{ab}(\alpha,\beta) = g^2 \frac{-i\pi}{8\pi^2} T_a\cdot T_b \int_{\alpha^2}^{\beta^2} \frac{dk_\perp}{k_\perp^2}$$

$$T_a\cdot T_b = (T_a\cdot T_b)^{\dagger}$$

$$C = -C^{\dagger}$$

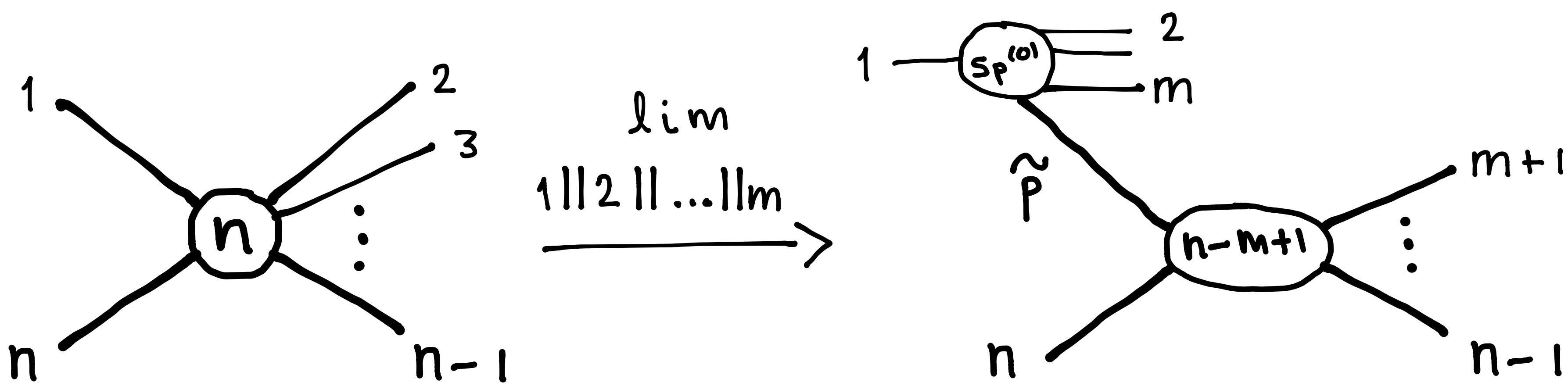
Observable ϕ

$$\Rightarrow \sigma_0 + \sigma_1 = \langle n | \int^{Q} d[q]\, J(q)\cdot J(q) \left[\phi(q) - 1\right] | n \rangle ,$$

Miscancellation induces logs:

$$\sim \alpha_s \ln^{\gamma}(V), \qquad \gamma = \{0,1,2\}.$$

Tree level collinear factorisation



$$|n\rangle \simeq Sp^{(0)}_{p}\, |n-m+1\rangle$$

Depends only on collinear partons

Depends only on non-collinear partons

$$Sp^{(0)}_{p} = Sp^{(0)}_{p}(\tilde{p}, 1, \ldots, m)$$

$$|n-m+1(\tilde{p}, m+1, \ldots, n)\rangle$$

Hence, universal (process independent) contributions at cross section level

$$\langle n|n\rangle \simeq \langle n-m+1|\, Sp^{(0)\dagger}_{p}\, Sp^{(0)}_{p}\, |n-m+1\rangle,$$

$$\equiv \langle n-m+1|\, P^{(0)}\, |n-m+1\rangle.$$

PDF's

Collinear factorisation beyond tree level

$$\mathrm{Sp}_P^{(\ell)} \, |n-m+1\rangle \,, \quad \ell-\text{loops}$$

but, in general

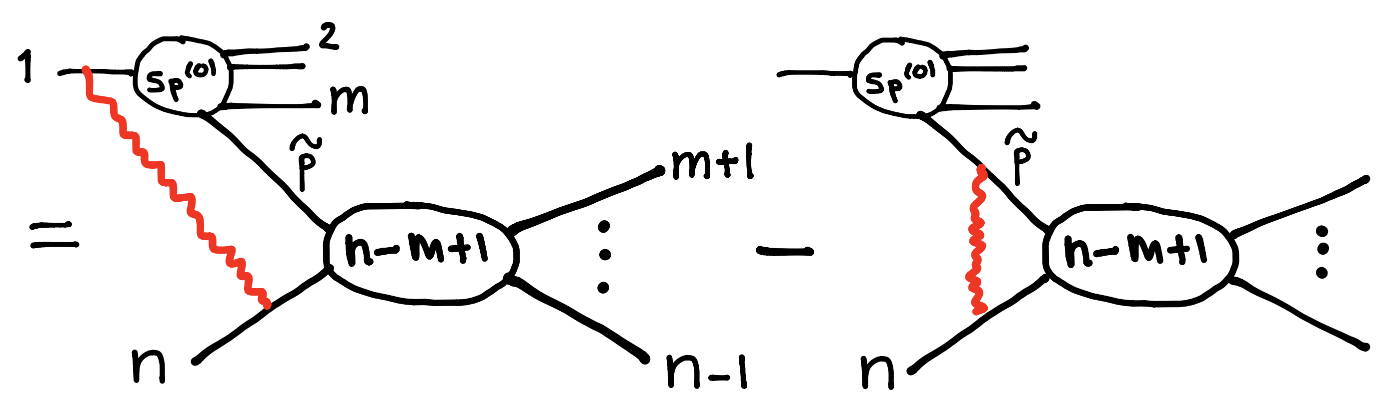
$$\mathrm{Sp}_P^{(\ell)} = \mathrm{Sp}_P^{(\ell)}(\tilde{P}, 1, \ldots, m, m+1, \ldots, n\rangle$$

Exceptions:

- 0 incoming partons (e.g. e+e-).
- 1 incoming parton (e.g. DIS).
- Sufficiently inclusive observables.

The problem first seeds at one loop order

$$\mathrm{Sp}_{P_{n.f.}}^{(1)} \, |n-m+1\rangle =$$



$$= [C^{1n} - C^{\tilde{P}n}] \, \mathrm{Sp}_P^{(0)} \, |n-m+1\rangle$$

Pole part

These contributions trivially cancel at cross section level

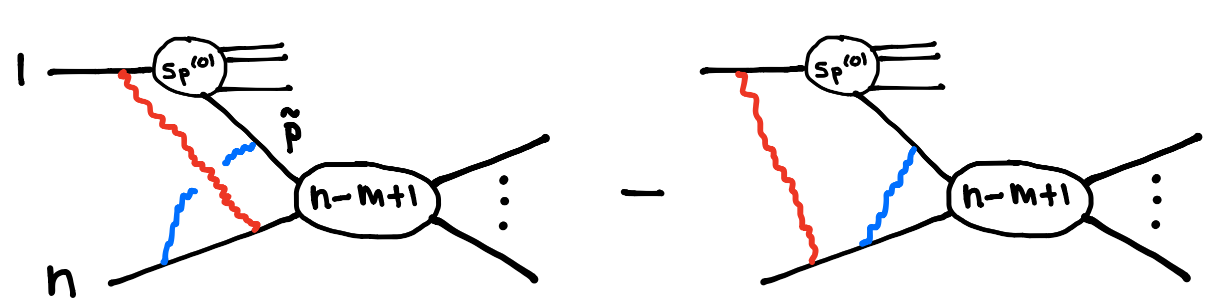
$$P_{n.f.}^{(1)} = \mathrm{Sp}_P^{(0)\dagger} \, \mathrm{Sp}_{P_{n.f.}}^{(1)} + h.c = 0. \quad \checkmark$$

Collinear factorisation at two loops

Between $\{1, ..., .m\}$
& between $\{m+1, ..., .n, \tilde{P}\}$

$$\langle n-m+1 | \left\{ P^{(2)}_{n.f.} \sim Sp^{(0)\dagger} \left[\overbrace{E^{n}}, \left[C^{1n} - C^{\tilde{P}n} \right] \right] Sp^{(0)} + \dots \right\} | n-m+1 \rangle$$

Pole part



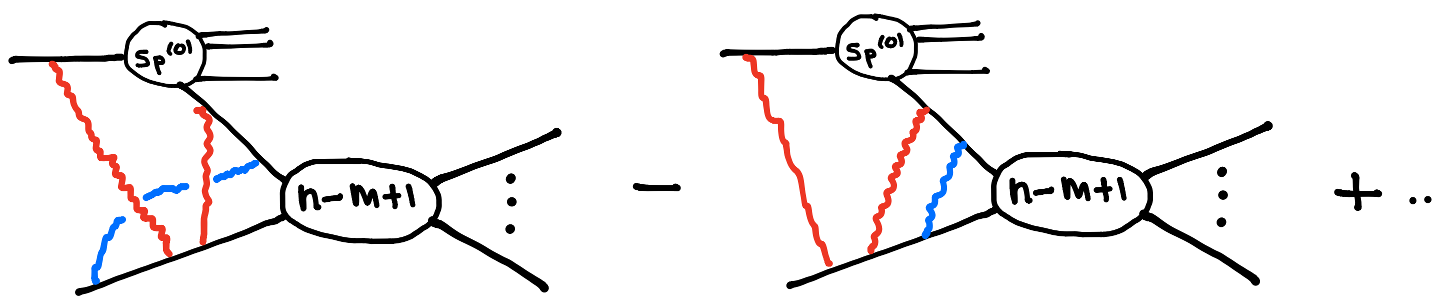
$$d\sigma = \mathrm{Tr}\left(\underbrace{Sp^{(0)} | n-m+1 \rangle \langle n-m+1 | Sp^{(0)\dagger}} \left[T_a \cdot T_b , T_e \cdot T_d \right] \right) = 0 . \checkmark$$

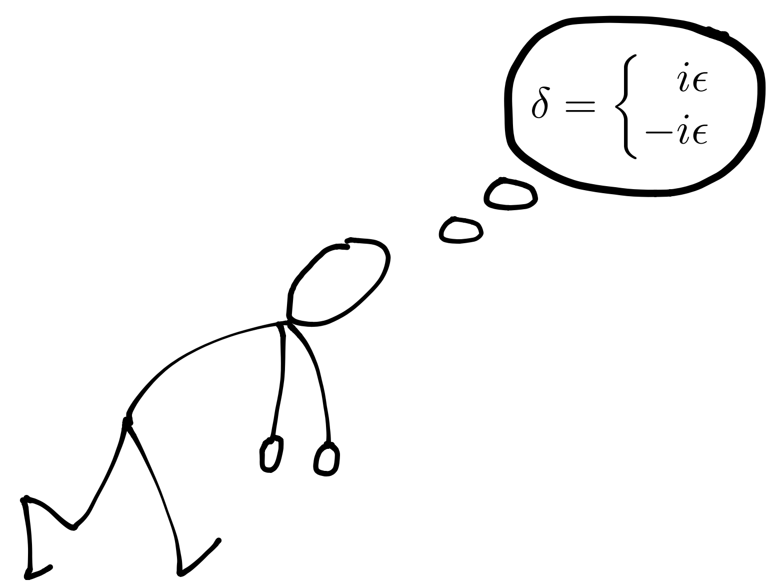
Real matrices (QCD)
Seymour & Sjödahl
hep/ 0810.5756

Three loops

$$\langle n-m+1 | \left\{ P^{(3)}_{n.f} \sim S_P^{(0)\dagger} \left[\left[E^{n}, C^{1n} \right], C^{\tilde{P}n} \right] S_P^{(0)} + \ldots \right\} | n-m+1 \rangle \neq 0. \;\; \times$$

Pole part

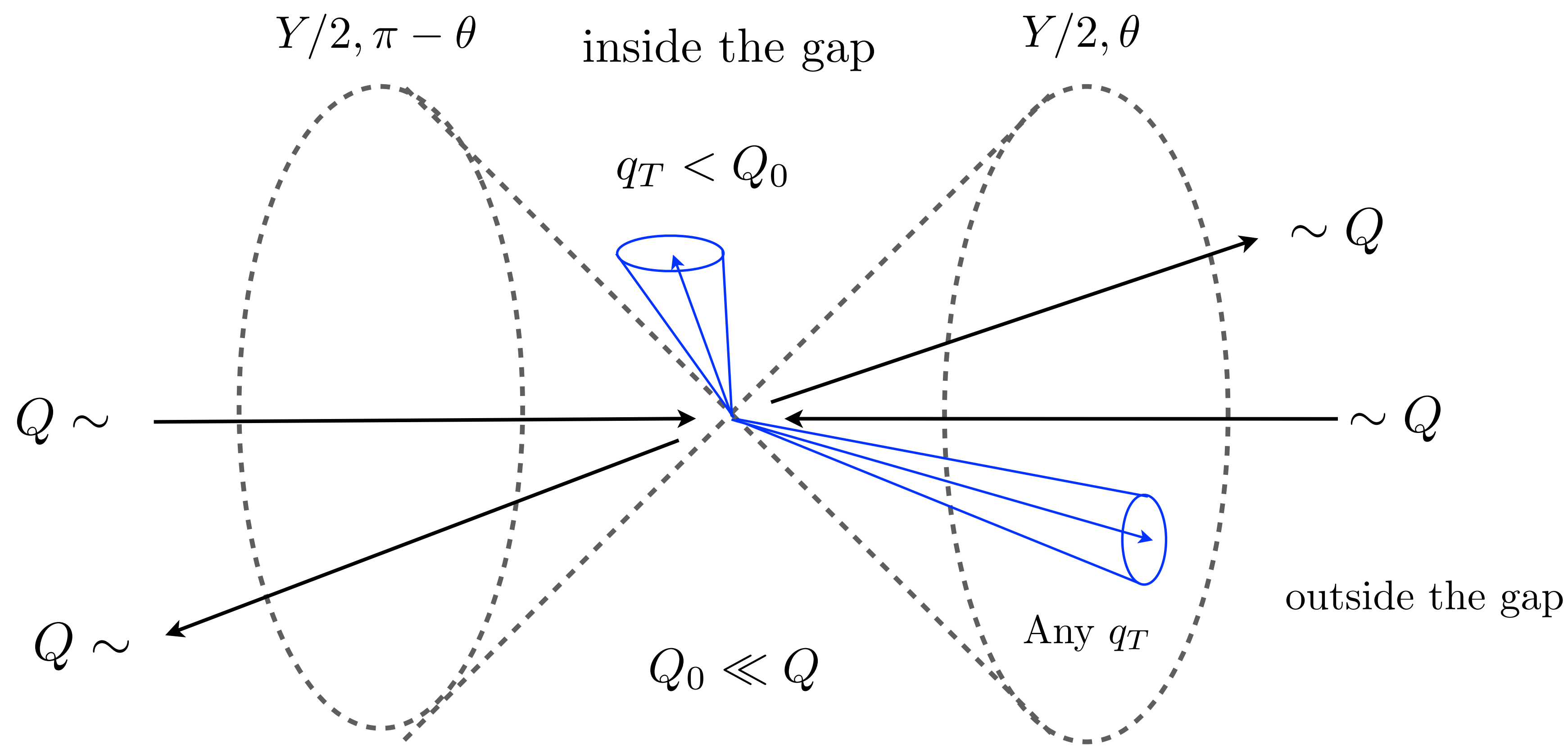




Concrete example:
Gaps between jets and the ordering problem

Forshaw, Kyrieleis & Seymour hep /0604094 ; /0808.1269

Concrete example: Gaps between jets



Soft corrections | Leading logs | Super-leading logs (2,3-loops)

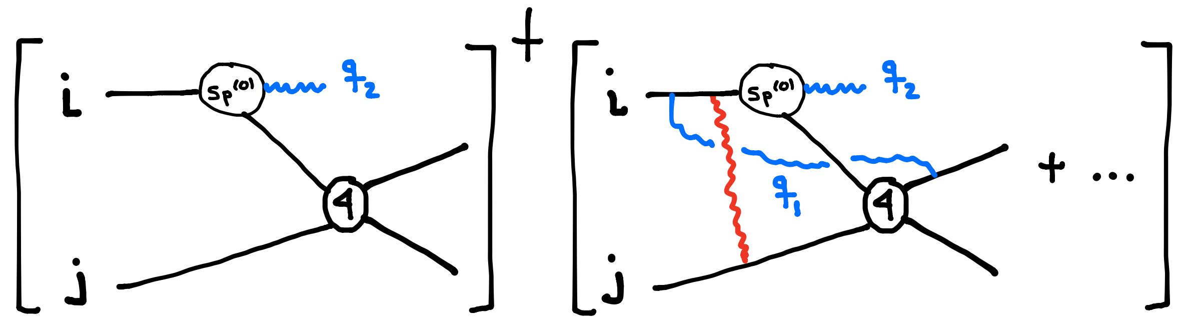
$$d\sigma_m = |\mathcal{M}(q_1, ...q_m)|^2 \, \phi \, dPS$$

$$\sim \alpha_s^n \ln^n \left(\frac{Q^2}{Q_0^2}\right)$$

$$\sim \alpha_s^3 \ln^4 \left(\frac{Q^2}{Q_0^2}\right), \alpha_s^4 \ln^5 \left(\frac{Q^2}{Q_0^2}\right)$$

Origen or super-leading logs in GBJs

$$\sigma_1+\sigma_2 \sim \int^{Q} d[q_1]\,d[q_2]$$



$$\times\, \Theta(q_{2T}-q_{1T})\,[\phi(q_2)-\phi(q_1,q_2)],$$

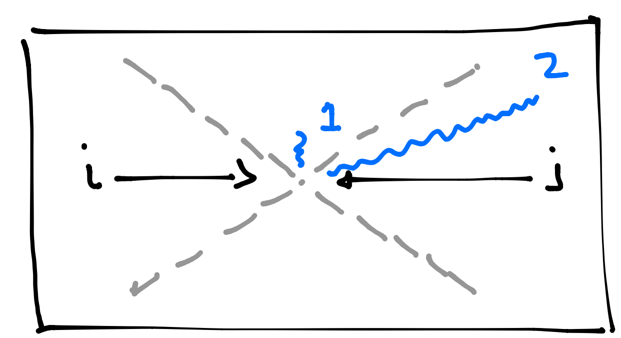
2-loops
1-real

1-loop
2-real

$$= \Theta(q_{2T}-q_{1T})\,\left[\Theta_{out}(2)\,\Theta_{in}(1)\,\Theta(q_{1T}-Q_0)+\cancel{\Theta_{in}\,\Theta_{in}\,\Theta(q_{2T}+q_{1T}-Q_0)}\right].$$

Sub

$$q_2 \parallel i \Rightarrow S_p^{(0)} \sim \frac{T_i}{\sqrt{p_i\cdot q_2}\;z},$$



$$\sim \underbrace{\alpha^3\,\ln^4\frac{Q^2}{Q_0^2}}_{\text{Super log!}}\,\langle 4|\,T_i^{\dagger}\,[T_t^2,\,T_i\cdot T_j]\,T_i\,|4\rangle + \mathcal{O}\!\left(\alpha_s^3\,\ln^3\frac{Q}{Q_0}\right).$$

Generalised Forshaw - Kyrielies - Seymour (FKS) algorithm for the leading Eikonal and Coulomb gluons

The corrections due to 0,1,2,… real emissions are

No emissions: $\sigma_0 = \langle n | V_{0Q}^{\dagger} V_{0Q} | n \rangle = | V_{0Q} | n \rangle |^2 \to$ Catani 2-loops hep/9802439

1 real emission: $d\sigma_1 = | V_{01} \mathbf{J}(1) V_{1Q} | n \rangle |^2 \, \phi \, d[1]$

2 real emissions: $d\sigma_2 = | V_{02} \mathbf{J}(2) V_{21} \mathbf{J}(1) V_{1Q} | n \rangle |^2 \, \phi \, \Theta(q_{2T} < q_{1T}) \, d[1] \, d[2]$

etc

V is the operator that inserts virtual exchanges

$$V_{ab} = \exp\left\{ -\frac{1}{2} \int d[K] \, \mathbf{J}(K) \cdot \mathbf{J}(K) \, \Theta(q_{aT} < k_T < q_{bT}) + \frac{1}{2} \sum_{\substack{a \neq b \\ a,b \in \text{in} \\ \text{or out.}}}^{l \; m} C(q_{aT}, q_{bT}) \right\}$$

$$\boxed{C^{ab}(\alpha, \beta) = g^2 \frac{-i\pi}{8\pi^2} \mathbf{T}_a \cdot \mathbf{T}_b \int_{\alpha^2}^{\beta^2} \frac{dk_T^2}{k_T^2}}$$

Based on soft gluon insertion technique.

Is it right to order emissions w.r.t the qT?

The ordering problem

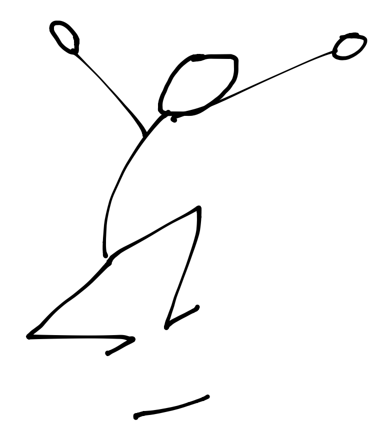
Banfi, Salam, Zanderighi hep/1001.4082

The coefficient of super leading log in GBJ are different for different ordering variables (coherence breakdown):

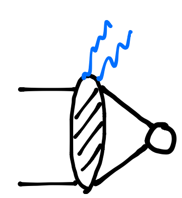
- Angular ordering: Coefficient vanishes.
- Energy ordering: Coefficient diverges.
- Transverse momentum ordering: Coefficient is finite.

Testing FKS algorithm: fixed calculation of one loop amplitudes with 1 and 2 emissions:

$$\mathrm{Im}\left\{ V_{01}\, J(1)\, V_{1Q}\, |2\rangle \right\} \simeq \left[C^{ij}(0,q_{1T})\, J(q_1) + J(q_1)\, C^{ij}(q_{1T},Q) \right] |2\rangle + \mathcal{O}(2\ \text{loops})$$



$$\mathrm{Im}\left\{ V_{02}\, J(2)\, V_{21}\, J(1)\, V_{1Q}\, |2\rangle \right\} \longrightarrow \mathrm{Im}$$



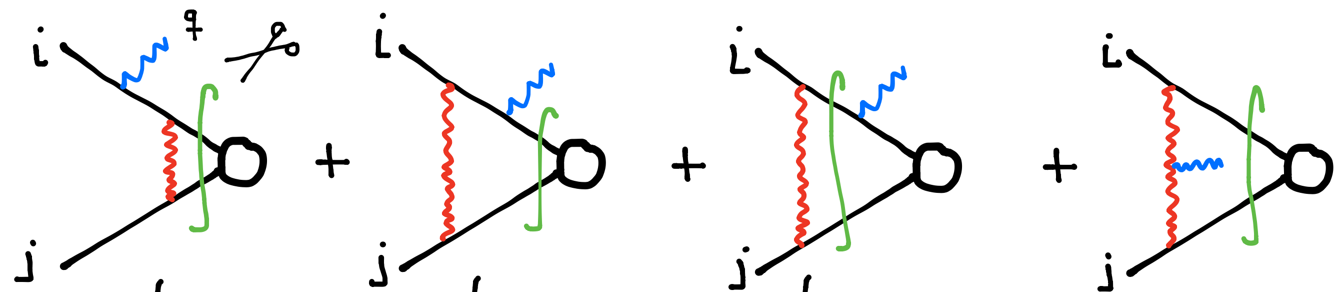
$$+\ \mathcal{O}(2\text{-loops})$$

$(\sim 130\ \text{graphs})$

Our one loop calculation is exact within the Eikonal approximation, i.e. no assumptions about ordering between the momenta of different soft emissions.

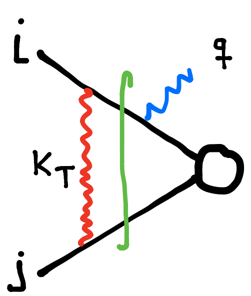
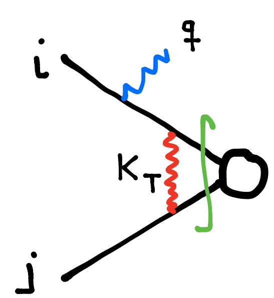
kT ordering is correct at the first two non-trivial orders.

One real emissions and one virtual exchange

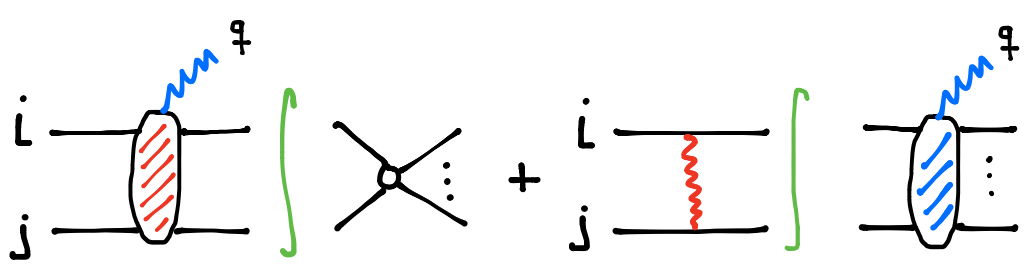


$$-i\pi\,\frac{p_i\cdot\varepsilon}{p_i\cdot q}\int_0^{Q^2}\frac{dk_T^2}{k_T^2}\left\{T_i\,(T_i\cdot T_j) - T_i\cdot T_j\;T_i + T_i\cdot T_j\;T_i\right\}|2\rangle$$

$$-i\pi\,\frac{p_i\cdot\varepsilon}{p_i\cdot q}\int\underbrace{\frac{dk_T^2}{k_T^2}\,\frac{q_T^2}{k_T^2+q_T^2}}_{\simeq\int_0^{q_T^2}\frac{dk_T^2}{k_T^2}}\left[T_i\cdot T_j\;T_i - T_i\;T_i\cdot T_j\right]|2\rangle$$

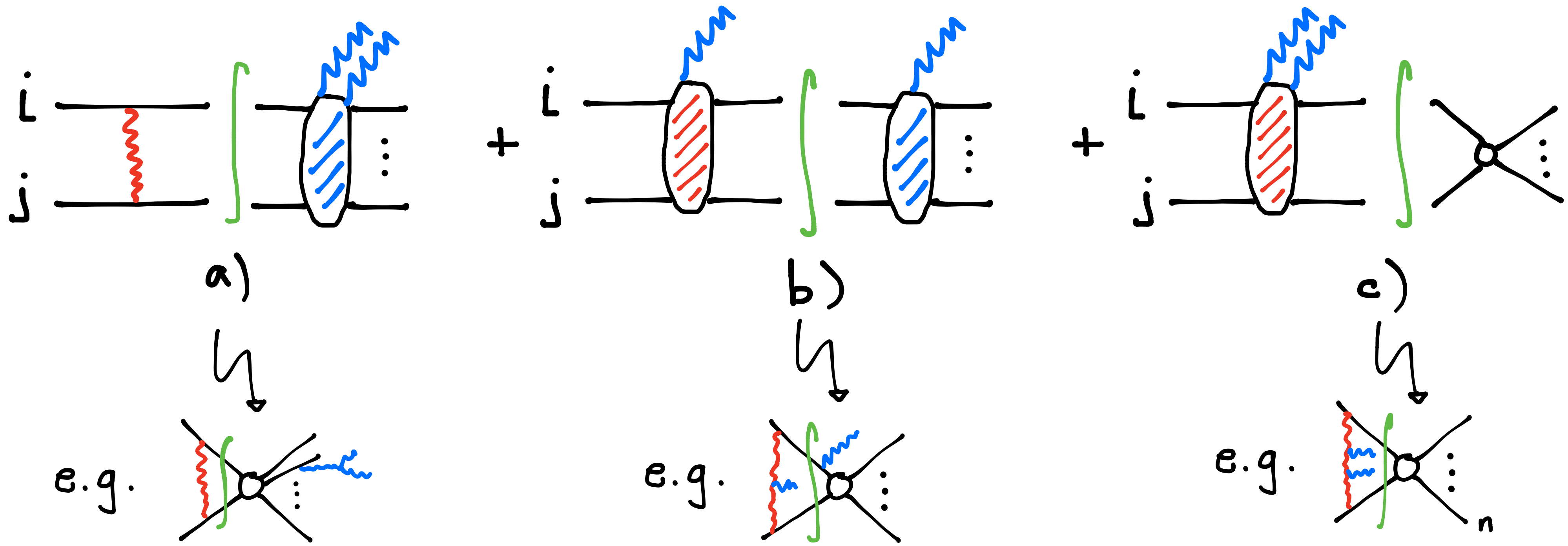
$\overset{!}{\simeq}\int_0^{q_T^2}\frac{dk_T^2}{k_T^2}$  $+\int_{q_T^2}^{Q^2}\frac{dk_T^2}{k_T^2}$ 

This proof generalises for a general hard process



$$\simeq\left[C^{ij}(0,q_T)\,J(q) + J(q)\,C^{ij}(q_T,Q)\right]|n\rangle$$

Two emissions case (paper in preparation)

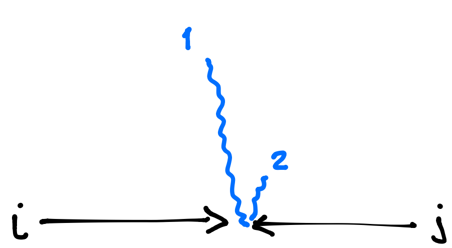


For strongly order emissions:

$$\simeq \Big[C^{ij}(0,q_{2T})\, J(q_2)\, J(q_1) + J(q_2)\, C^{ij}(q_{2T},q_{1T})\, J(q_1) + J(q_2)\, J(q_1)\, C^{ij}(q_{1T},Q) \Big] |n\rangle$$

$p_i \cdot p_j = p_i^+ p_j^-$,

$q_1^\pm \sim q_{1T} \gg q_2^\pm \sim q_{2T}$



In agreement with the FKS algorithm.

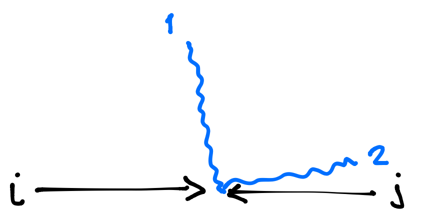
Collinear limits

Emission q_2 collinear with p_i by virtue of its small transverse momentum and q_1 wide angle:

$$\simeq \left[C^{ij}(0,q_{2T})\,\tilde{J}(q_2)\,J(q_1) + \tilde{J}(q_2)\,C^{ij}(q_{2T},q_{1T})\,J(q_1) + \tilde{J}(q_2)\,J(q_1)\,C^{ij}(q_{1T},Q) \right] |n\rangle$$

$q_1^{\pm} \sim q_{1T} >> q_{2T}$

$\&\ \ q_2^{+} >> q_{2T}$



In agreement with the FKS algorithm

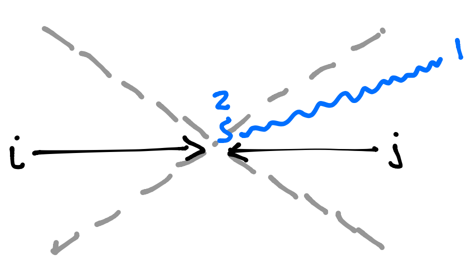
$$\left(\tilde{J}(q) \equiv g\, T_i \frac{p_i \cdot \varepsilon}{p_i \cdot q} \right)$$

Emission q_1 collinear with p_i by virtue of its high energy and q_2 wide angle:

$$\simeq \left[C^{ij}(0,q_{2T})\,J(q_2)\,\tilde{J}(q_1) + J(q_2)\,C^{ij}(q_{2T},q_{1T})\,\tilde{J}(q_1) + J(q_2)\,\tilde{J}(q_1)\,C^{ij}(q_{1T},Q) \right] |n\rangle$$

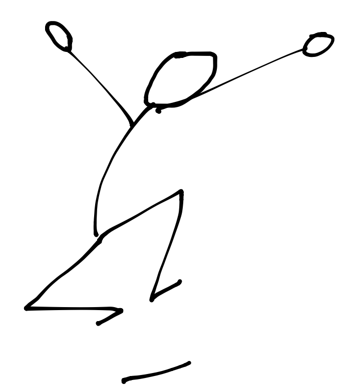
$q_{1T} >> q_2^{+} \sim q_{2T}$

$q_1^{+} >> q_{1T}$

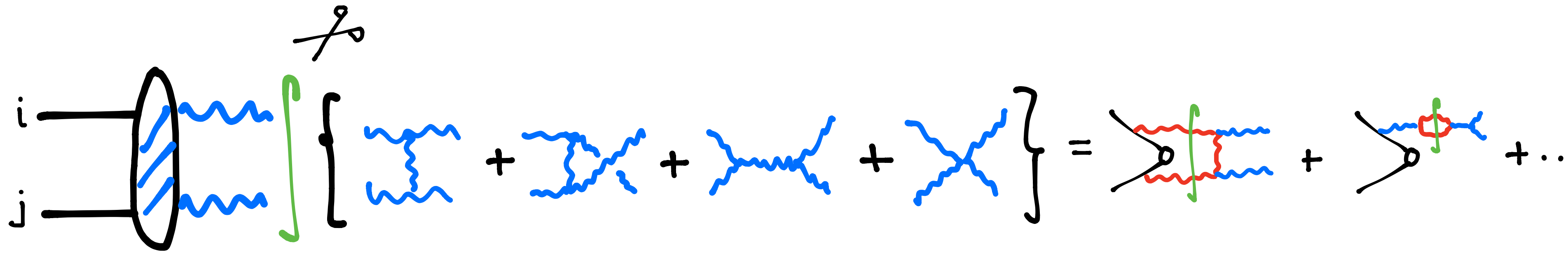


Limit of the SL-logs

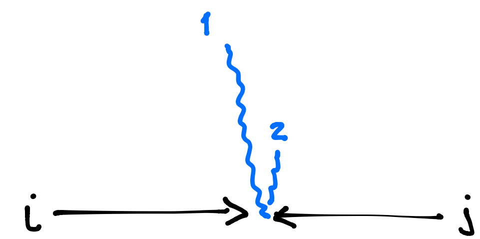
In agreement with the FKS algorithm



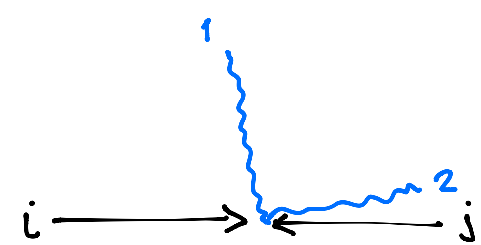
At this order, there are Coulomb exchanges between the emitted gluons



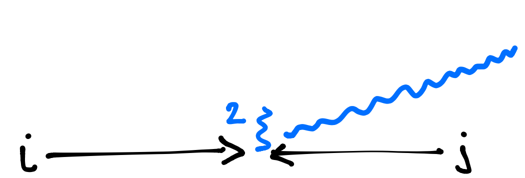
$$\simeq C^{q_1 q_2}(0, q_{2T})\, J(q_2)\, J(q_1)\, |2\rangle$$



$$\simeq C^{q_1 q_2}(0, q_{2T})\, \tilde{J}(q_2)\, J(q_1)\, |2\rangle$$



$$\simeq C^{q_1 q_2}(0, q_{2T})\, J(q_2)\, \tilde{J}(q_1)\, |2\rangle$$



This proves the FKS algorithm for $|2\rangle$ at one loop order

$$\mathrm{Im}\left\{ V_{02} J(2) V_{21} J(1) V_{1Q} |2\rangle \right\} \simeq$$

$$\left[\left(C^{ij}(0, q_{2T}) + C^{q_1 q_2}(0, q_{2T}) \right) J(q_2)\, J(q_1) + J(q_2)\, C^{ij}(q_{2T}, q_{1T})\, J(q_1) + J(q_2)\, J(q_1)\, C^{ij}(q_{1T}, Q) \right] |2\rangle$$

$$+\, \mathcal{O}(2\text{-loops})\,.$$

$|n\rangle$ in progress. (Already integrated. Stay tuned)

Conclusions

- We studied the soft corrections to hard process due to one virtual gluon and one or two real gluon emissions. We focused on the Coulomb exchanges and computed their leading behaviour in the strongly ordered regime and in the single collinear limits described above.

- For a general 2->0 hard process, the leading behaviour can be written as products of currents and kT ordered Coulomb operators. For a general 2->m hard process, this is also true for the Coulomb gluons exchanged between the incoming partons. The non-abelian nature of QCD plays a central role in engineering kT ordering.

- These results constitute non-trivial evidence that the the FKS algorithm correctly predicts the leading soft corrections to a hard process. In particular, the leading soft corrections to the GBJ.

- Studying the leading behaviour of Coulomb interactions is an important step towards including factorisation breaking effects and perhaps knowing the definition of a sufficiently inclusive observable.