

High-energy QCD and rapidity evolution of Wilson lines

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JLAB & ODU

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1 Introduction: BFKL pomeron in high-energy pQCD

- Regge limit in QCD.
- Perturbative QCD at high energies.
- BFKL and collider physics

2 High-energy scattering and Wilson lines

- High-energy scattering and Wilson lines.
- Evolution equation for color dipoles.
- Light-ray vs Wilson-line operator expansion.
- Leading order: BK equation.

3 NLO high-energy amplitudes

- Conformal composite dipoles and NLO BK kernel in $\mathcal{N} = 4$.
- Photon impact factor.
- NLO BK kernel in QCD.
- rcBK.
- NLO hierarchy of Wilson-lines evolution.
- Conclusions

1 Rapidity evolution of gluon TMD

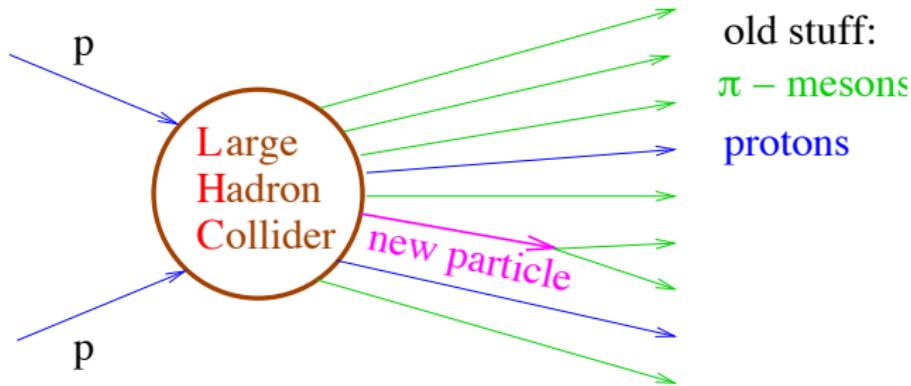
- Definition.
- One loop: real corrections
- Virtual corrections.
- One-loop result
- Conclusions

High-energy scattering

Heisenberg uncertainty principle: $\Delta x = \frac{\hbar}{p} = \frac{\hbar c}{E}$

LHC: $E=7 \rightarrow 14 \text{ TeV} \Leftrightarrow \text{distances} \sim 10^{-18} \text{ cm}$

(Planck scale is 10^{-33} cm - a long way to go!)



To separate a “new physics signal” from the “old” background one needs to understand the behavior of QCD cross sections at large energies

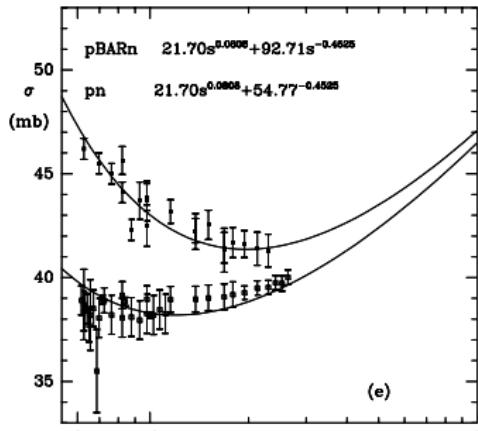
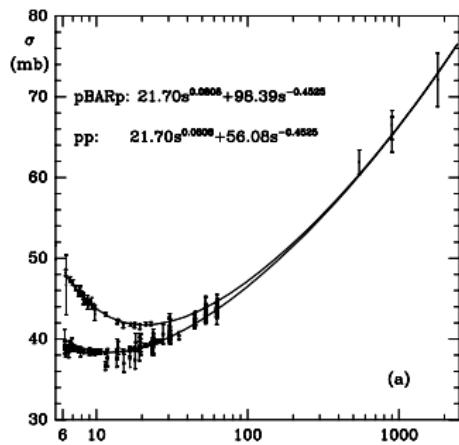
Strong interactions at asymptotic energies: Froissart bound

Regge limit: $E \gg$ everything else

$$\left. \begin{array}{c} \text{Causality} \\ \text{Unitarity} \end{array} \right\} \Rightarrow \sigma_{\text{tot}} \stackrel{E \rightarrow \infty}{\leq} \ln^2 E \quad \text{Froissart, 1962}$$

Long-standing problem - not explained in any quantum field theory (or string theory) in 50 years!

Experiment: $\sigma_{\text{tot}} \sim s^{0.08}$ ($s \equiv 4E_{\text{c.m.}}^2$). Numerically close to $\ln^2 E$.



Deep inelastic scattering in QCD

$D_q(x_B) \rightarrow D_q(x_B, Q^2)$ - “scaling violations”

DGLAP evolution (LLA(Q^2)

$$Q \frac{d}{dQ} D_q(x, Q^2) = \int_x^1 dx' K_{\text{DGLAP}}(x, x') D_q(x', Q^2)$$

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-77

$$K_{\text{DGLAP}} = \alpha_s(Q) K_{\text{LO}} + \alpha_s^2(Q) K_{\text{NLO}} + \alpha_s^3(Q) K_{\text{NNLO}} \dots$$

The DGLAP equation sums up logs of $\frac{Q^2}{m_N^2}$

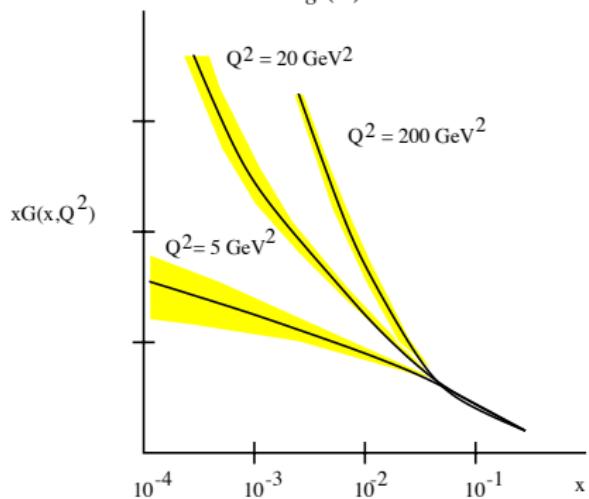
$$D_q(x, Q^2) = \sum_n \left(\alpha_s \ln \frac{Q^2}{m_N^2} \right)^n [a_n(x) + \alpha_s b_n(x) + \alpha_s^2 c_n(x) + \dots]$$

One fit at low $Q_0^2 \sim 1 \text{ GeV}^2$ describes all the experimental data on DIS!

Deep inelastic scattering at small x_B

Regge limit in DIS: $E \gg Q \equiv x_B \ll 1$

HERA data for $x D_g(x)$



DGLAP evolution $\equiv Q^2$ evolution

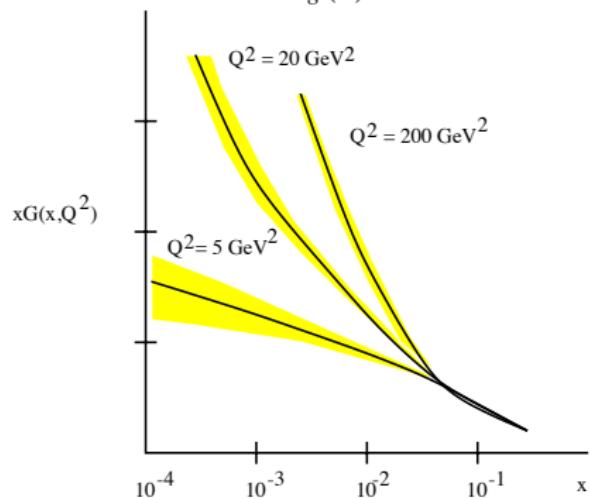
$$Q \frac{d}{dQ} D_g(x_B, Q^2) = K_{\text{DGLAP}} D_g(x_B, Q^2)$$

Not really a theory -
needs the x -dependence of the input at
 $Q_0^2 \sim 1 \text{ GeV}^2$

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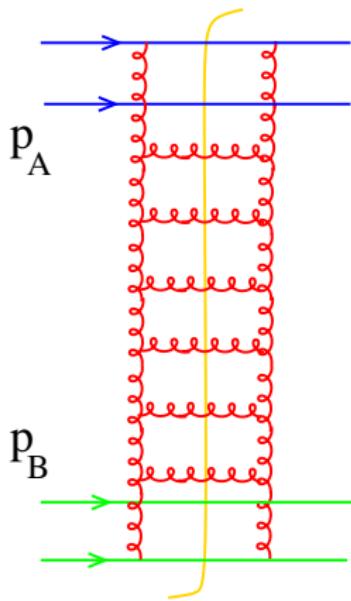
BFKL evolution $\equiv x_B$ evolution
(Balitsky, Fadin, Kuraev, Lipatov,
1975-78)

$$\frac{d}{dx_B} D_g(x_B, Q^2) = K_{\text{BFKL}} D_g(x_B, Q^2)$$

Theory, but with problems

In pQCD: Leading Log Approximation \Rightarrow BFKL pomeron

$$s = (p_A + p_B)^2 \simeq 4E^2$$

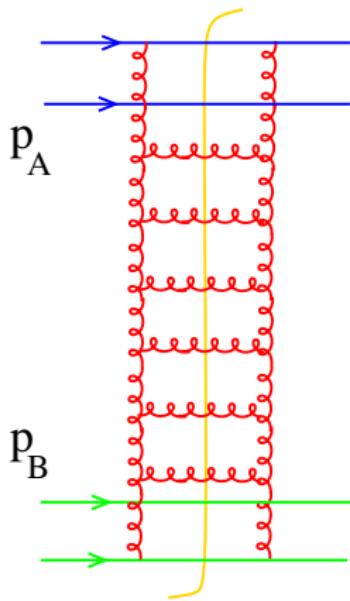


Leading Log Approximation (LLA(x)):

$$\alpha_s \ll 1, \quad \alpha_s \ln s \sim 1$$

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Leading Log Approximation (LLA(x)):

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The sum of gluon ladder diagrams gives

$$\sigma_{\text{tot}} \sim s^{12 \frac{\alpha_s}{\pi} \ln 2} \quad \text{BFKL pomeron}$$

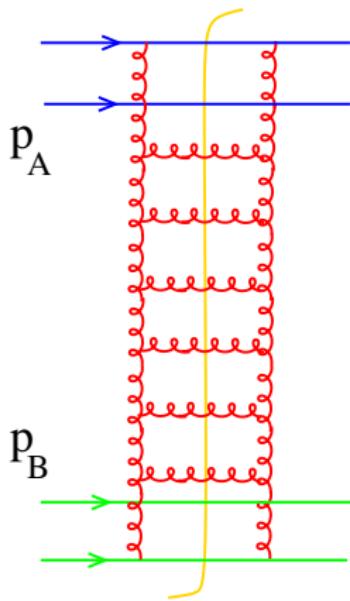
Numerically: for DIS at HERA

$$\sigma \sim s^{0.3} = x_B^{-0.3}$$

- qualitatively OK

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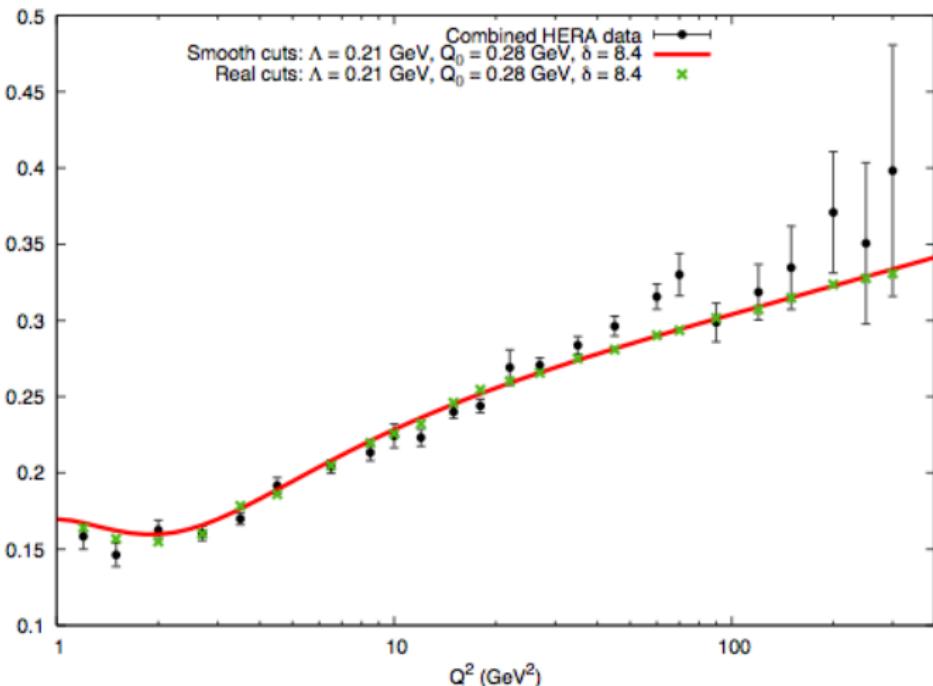
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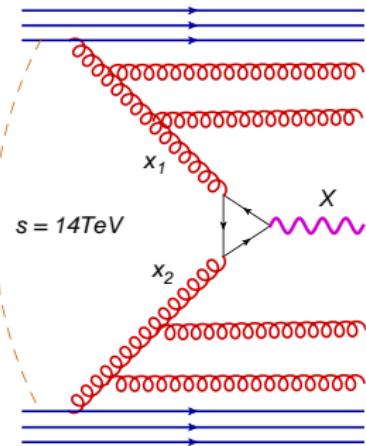
BFKL vs HERA data

$$F_2(x_B, Q^2) = c(Q^2)x_B^{-\lambda(Q^2)}$$



M.Hentschinski, A. Sabio Vera and C. Salas, 2010

DGLAP vs BFKL in particle production

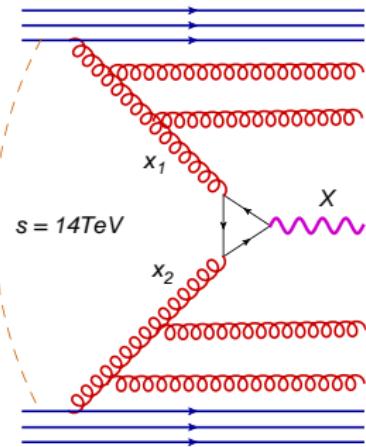


Collinear factorization (LLA(Q^2)):

$$\sigma_{pp \rightarrow X} = \int_0^1 dx_1 dx_2 D_g(x_1, m_X) D_g(x_2, m_X) \sigma_{gg \rightarrow X}$$

sum of the logs $\left(\alpha_s \ln \frac{m_X^2}{m_N^2}\right)^n$, $\ln \frac{s}{m_X^2} \sim 1$

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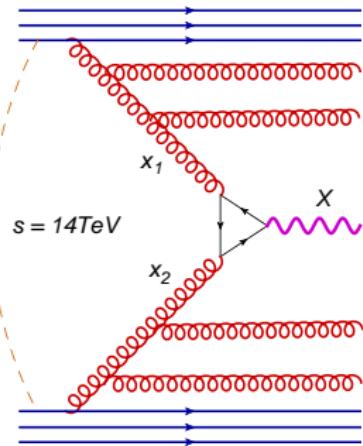
LLA(x): k_T -factorization

$$\sigma_{pp \rightarrow X} = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg \rightarrow X}$$

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Much less understood theoretically.

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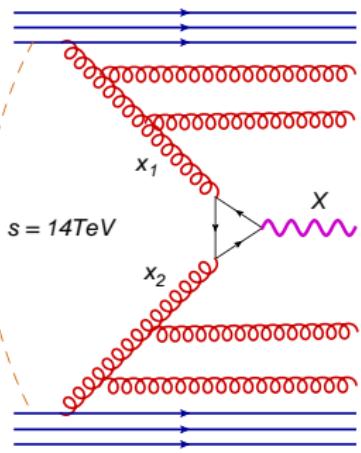
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Much less understood theoretically.

For Higgs production in the central rapidity region $x_{1,2} \sim \frac{m_H}{\sqrt{s}} \simeq 0.01$ and we know from DIS experiments that at such x_B the DGLAP formalism works pretty well \Rightarrow no need for BFKL resummation

DGLAP vs BFKL in particle production



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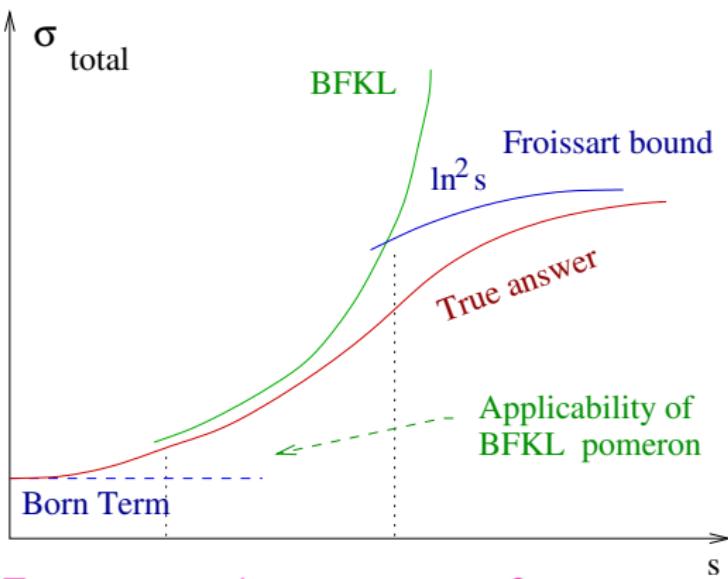
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Much less understood theoretically.

For $m_X \sim 10\text{GeV}$ (like $\bar{b}b$ pair or mini-jet) collinear factorization does not seem to work well \Rightarrow some kind of BFKL resummation is needed.

Towards the high-energy QCD



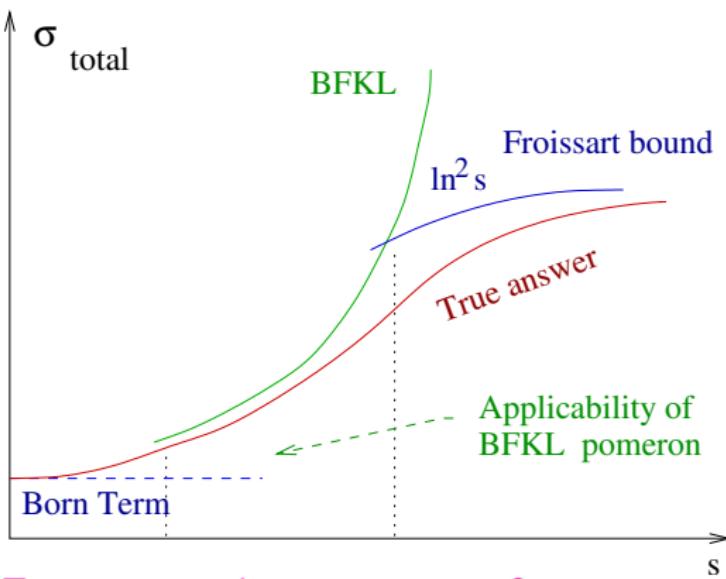
$\sigma_{\text{tot}} \sim s^{12 \frac{\alpha_s}{\pi} \ln 2}$ violates
Froissart bound $\sigma_{\text{tot}} \leq \ln^2 s$
 \Rightarrow pre-asymptotic behavior.

True asymptotics as $E \rightarrow \infty = ?$

Possible approaches:

- Sum all logs $\alpha_s^m \ln^n s$
- Reduce high-energy QCD to 2 + 1 effective theory

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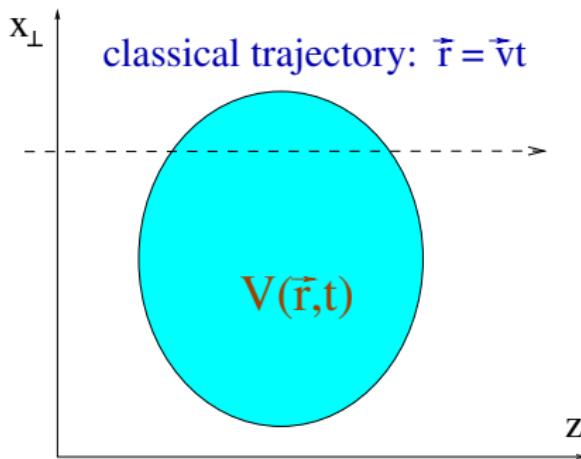
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This talk: leading order + NLO corrections $\alpha_s^{n+1} \ln^n s$

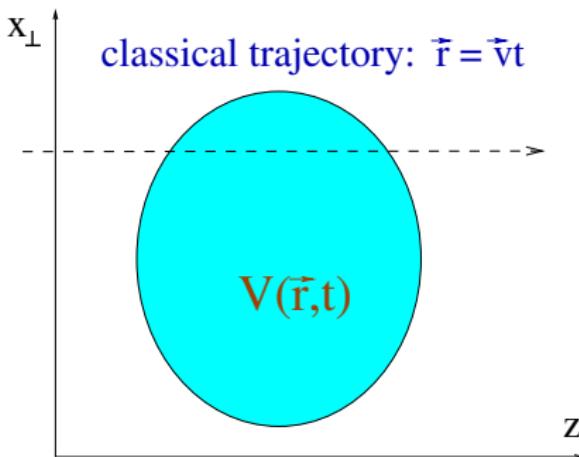
High-energy scattering and “Wilson lines” in quantum mechanics



WKB approximation: $\Psi \sim e^{\frac{i}{\hbar}S}$

$$\begin{aligned} S &= \int (pdz - Edt) \\ &= -Et + \int dz' \sqrt{2m(E - V(z'))} \end{aligned}$$

High-energy scattering and “Wilson lines” in quantum mechanics



classical trajectory: $\vec{r} = \vec{v}t$

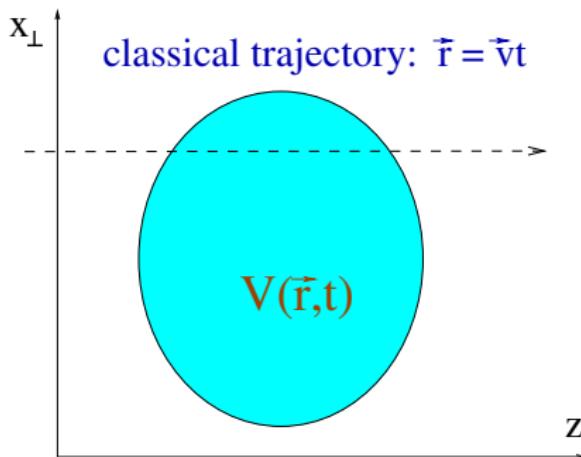
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$$\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar}(Et - kx)} e^{-\frac{i}{\hbar} \int_{-\infty}^z dz' V(z')}$$

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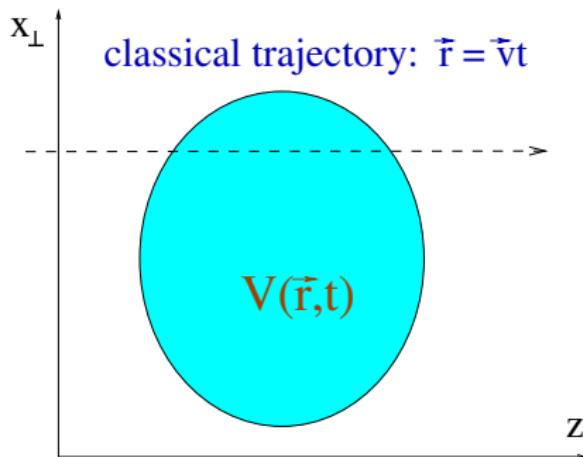
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Ψ at high energy = free wave \times phase factor ordered along the line $\parallel \vec{v}$.

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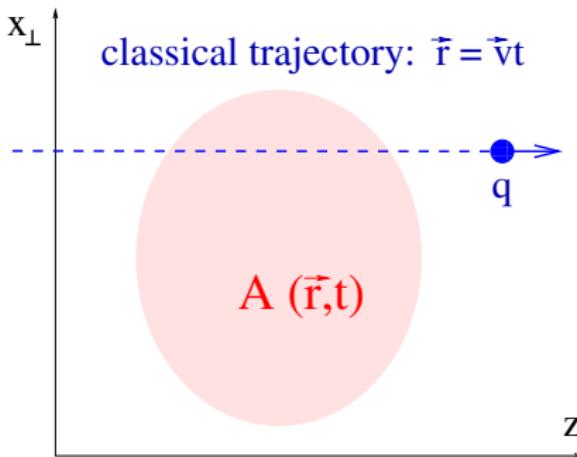
Ψ at high energy = free wave \times phase factor ordered along the line $\parallel \vec{v}$.

The scattering amplitude is proportional to $\Psi(t = \infty)$ defined by

$$U(x_{\perp}) = e^{-\frac{i}{\hbar} \int_{-\infty}^{\infty} dz' V(z' + x_{\perp})}$$

Glauber formula: $\sigma_{\text{tot}} = 2 \int d^2 x_{\perp} [1 - \Re U(x_{\perp})]$

High-energy phase factor in QED and QCD



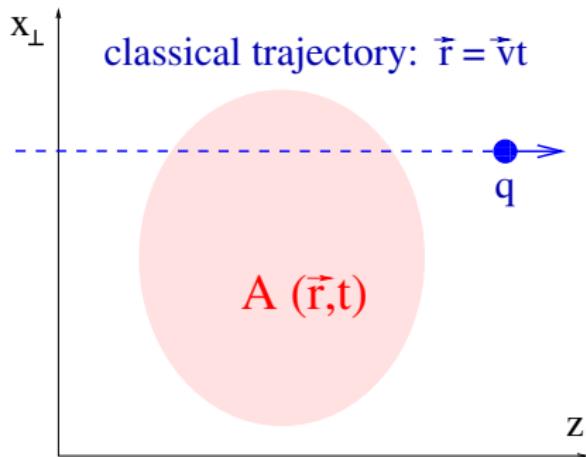
classical trajectory: $\vec{r} = \vec{v}t$

$$\begin{aligned} S_e &= \int dt \left\{ -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A} \right\} \\ &= S_{\text{free}} + \int dt (-e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A}) \end{aligned}$$

⇒ phase factor for the high-energy scattering is

$$\begin{aligned} U(x_\perp) &= e^{-\frac{ie}{\hbar c} \int_{-\infty}^{\infty} dt (-e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A})} \\ &= e^{-\frac{ie}{\hbar c} \int_{-\infty}^{\infty} dt \dot{x}_\mu A^\mu(x(t))} \end{aligned}$$

High-energy phase factor in QED and QCD



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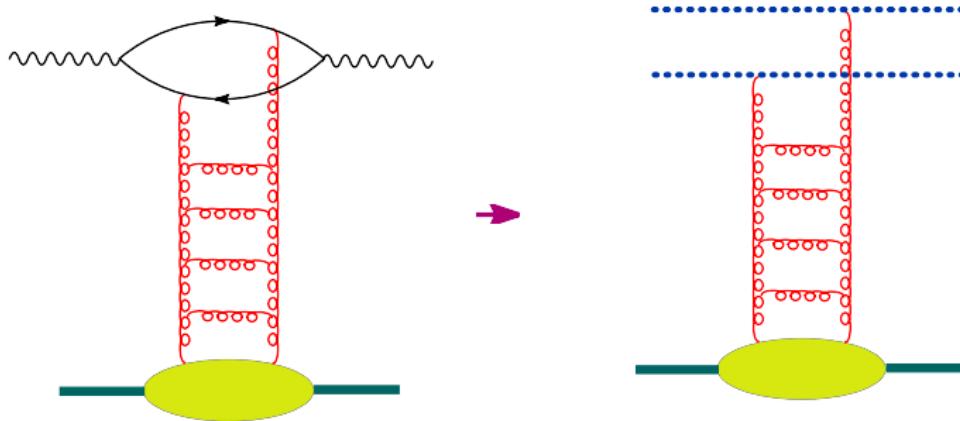
In QCD $e \rightarrow -g$, $A_{\mu} \rightarrow A_{\mu} \equiv A_{\mu}^a t^a$ t^a - color matrices

$$\Rightarrow U(x_{\perp}, v) = P \exp \left\{ \frac{ig}{\hbar c} \int_{-\infty}^{\infty} dt \dot{x}_{\mu} A^{\mu}(x(t)) \right\} \quad \text{Wilson - line operator}$$

(Later $\hbar = c = 1$)

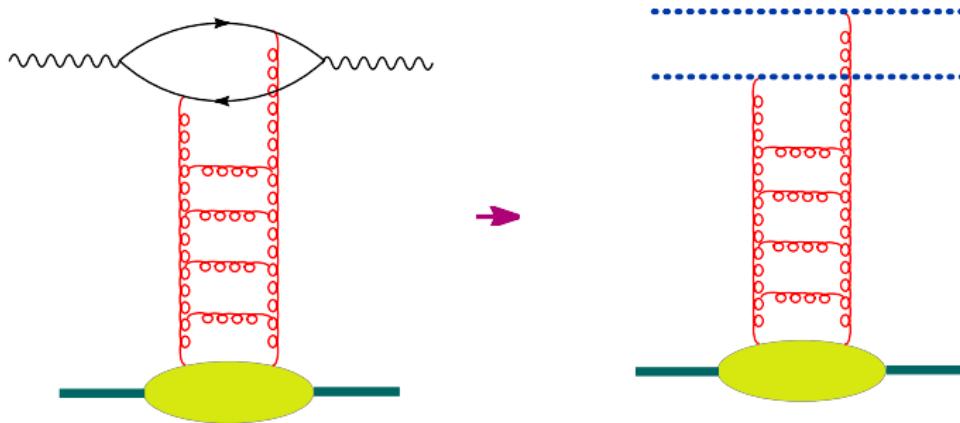
DIS at high energy

- At high energies, particles move along straight lines \Rightarrow the amplitude of $\gamma^*A \rightarrow \gamma^*A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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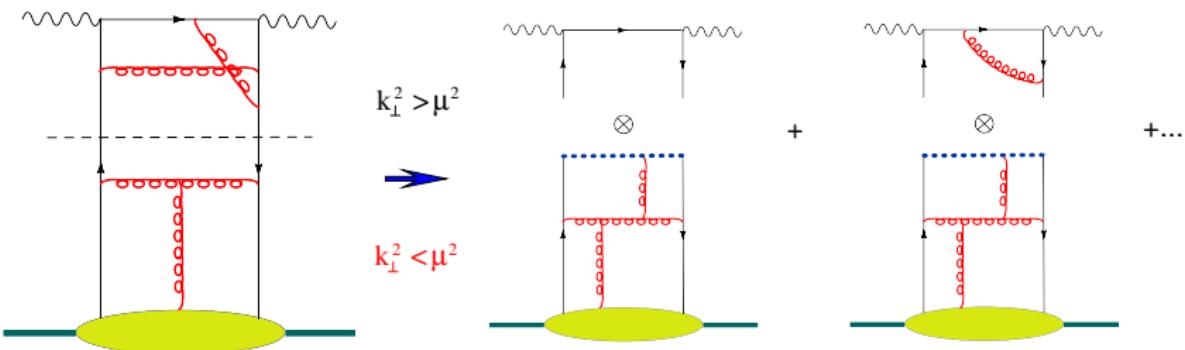
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$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle$$

Formally, \rightarrow means the operator expansion in Wilson lines

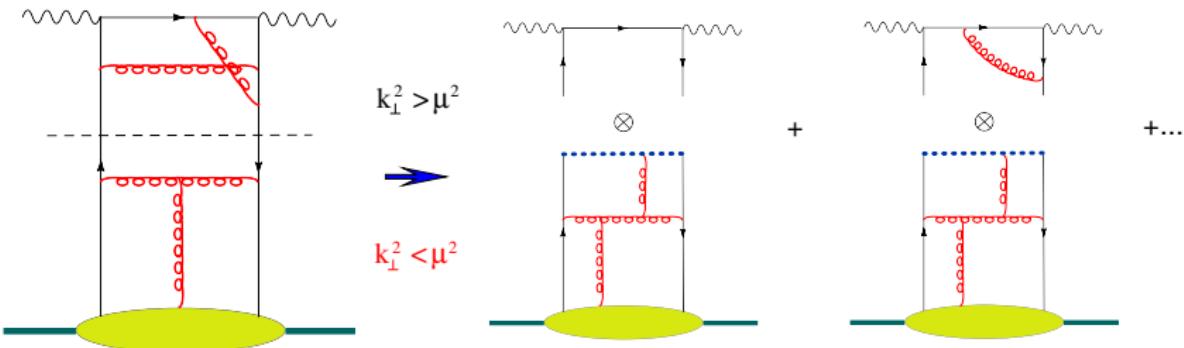
Light-cone expansion and DGLAP evolution in the NLO



$k_{\perp}^2 > \mu^2$ - coefficient functions

$k_{\perp}^2 < \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

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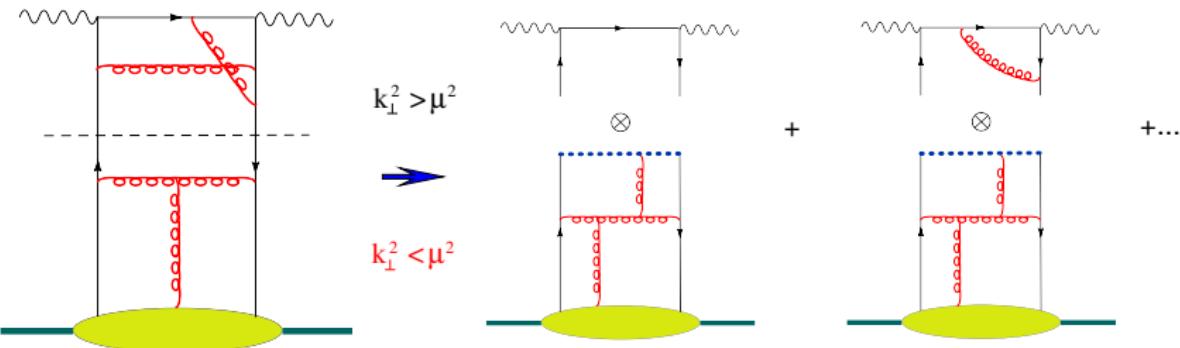
OPE in light-ray operators

$$(x - y)^2 \rightarrow 0$$

$$T\{j_\mu(x)j_\nu(0)\} = \frac{x_\xi}{2\pi^2 x^4} \left[1 + \frac{\alpha_s}{\pi} (\ln x^2 \mu^2 + C) \right] \bar{\psi}(x) \gamma_\mu \gamma^\xi \gamma_\nu [x, 0] \psi(0) + O(\frac{1}{x^2})$$

$$[x, y] \equiv P e^{ig \int_0^1 du (x-y)^\mu A_\mu(ux+(1-u)y)} - \text{gauge link}$$

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Renorm-group equation for light-ray operators \Rightarrow DGLAP evolution of parton densities
 $(x-y)^2 = 0$

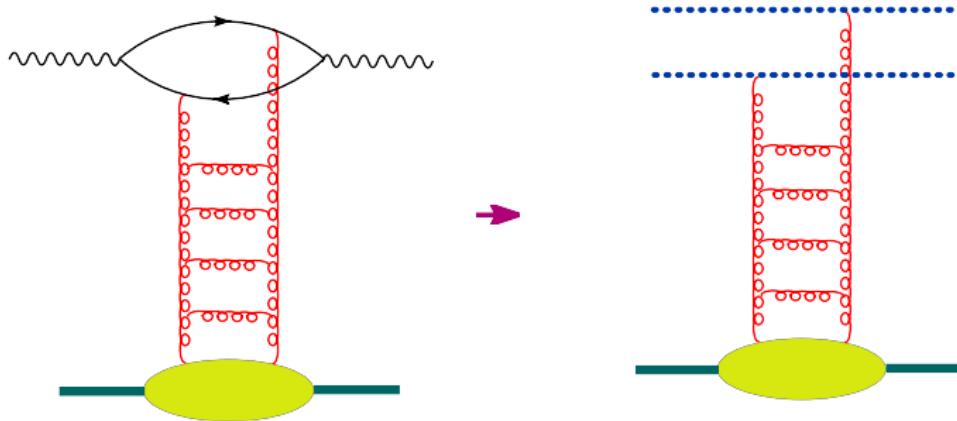
$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x, y] \psi(y) = K_{\text{LO}} \bar{\psi}(x)[x, y] \psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x, y] \psi(y)$$

Four steps of an OPE

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

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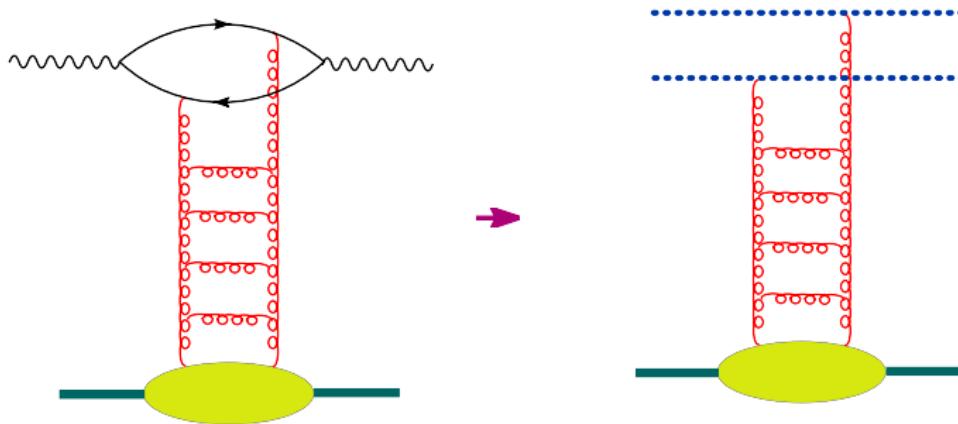
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$$U(x_\perp) = \text{Pexp} \left[ig \int_{-\infty}^{\infty} du n^\mu A_\mu(un + x_\perp) \right]$$

Wilson line

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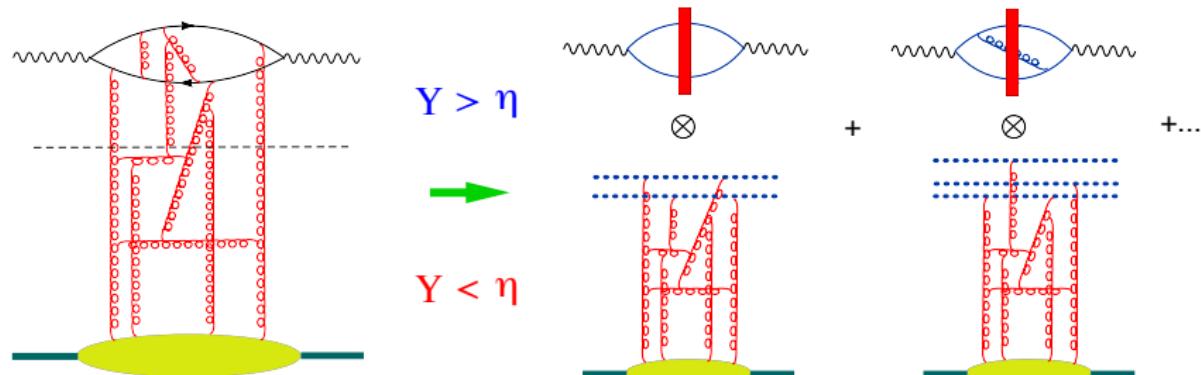


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Formally, \rightarrow means the operator expansion in Wilson lines

Rapidity factorization



η - rapidity factorization scale

Rapidity $Y > \eta$ - coefficient function ("impact factor")

Rapidity $Y < \eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp} \left[ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp) \right]$$

$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

Spectator frame: propagation in the shock-wave background.



Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space \Rightarrow we can replace the gauge factor along the actual path with the one along the straight-line path.

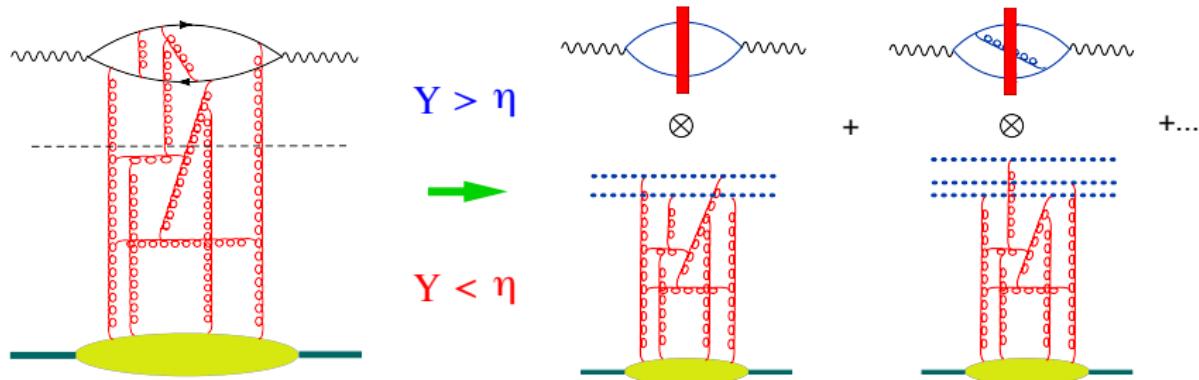


[$x \rightarrow z$: free propagation] \times

[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave] \times

[$z \rightarrow y$: free propagation]

High-energy expansion in color dipoles

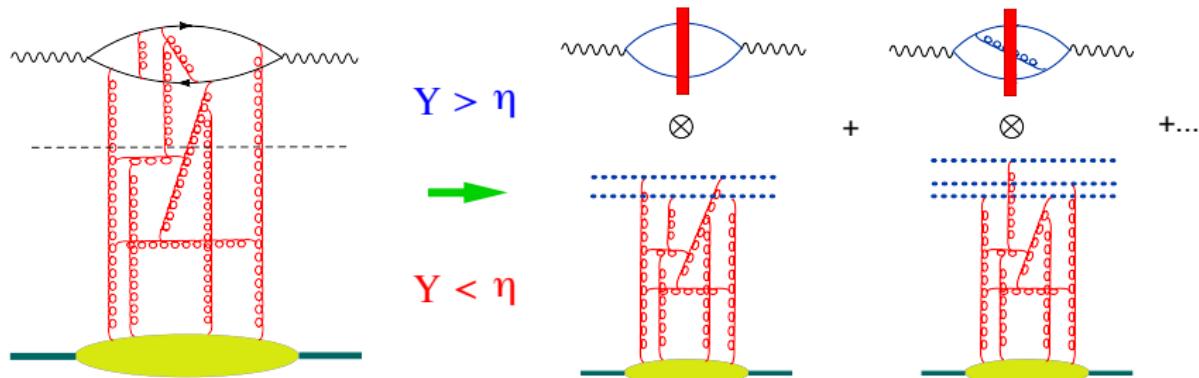


The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$

+ NLO contribution

High-energy expansion in color dipoles



η - rapidity factorization scale

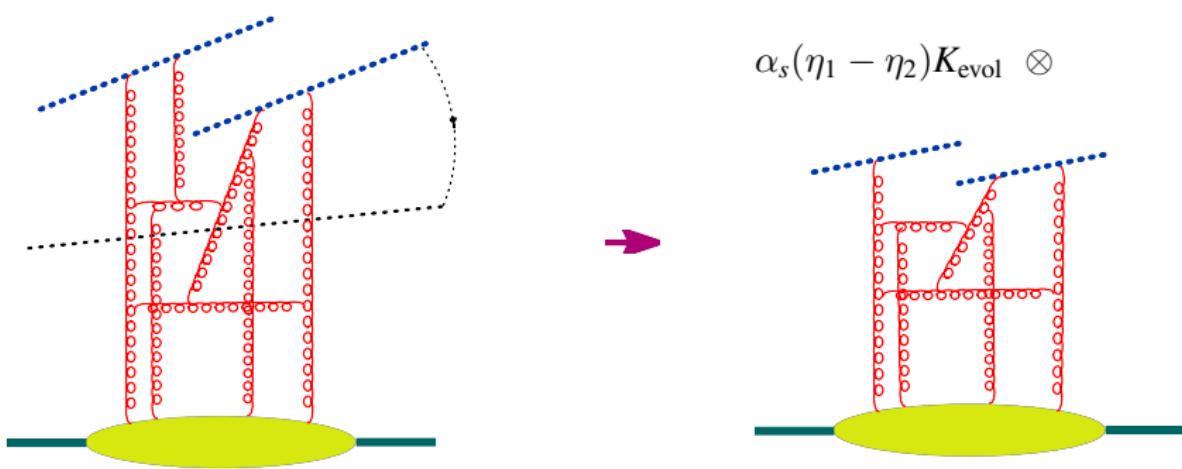
Evolution equation for color dipoles

$$\begin{aligned} \frac{d}{d\eta} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \frac{(x-y)^2}{(x-z)^2(y-z)^2} [\text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \\ &\quad - N_c \text{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\text{NLO}} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2) \end{aligned}$$

(Linear part of $K_{\text{NLO}} = K_{\text{NLO BFKL}}$)

Evolution equation for color dipoles

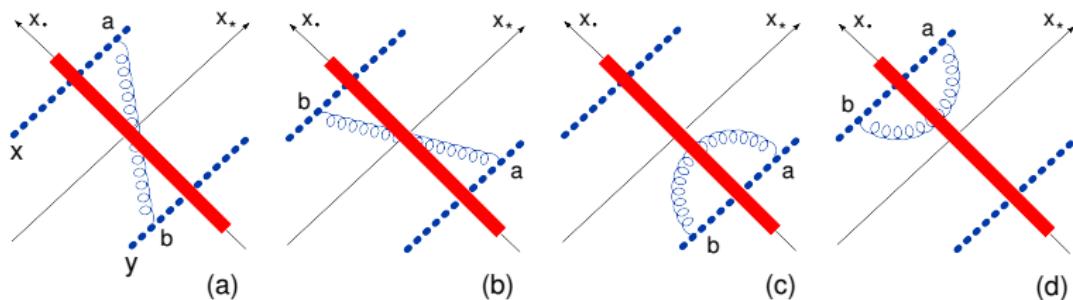
To get the evolution equation, consider the dipole with the rapidities up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



Evolution equation in the leading order

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

⇒ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$)

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I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$)

LLA for DIS in sQCD \Rightarrow BK eqn

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To check conformal invariance of the NLO BK equation(in $\mathcal{N}=4$ SYM)
- To determine the argument of the coupling constant of the BK equation(in QCD).
- To get the region of application of the leading order evolution equation.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \rightarrow x_\perp/x_\perp^2$ and $x^+ \rightarrow x^+/x_\perp^2$

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\Rightarrow The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x-y)^2}{(x-z)^2(z-y)^2} [\text{Tr}\{V_x V_z^\dagger\} \text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z})$$

$$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z, \bar{z})$$

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Conformal invariance of the evolution kernel

$$\begin{aligned} \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] &= \frac{\alpha_s N_c}{2\pi^2} \int dz K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\}] \\ &\Rightarrow \left[x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \end{aligned}$$

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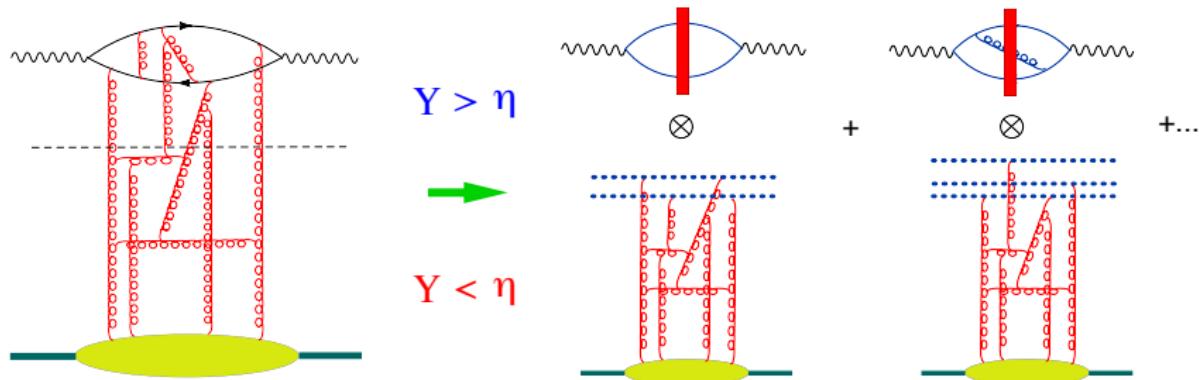
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In the leading order - OK. In the NLO - ?

Expansion of the amplitude in color dipoles in the NLO



The high-energy operator expansion is

$$\mathcal{O} \equiv \text{Tr}\{Z^2\}$$

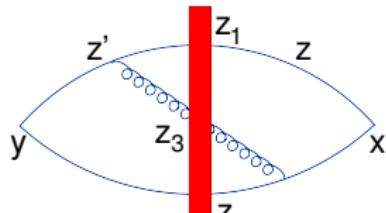
$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2 z_1 d^2 z_2 I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\} \\ + \int d^2 z_1 d^2 z_2 d^2 z_3 I^{\text{NLO}}(z_1, z_2, z_3) [\frac{1}{N_c} \text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger \eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger \eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\}]$$

In the leading order - conf. invariant impact factor

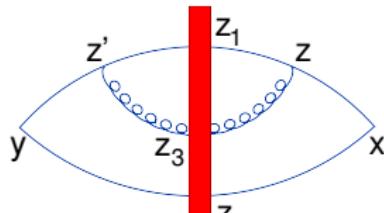
$$I_{\text{LO}} = \frac{x_+^{-2} y_+^{-2}}{\pi^2 \mathcal{Z}_1^2 \mathcal{Z}_2^2}, \quad \mathcal{Z}_i \equiv \frac{(x - z_i)_\perp^2}{x_+} - \frac{(y - z_i)_\perp^2}{y_+}$$

CCP, 2007

NLO impact factor



(a)



(b)

$$I^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) = -I^{\text{LO}} \times \frac{\lambda}{\pi^2} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \left[\ln \frac{\sigma s}{4} \mathcal{Z}_3 - \frac{i\pi}{2} + C \right]$$

The NLO impact factor is not Möbius invariant \Leftarrow the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\begin{aligned} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} &= \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &+ \frac{\lambda}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\lambda^2) \end{aligned}$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2 I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3 I^{\text{NLO}}(z_1, z_2, z_3) \left[\frac{1}{N_c} \text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right]$$
$$I^{\text{NLO}} = -I^{\text{LO}} \frac{\lambda}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\ln \frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2} \mathcal{Z}_3^2 - i\pi + 2C \right]$$

The new NLO impact factor is conformally invariant
⇒ $\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbation theory.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)$$

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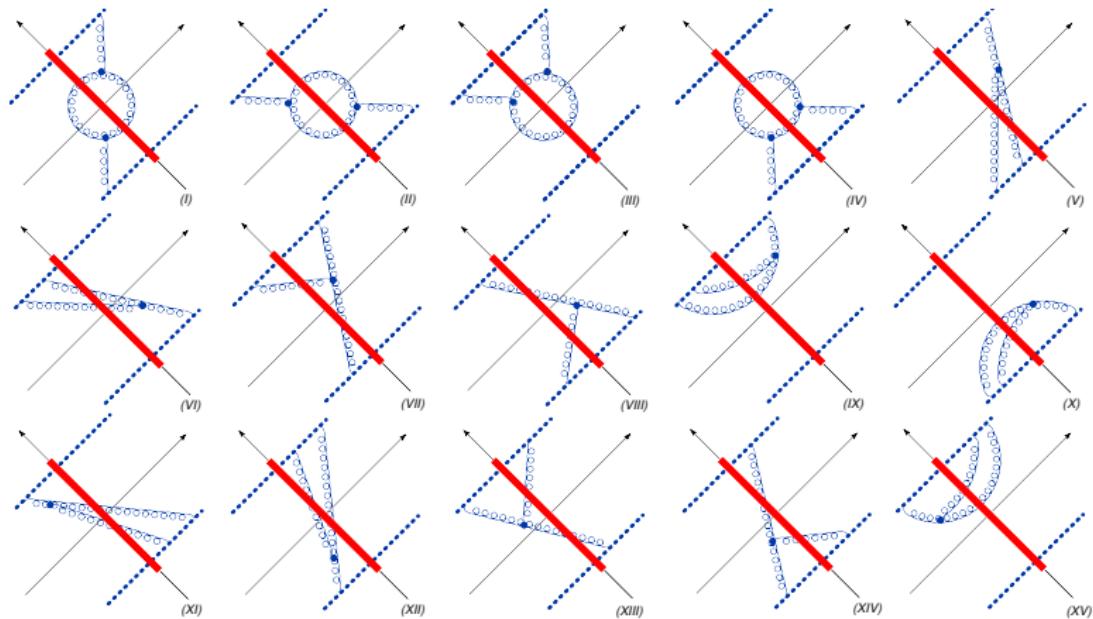
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow \left[\frac{1}{v} \right]_+$ prescription in the integrals over Feynman parameter v

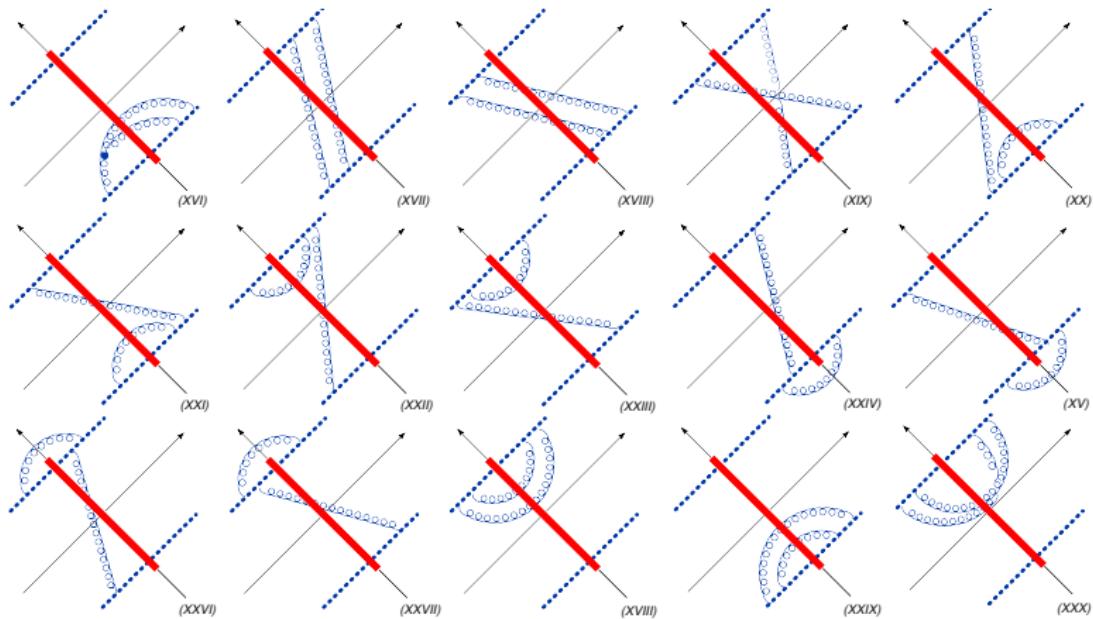
Typical integral

$$\int_0^1 dv \frac{1}{(k-p)_\perp^2 v + p_\perp^2 (1-v)} \left[\frac{1}{v} \right]_+ = \frac{1}{p_\perp^2} \ln \frac{(k-p)_\perp^2}{p_\perp^2}$$

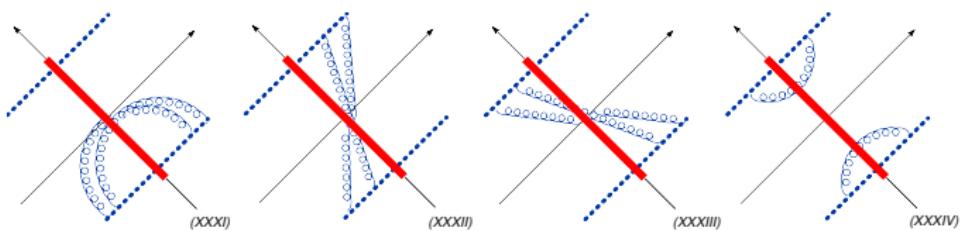
Gluon part of the NLO BK kernel: diagrams



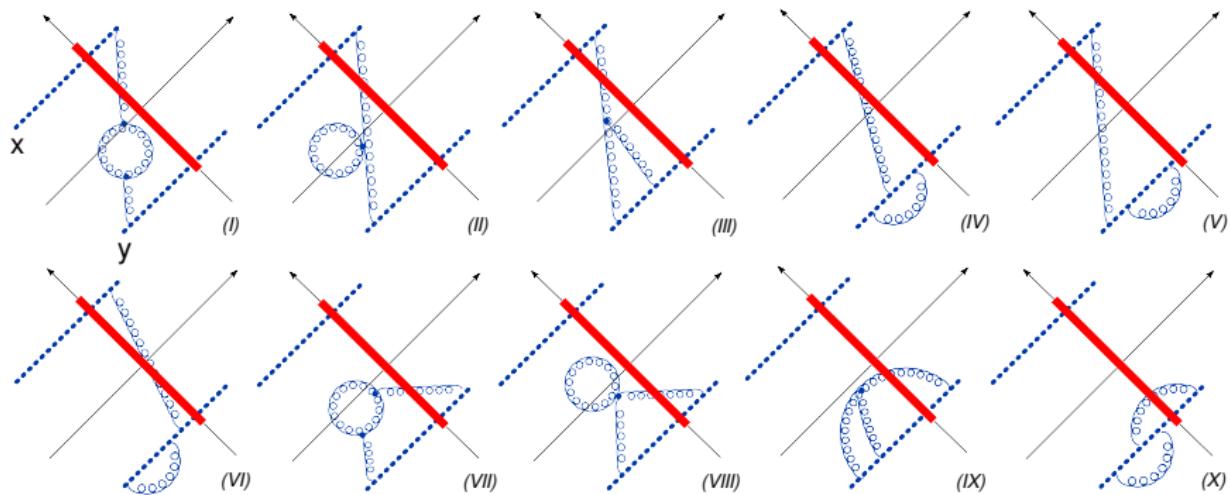
Diagrams for $1 \rightarrow 3$ dipoles transition



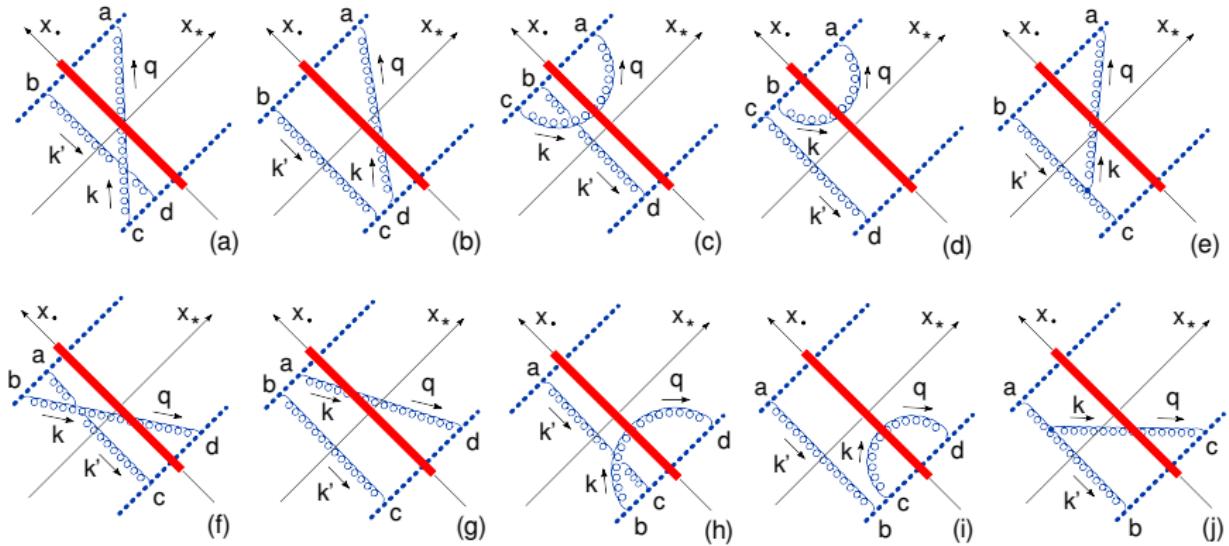
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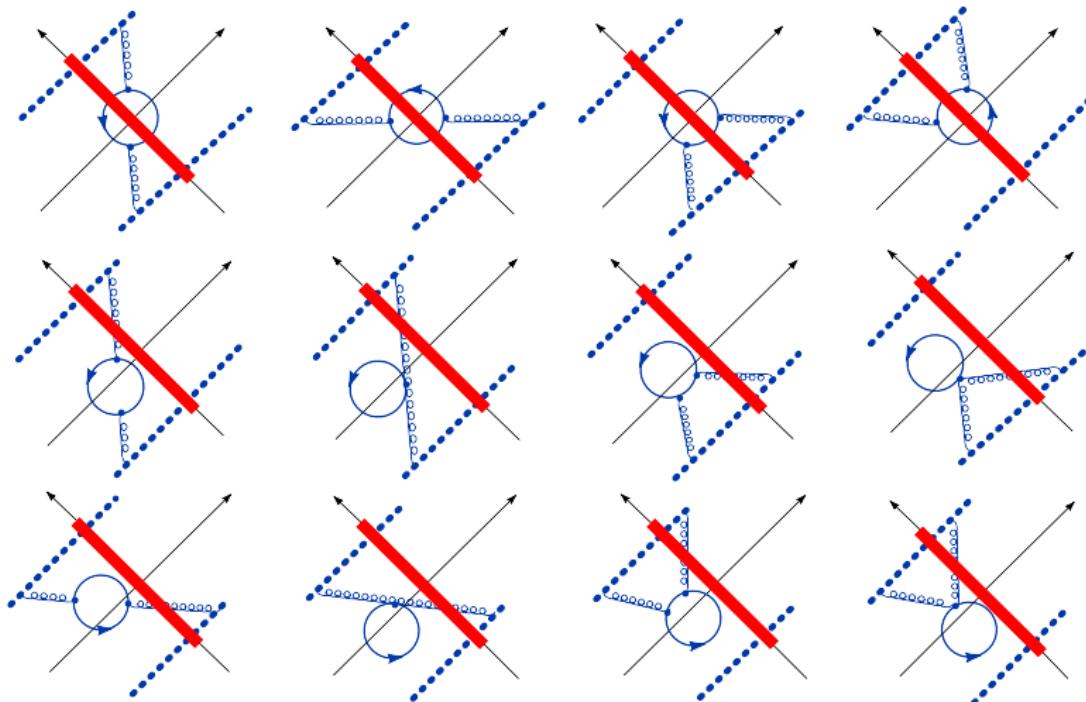
"Running coupling" diagrams



$1 \rightarrow 2$ dipole transition diagrams



Gluino and scalar loops



$$\begin{aligned}
& \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
&= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
&\times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\
&- \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
&\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta)^{bb'}
\end{aligned}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

$$\begin{aligned}
& \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
&= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
&\times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\
&- \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
&\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta)^{bb'}
\end{aligned}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

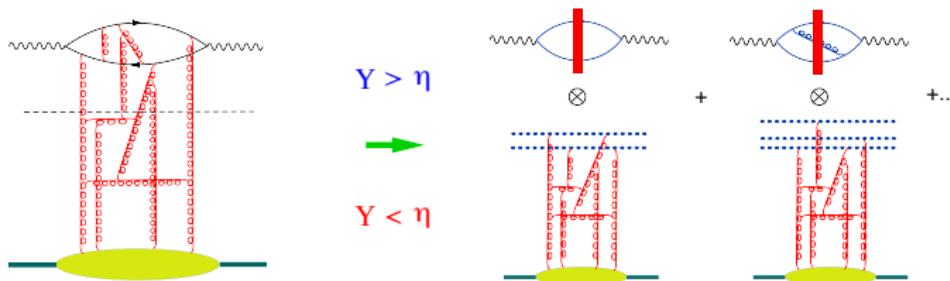
For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$

$$\begin{aligned} & \frac{d}{d\eta} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3} \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ & - \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} [(\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta)^{bb'} - (z_4 \rightarrow z_3)] \end{aligned}$$

Now Möbius invariant!

In QCD



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small- x evolution

Photon impact factor:

$$(x-y)^4 T\{\bar{\psi}(x)\gamma^\mu \hat{\psi}(x)\bar{\psi}(y)\gamma^\nu \hat{\psi}(y)\} = \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{\mu\nu}(z_1, z_2) \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$

$$I^{\mu\nu}(q, k_\perp) = \frac{N_c}{32} \int \frac{d\nu}{\pi\nu} \frac{\sinh \pi\nu}{(1+\nu^2) \cosh^2 \pi\nu} \left(\frac{k_\perp^2}{Q^2}\right)^{\frac{1}{2}-i\nu} \times \left\{ \left[\left(\frac{9}{4} + \nu^2\right) \left(1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} \mathcal{F}_1(\nu)\right) P_1^{\mu\nu} + \left(\frac{11}{4} + 3\nu^2\right) \left(1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} \mathcal{F}_2(\nu)\right) P_2^{\mu\nu} \right] \right.$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \quad P_2^{\mu\nu} = \frac{1}{q^2} \left(q^\mu - \frac{p_2^\mu q^2}{q \cdot p_2} \right) \left(q^\nu - \frac{p_2^\nu q^2}{q \cdot p_2} \right)$$

$\mathcal{F}_1(\nu), \mathcal{F}_2(\nu)$ - simple transcedentality-2 functions (G.A. Chirilli and I.B., 2013)

NLO evolution of composite “conformal” dipoles in QCD

I. B. and G. Chirilli

$$\begin{aligned}
 a \frac{d}{da} [\text{tr}\{U_{z_1} U_{z_2}^\dagger\}]_a^{\text{comp}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left([\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_2}^\dagger\} - N_c \text{tr}\{U_{z_1} U_{z_2}^\dagger\}]_a^{\text{comp}} \right. \\
 &\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
 &+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_{34}^4} \left\{ \left[-2 + \frac{z_{23}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{23}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{23}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right] \right. \\
 &\times [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_4}^\dagger\} \{U_{z_4} U_{z_2}^\dagger\} - \text{tr}\{U_{z_1} U_{z_3}^\dagger U_{z_4} U_{z_2}^\dagger U_{z_3} U_{z_4}^\dagger\} - (z_4 \rightarrow z_3)] \\
 &+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[2 \ln \frac{z_{12}^2 z_{34}^2}{z_{23}^2 z_{23}^2} + \left(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{23}^2 z_{23}^2} \right] \\
 &\times [\text{tr}\{U_{z_1} U_{z_3}^\dagger\} \text{tr}\{U_{z_3} U_{z_4}^\dagger\} \text{tr}\{U_{z_4} U_{z_2}^\dagger\} - \text{tr}\{U_{z_1} U_{z_4}^\dagger U_{z_3} U_{z_2}^\dagger U_{z_4} U_{z_3}^\dagger\} - (z_4 \rightarrow z_3)] \Big\} \\
 b &= \frac{11}{3} N_c - \frac{2}{3} n_f
 \end{aligned}$$

$K_{\text{NLO BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
+ Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\text{NLO BK}}$ reproduces the known result for the forward NLO BFKL kernel.

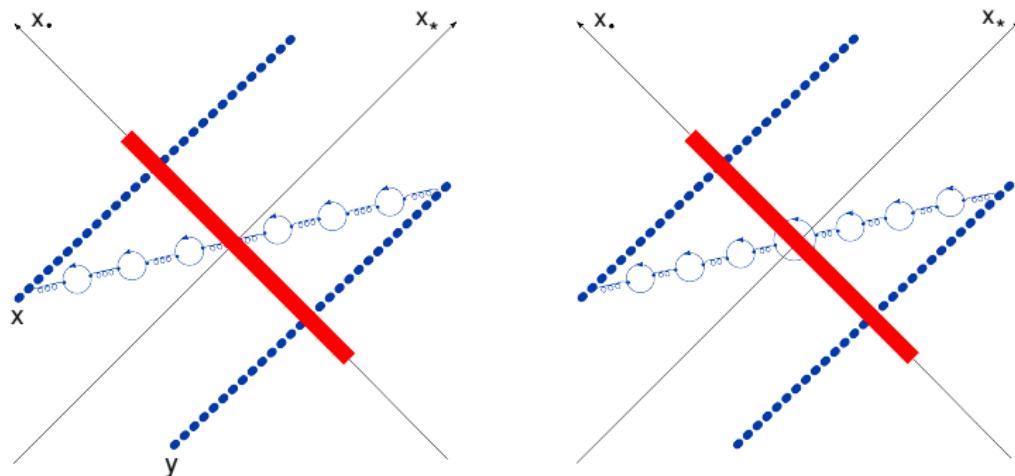
Argument of coupling constant

$$\frac{d}{d\eta} \hat{\mathcal{U}}(z_1, z_2) = \frac{\alpha_s(?_\perp) N_c}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ \hat{\mathcal{U}}(z_1, z_3) + \hat{\mathcal{U}}(z_3, z_2) - \hat{\mathcal{U}}(z_1, z_2) - \hat{\mathcal{U}}(z_1, z_3) \hat{\mathcal{U}}(z_3, z_2) \right\}$$

Argument of coupling constant

$$\frac{d}{d\eta} \hat{\mathcal{U}}(z_1, z_2) = \frac{\alpha_s(?_\perp) N_c}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ \hat{\mathcal{U}}(z_1, z_3) + \hat{\mathcal{U}}(z_3, z_2) - \hat{\mathcal{U}}(z_1, z_2) - \hat{\mathcal{U}}(z_1, z_3) \hat{\mathcal{U}}(z_3, z_2) \right\}$$

Renormalon-based approach: summation of quark bubbles



$$-\frac{2}{3} n_f \rightarrow b = \frac{11}{3} N_c - \frac{2}{3} n_f$$

Argument of coupling constant (rcBK)

$$\begin{aligned} \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\} &= \frac{\alpha_s(z_{12}^2)}{2\pi^2} \int d^2 z [\text{Tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{Tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{Tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}] \\ &\times \left[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2} \left(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1 \right) + \frac{1}{z_{23}^2} \left(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1 \right) \right] + \dots \end{aligned}$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

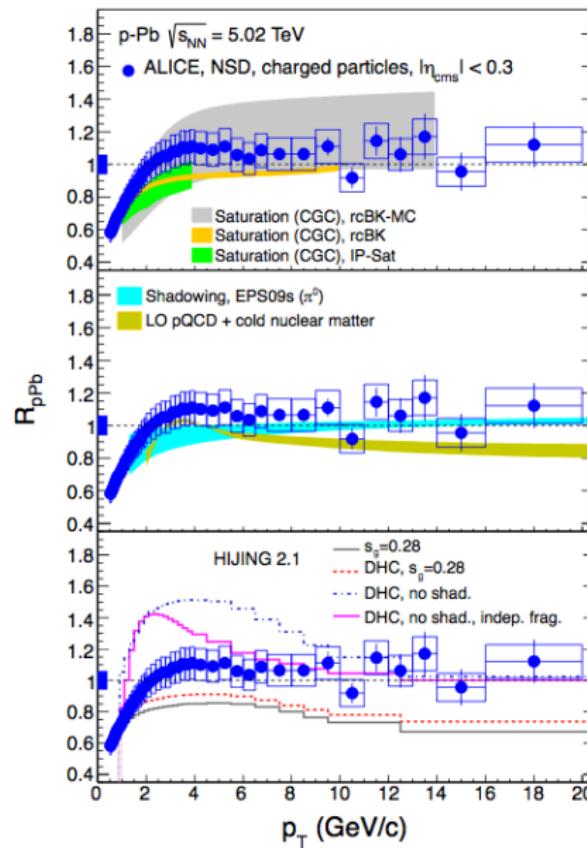
When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2} \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \quad |z_{12}| \ll |z_{13}|, |z_{23}|$$

$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \quad |z_{13}| \ll |z_{12}|, |z_{23}|$$

$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \quad |z_{23}| \ll |z_{12}|, |z_{13}|$$

⇒ the argument of the coupling constant is given by the size of the smallest dipole.



ALICE arXiv:1210.4520

Nuclear modification factor

$$R^{pPb}(p_T) = \frac{d^2 N_{\text{ch}}^{pPb} / d\eta dp_T}{\langle T_{pPb} \rangle d^2 \sigma_{\text{ch}}^{\text{pp}} / d\eta dp_T}$$

$N^{pPb} \equiv$ charged particle yield in p-Pb collisions.

NLO hierarchy of evolution of Wilson lines (G.A.C. and I.B., 2013)

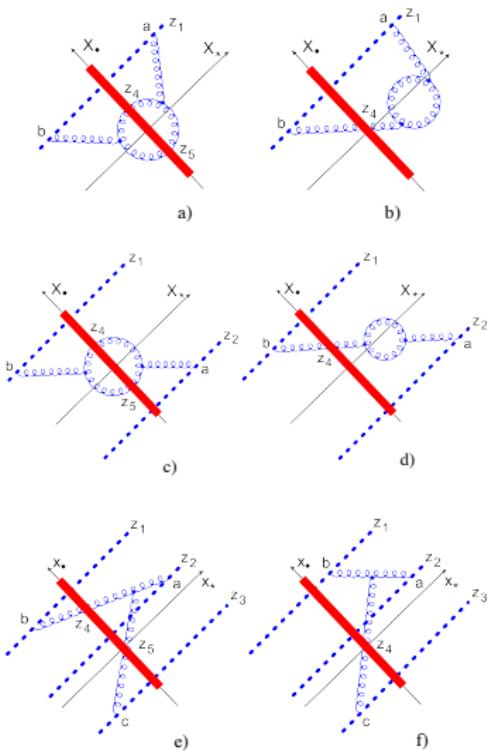
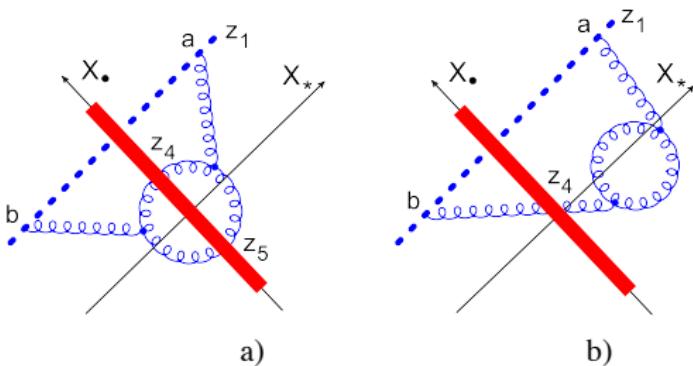


Figure : Typical NLO diagrams: self-interaction (a,b), pairwise interactions (c,d), and triple interaction (e,f)

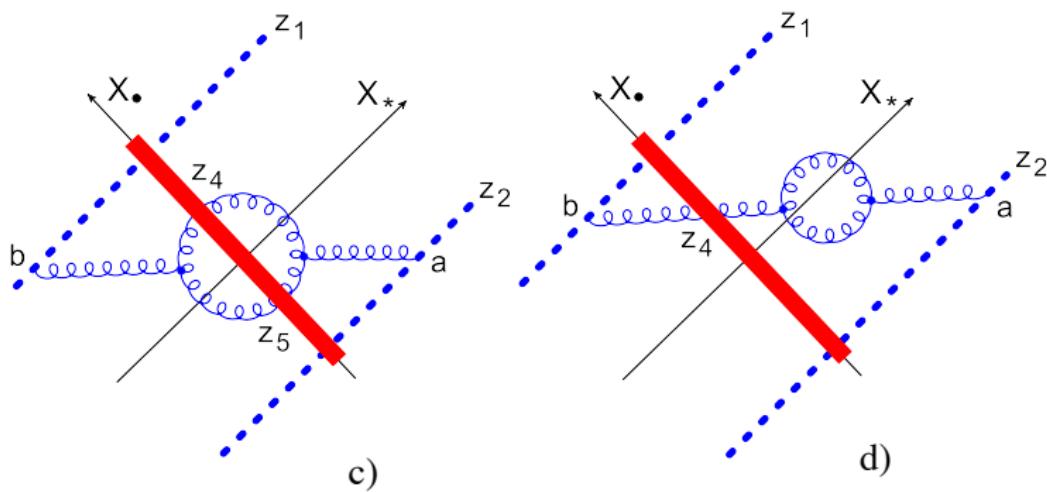
Self-interaction (gluon reggeization)



$$\begin{aligned} \frac{d}{d\eta}(U_1)_{ij} = & \frac{\alpha_s^2}{8\pi^4} \int \frac{d^2 z_4 d^2 z_5}{z_{45}^2} \left\{ U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \right. \\ & \times \left(\left[2I_1 - \frac{4}{z_{45}^2} \right] f^{ade} f^{bd'e'} (t^a U_1 t^b)_{ij} + \frac{(z_{14}, z_{15})}{z_{14}^2 z_{15}^2} \ln \frac{z_{14}^2}{z_{15}^2} \left[i f^{ad'e'} (\{t^d, t^e\} U_1 t^a)_{ij} - i f^{ade} (t^a U_1 \{t^{d'}, t^{e'}\})_{ij} \right] \right) \Big\} \\ & + \frac{\alpha_s^2 N_c}{4\pi^3} \int \frac{d^2 z_4}{z_{14}^2} (U_4^{ab} - U_1^{ab}) (t^a U_1 t^b)_{ij} \left\{ \left[\frac{11}{3} \ln z_{14}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \right. \end{aligned}$$

$$I_1 \equiv I(z_1, z_4, z_5) = \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2} \left[\frac{z_{14}^2 + z_{15}^2}{z_{45}^2} - \frac{(z_{14}, z_{15})}{z_{14}^2} - \frac{(z_{14}, z_{15})}{z_{15}^2} - 2 \right]$$

Pairwise interaction



$$\frac{d}{d\eta} (U_1)_{ij} (U_2)_{kl} = \frac{\alpha_s^2}{8\pi^4} \int d^2 z_4 d^2 z_5 (\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3) + \frac{\alpha_s^2}{8\pi^3} \int d^2 z_4 (\mathcal{B}_1 + N_c \mathcal{B}_2)$$

Pairwise interaction

$$\begin{aligned}\mathcal{A}_1 &= [(t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl}] \\ &\times \left[f^{ade} f^{bd'e'} U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \left(-K - \frac{4}{z_{45}^4} + \frac{I_1}{z_{45}^2} + \frac{I_2}{z_{45}^2} \right) \right]\end{aligned}$$

$K = \text{NLO BK kernel for } \mathcal{N} = 4 \text{ SYM}$

$$\begin{aligned}\mathcal{A}_2 &= 4(U_4 - U_1)^{dd'} (U_5 - U_2)^{ee'} \\ &\left\{ i \left[f^{ad'e'} (t^d U_1 t^a)_{ij} (t^e U_2)_{kl} - f^{ade} (t^a U_1 t^{d'})_{ij} (U_2 t^{e'})_{kl} \right] J_{1245} \ln \frac{z_{14}^2}{z_{15}^2} \right. \\ &\left. + i \left[f^{ad'e'} (t^d U_1)_{ij} (t^e U_2 t^a)_{kl} - f^{ade} (U_1 t^{d'})_{ij} (t^a U_2 t^{e'})_{kl} \right] J_{2154} \ln \frac{z_{24}^2}{z_{25}^2} \right\}\end{aligned}$$

$$J_{1245} \equiv J(z_1, z_2, z_4, z_5) = \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} - 2 \frac{(z_{15}, z_{45})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{25}^2 z_{45}^2} + 2 \frac{(z_{25}, z_{45})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

Pairwise interaction

$$\begin{aligned}\mathcal{A}_3 = & \ 2U_4^{dd'} \left\{ i \left[f^{ad'e'} (U_1 t^a)_{ij} (t^d t^e U_2)_{kl} - f^{ade} (t^a U_1)_{ij} (U_2 t^{e'} t^{d'})_{kl} \right] \right. \\ & \times \left[\mathcal{J}_{1245} \ln \frac{z_{14}^2}{z_{15}^2} + (J_{2145} - J_{2154}) \ln \frac{z_{24}^2}{z_{25}^2} \right] (U_5 - U_2)^{ee'} \\ & + i \left[f^{ad'e'} (t^d t^e U_1)_{ij} (U_2 t^a)_{kl} - f^{ade} (U_1 t^{e'} t^{d'})_{ij} (t^a U_2)_{kl} \right] \\ & \times \left. \left[\mathcal{J}_{2145} \ln \frac{z_{24}^2}{z_{25}^2} + (J_{1245} - J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2} \right] (U_5 - U_1)^{ee'} \right\}\end{aligned}$$

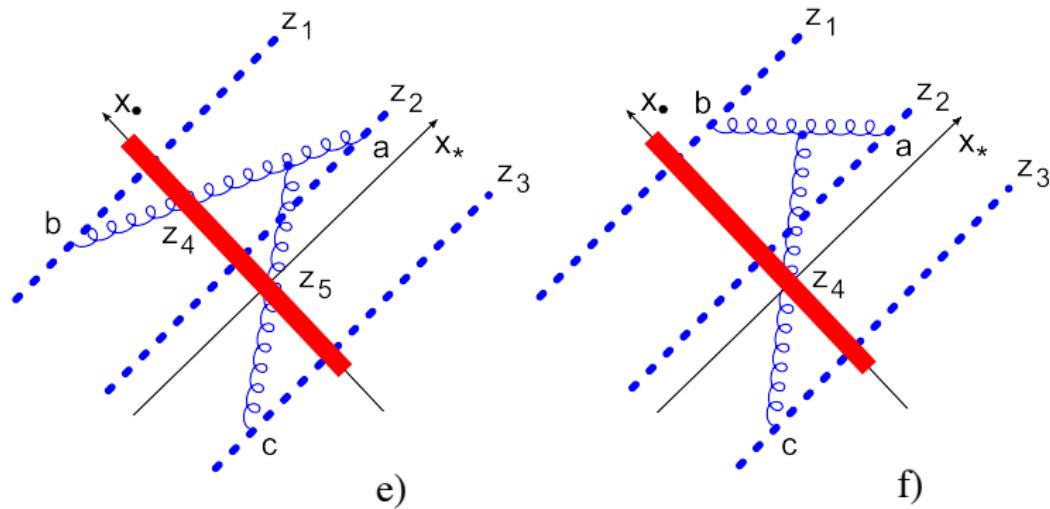
$$\begin{aligned}\mathcal{J}_{1245} &\equiv \mathcal{J}(z_1, z_2, z_4, z_5) \\ &= \frac{(z_{24}, z_{25})}{z_{24}^2 z_{25}^2 z_{45}^2} - \frac{2(z_{24}, z_{45})(z_{15}, z_{25})}{z_{24}^2 z_{25}^2 z_{15}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{24})}{z_{14}^2 z_{24}^2 z_{25}^2 z_{45}^2} - 2 \frac{(z_{14}, z_{24})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{24}^2 z_{25}^2}\end{aligned}$$

Pairwise interaction

$$\begin{aligned}\mathcal{B}_1 = & 2 \ln \frac{z_{14}^2}{z_{12}^2} \ln \frac{z_{24}^2}{z_{12}^2} \\ & \times \left\{ (U_4 - U_1)^{ab} i [f^{bde} (t^a U_1 t^d)_{ij} (U_2 t^e)_{kl} + f^{ade} (t^e U_1 t^b)_{ij} (t^d U_2)_{kl}] \left[\frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{14}^2} \right] \right. \\ & \left. + (U_4 - U_2)^{ab} i [f^{bde} (U_1 t^e)_{ij} (t^a U_2 t^d)_{kl} + f^{ade} (t^d U_1)_{ij} (t^e U_2 t^b)_{kl}] \left[\frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{24}^2} \right] \right\}\end{aligned}$$

$$\begin{aligned}\mathcal{B}_2 = & [2U_4^{ab} - U_1^{ab} - U_2^{ab}] [(t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl}] \\ & \times \left\{ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} \left[\frac{11}{3} \ln z_{12}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] + \frac{11}{3} \left[\frac{1}{2z_{14}^2} \ln \frac{z_{24}^2}{z_{12}^2} + \frac{1}{2z_{24}^2} \ln \frac{z_{14}^2}{z_{12}^2} \right] \right\}\end{aligned}$$

Triple interaction



$$\begin{aligned} \mathcal{J}_{12345} \equiv \mathcal{J}(z_1, z_2, z_3, z_4, z_5) &= -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2} \\ &- \frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{35}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} \end{aligned}$$

Triple interaction

$$\begin{aligned} & \frac{d}{d\eta} (U_1)_{ij} (U_2)_{kl} (U_3)_{mn} \\ &= i \frac{\alpha_s^2}{2\pi^4} \int d^2 z_4 d^2 z_5 \left\{ \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \right. \\ & \quad \times f^{cde} \left[(t^a U_1)_{ij} (t^b U_2)_{kl} (U_3 t^c)_{mn} (U_4 - U_1)^{ad} (U_5 - U_2)^{be} \right. \\ & \quad \left. - (U_1 t^a)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{da} (U_5 - U_2)^{eb} \right] \\ & \quad + \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} \\ & \quad \times f^{ade} \left[(U_1 t^a)_{ij} (t^b U_2)_{kl} (t^c U_3)_{mn} (U_4 - U_3)^{cd} (U_5 - U_2)^{be} \right. \\ & \quad \left. - (t^a U_1)_{ij} \otimes (U_2 t^b)_{kl} (U_3 t^c)_{mn} (U_4^{dc} - U_3^{dc}) (U_5^{eb} - U_2^{eb}) \right] \\ & \quad + \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} \\ & \quad \times f^{bde} \left[(t^a U_1)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{ad} (U_5 - U_3)^{ce} \right. \\ & \quad \left. - (U_1 t^a)_{ij} (t^b U_2)_{kl} (U_3 t^c)_{mn} (U_4 - U_1)^{da} (U_5 - U_3)^{ec} \right] \end{aligned} \tag{1}$$

Rapidity evolution of the baryon “tripole”

Baryon operator

$$B_{123} = \varepsilon^{i'j'h'} \varepsilon_{ijh} U_{i'}^i(r_{1\perp}) U_{j'}^j(r_{1\perp}) U_{h'}^h(r_{3\perp}) \equiv U_1 \cdot U_2 \cdot U_3,$$

Evolution equation in the LO ([A. Grabovsky, 2013](#))

$$\begin{aligned} \frac{d}{d\eta} B_{123} &= \frac{\alpha_s 3}{8\pi^2} \int d\vec{r}_4 \left[\frac{\vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \ln \left(\frac{a \vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \right) \right. \\ &\times \left. (-B_{123} + \frac{1}{6} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right]. \end{aligned}$$

Composite “conformal” baryon operator in the NLO

General prescription:

$$O^{conf} = O + \frac{1}{2} \frac{\partial O}{\partial \eta} \Bigg|_{\frac{\vec{r}_{im}^2 \vec{r}_{in}^2}{\vec{r}_{im}^2 \vec{r}_{in}^2} \rightarrow \frac{\vec{r}_{im}^2 \vec{r}_{in}^2}{\vec{r}_{im}^2 \vec{r}_{in}^2} \ln \left(\frac{\vec{r}_{im}^2 a}{\vec{r}_{im}^2 \vec{r}_{in}^2} \right)}$$

(cf. “Conformal JIMWLK” by Kovner and Lublinsky, 2014)

Composite “conformal” baryon operator

$$B_{123}^{conf} = B_{123} + \frac{\alpha_s 3}{8\pi^2} \int d^2 \vec{r}_4 \left[\frac{\vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \ln \left(\frac{a \vec{r}_{12}^2}{\vec{r}_{41}^2 \vec{r}_{42}^2} \right) \right. \\ \left. \times (-B_{123} + \frac{1}{6} (B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214})) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right].$$

$$\begin{aligned}
\frac{dB_{123}^{conf}}{d\eta} = & \text{ LO} - \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left(\left\{ \tilde{L}_{12}^C (U_0 U_4^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 \right. \right. \\
& + L_{12}^C \left[(U_0 U_4^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 + \text{tr} (U_0 U_4^\dagger) (U_1 U_0^\dagger U_2) \cdot U_3 \cdot U_4 \right. \\
& \quad \left. \left. - \frac{3}{4} [B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124}] + \frac{1}{2} B_{123} \right] \right. \\
& + M_{12}^C \left[(U_0 U_4^\dagger U_3) \cdot (U_2 U_0^\dagger U_1) \cdot U_4 + (U_1 U_0^\dagger U_2) \cdot (U_3 U_4^\dagger U_0) \cdot U_4 \right] \\
& \quad \left. + Z_{12} B_{355} B_{125} + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4) \Big) \\
& - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_5 \left(\frac{11}{6} \left[\ln \left(\frac{\vec{r}_{15}^2}{\vec{r}_{25}^2} \right) \left(\frac{1}{\vec{r}_{25}^2} - \frac{1}{\vec{r}_{15}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{25}^2} \ln \left(\frac{\vec{r}_{12}^2}{\tilde{\mu}^2} \right) \right] \right. \\
& \quad \times \left. \left(\frac{3}{2} (B_{155} B_{235} + B_{255} B_{135} - B_{355} B_{125}) - 9 B_{123} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right).
\end{aligned}$$

NLO evolution kernels

Here

$$L_{12}^C = K_{12}^{\text{NLO BK}} + \frac{\vec{r}_{12}^2}{4\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{14}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right) + \frac{\vec{r}_{12}^2}{4\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{24}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right),$$

$$\tilde{L}_{12}^C = K_{12}^{\text{NLO BK}} + \frac{\vec{r}_{12}^2}{4\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{14}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right) - \frac{\vec{r}_{12}^2}{4\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{24}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right),$$

$$\begin{aligned} Z_{12} = & \frac{\vec{r}_{12}^2}{8\vec{r}_{15}^2\vec{r}_{25}^2} \left[\left(\frac{\vec{r}_{35}^2}{\vec{r}_{45}^2\vec{r}_{34}^2} - \frac{\vec{r}_{25}^2}{\vec{r}_{45}^2\vec{r}_{24}^2} \right) \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{14}^2}{\vec{r}_{45}^2\vec{r}_{12}^2} \right) \right. \\ & \left. + \frac{\vec{r}_{15}^2}{\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{34}^2}{\vec{r}_{35}^2\vec{r}_{24}^2} \right) + \frac{\vec{r}_{13}^2}{\vec{r}_{14}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{35}^2\vec{r}_{12}^2}{\vec{r}_{25}^2\vec{r}_{13}^2} \right) \right] - (1 \leftrightarrow 3), \end{aligned}$$

NLO evolution kernels

$$\begin{aligned} M_{12}^C = & \frac{\vec{r}_{12}^2}{16\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{25}^2\vec{r}_{34}^4}{\vec{r}_{35}^4\vec{r}_{14}^2\vec{r}_{24}^2} \right) + \frac{\vec{r}_{12}^2}{16\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{35}^4\vec{r}_{45}^4\vec{r}_{12}^4\vec{r}_{24}^2}{\vec{r}_{15}^2\vec{r}_{25}^6\vec{r}_{14}^2\vec{r}_{34}^4} \right) \\ & + \frac{\vec{r}_{23}^2}{16\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{15}^4\vec{r}_{35}^2\vec{r}_{24}^6\vec{r}_{34}^2}{\vec{r}_{25}^2\vec{r}_{45}^4\vec{r}_{14}^4\vec{r}_{23}^4} \right) + \frac{\vec{r}_{23}^2}{16\vec{r}_{35}^2\vec{r}_{45}^2\vec{r}_{24}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{35}^2\vec{r}_{14}^4}{\vec{r}_{15}^4\vec{r}_{24}^2\vec{r}_{34}^2} \right) \\ & + \frac{\vec{r}_{13}^2}{16\vec{r}_{35}^2\vec{r}_{45}^2\vec{r}_{14}^2} \ln \left(\frac{\vec{r}_{25}^4\vec{r}_{14}^2\vec{r}_{34}^2}{\vec{r}_{15}^2\vec{r}_{35}^2\vec{r}_{24}^4} \right) + \frac{\vec{r}_{13}^2}{16\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{25}^4\vec{r}_{14}^2\vec{r}_{34}^2}{\vec{r}_{15}^2\vec{r}_{35}^2\vec{r}_{24}^4} \right) \\ & + \frac{\vec{r}_{35}^2\vec{r}_{12}^2}{8\vec{r}_{15}^2\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{35}^2\vec{r}_{24}^4}{\vec{r}_{25}^2\vec{r}_{45}^2\vec{r}_{12}^2\vec{r}_{34}^2} \right) + \frac{\vec{r}_{23}^2\vec{r}_{12}^2}{8\vec{r}_{15}^2\vec{r}_{25}^2\vec{r}_{24}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{25}^2\vec{r}_{12}^2\vec{r}_{34}^2}{\vec{r}_{15}^2\vec{r}_{23}^2\vec{r}_{24}^2} \right) \\ & + \frac{\vec{r}_{14}^2\vec{r}_{23}^2}{8\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{24}^2\vec{r}_{34}^2} \ln \left(\frac{\vec{r}_{15}^2\vec{r}_{45}^2\vec{r}_{23}^2\vec{r}_{24}^2}{\vec{r}_{25}^4\vec{r}_{14}^2\vec{r}_{34}^2} \right) \end{aligned}$$

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL equation.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- It gives the anomalous dimensions of gluon light-ray operators at “the BFKL point” $j \rightarrow 1$
- NLO photon impact factor is calculated.
- NLO hierarchy of Wilson-line evolution is derived.
- NLO evolution of baryon operator (\ni odderon contribution) is obtained.

Rapidity evolution of gluon TMD from low to moderate x_B

At small x - Weizsäcker-Williams unintegrated gluon distribution

$$\sum_X \text{tr} \langle p | U \partial^i U^\dagger(z_\perp) | X \rangle \langle X | U \partial_i U^\dagger(0_\perp) \} | p \rangle$$

Rapidity factorization: each gluon has rapidity $\leq \ln x_B$.

Rewrite (later $n \equiv p_1$)

$$\alpha_s \mathcal{D}(x_B, z_\perp) = -\frac{\alpha_s}{2\pi(p \cdot n)x_B} \int du \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle$$

$$\mathcal{F}_\xi^a(z_\perp + un) \equiv [\infty n + z_\perp, un + z_\perp]^{am} n^\mu F_{\mu\xi}^m(un + z_\perp)$$

$$\tilde{\mathcal{F}}_\xi^a(z_\perp + un) \equiv n^\mu F_{\mu\xi}^m(un + z_\perp) [un + z_\perp, \infty n + z_\perp]^{ma}$$

and define the “WW unintegrated gluon distribution”

$$\mathcal{D}(x_B, k_\perp) = \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp) \quad x_B s \gg k_\perp^2 \gg \Lambda_{\text{QCD}}^2$$

NB: $\alpha_s \mathcal{D}(x_B, z_\perp)$ is renorm-invariant.

$$\begin{aligned}\mathcal{D}(x_B, k_\perp, \eta) &= \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp, \eta), \\ \alpha_s \mathcal{D}(x_B, z_\perp, \eta) &= \frac{-x_B^{-1} \alpha_s}{2\pi(p \cdot n)} \int du e^{-ix_B u(pn)} \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle\end{aligned}$$

There are more involved definitions with the above TMD multiplied by some Wilson-line factors but we will discuss the “primordial” TMD.

$$\begin{aligned} \mathcal{D}(x_B, k_\perp, \eta) &= \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp, \eta), \\ \alpha_s \mathcal{D}(x_B, z_\perp, \eta) &= \frac{-x_B^{-1} \alpha_s}{2\pi(p \cdot n)} \int du e^{-ix_B u(pn)} \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle \end{aligned}$$

There are more involved definitions with the above TMD multiplied by some Wilson-line factors but we will discuss the “primordial” TMD.

Now x_B is introduced explicitly in the definition of gluon TMD.
 However, because light-like Wilson lines exhibit rapidity divergencies, we need a separate cutoff η (not necessarily equal to $\ln x_B$) for the rapidity of the gluons emitted by Wilson lines.

$$\begin{aligned} \mathcal{D}(x_B, k_\perp, \eta) &= \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp, \eta), \\ \alpha_s \mathcal{D}(x_B, z_\perp, \eta) &= \frac{-x_B^{-1} \alpha_s}{2\pi(p \cdot n)} \int du e^{-ix_B u(pn)} \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle \end{aligned}$$

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Now x_B is introduced explicitly in the definition of gluon TMD. However, because light-like Wilson lines exhibit rapidity divergencies, we need a separate cutoff η (not necessarily equal to $\ln x_B$) for the rapidity of the gluons emitted by Wilson lines.

The above TMD will have double-logarithmic contributions of the type $(\alpha_s \eta \ln x_B)^n$ while the WW distribution has only single-log terms $(\alpha_s \ln x_B)^n$ described by the BK evolution.

Reconciliation

Some definitions

$$k = \alpha p_1 + \beta p_2 + k_{\perp} \quad \text{Sudakov variables}$$

$$\mathcal{F}_i^a(k_{\perp}, \beta_B) = \int d^2 z_{\perp} e^{-i(k,z)_{\perp}} \mathcal{F}_i^a(z_{\perp}, \beta_B),$$

$$\mathcal{F}_i^a(z_{\perp}, \beta_B) \equiv \frac{2}{s} \int dz_* e^{i\beta_B z_*} [\infty, z_*]_z^{am} F_{\bullet i}^m(z_*, z_{\perp})$$

and similarly

$$\tilde{\mathcal{F}}_i^a(k_{\perp}, \beta_B) = \int d^2 z_{\perp} e^{i(k,z)_{\perp}} \tilde{\mathcal{F}}_i^a(z_{\perp}, \beta_B),$$

$$\tilde{\mathcal{F}}_i^a(z_{\perp}, \beta_B) \equiv \frac{2}{s} \int dz_* e^{-i\beta_B z_*} F_{\bullet i}^m(z_*, z_{\perp}) [z_*, \infty]_z^{ma}$$

Double fun. interval for cross sections

$$\begin{aligned} \langle p | \tilde{\mathcal{F}}_i^a(k'_\perp, \beta'_B) \mathcal{F}^{ai}(k_\perp, \beta_B) | p \rangle &\equiv \sum_X \langle p | \tilde{\mathcal{F}}_i^a(k'_\perp, \beta'_B) | X \rangle \langle X | \mathcal{F}^{ai}(k_\perp, \beta_B) | p \rangle \\ &= -2\pi\delta(\beta_B - \beta'_B)(2\pi)^2\delta^{(2)}(k_\perp - k'_\perp) 2\pi x_B \mathcal{D}(\beta_B = x_B, k_\perp, \eta) \end{aligned}$$

Short-hand notation

$$\langle p | \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n | p \rangle \equiv \sum_X \langle p | \tilde{T}\{\tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m\} | X \rangle \langle X | T\{\mathcal{O}_1 \dots \mathcal{O}_n\} | p \rangle$$

This matrix element can be represented by a double functional integral

$$\begin{aligned} &\langle \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n \rangle \\ &= \int D\tilde{A} D\tilde{\bar{\psi}} D\tilde{\psi} e^{-iS_{\text{QCD}}(\tilde{A}, \tilde{\psi})} \int DA D\bar{\psi} D\psi e^{iS_{\text{QCD}}(A, \psi)} \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n \end{aligned}$$

The boundary condition $\tilde{A}(\vec{x}, t = \infty) = A(\vec{x}, t = \infty)$ (and similarly for quark fields) reflects the sum over all intermediate states X .

Gauge invariance

Due to the boundary condition $\tilde{A}(\vec{x}, t = \infty) = A(\vec{x}, t = \infty)$ the matrix element

$$\begin{aligned} & \langle \tilde{\mathcal{F}}_i^a(z'_\perp, \beta'_B)[z'_\perp + \infty p_1, z_\perp + \infty p_1] \mathcal{F}^{ai}(z_\perp, \beta_B) \rangle \\ &= \int D\tilde{A} D\tilde{\psi} D\tilde{\psi} e^{-iS_{\text{QCD}}(\tilde{A}, \tilde{\psi})} \int DA D\bar{\psi} D\psi e^{iS_{\text{QCD}}(A, \psi)} \\ & \quad \tilde{\mathcal{F}}_i^a(z'_\perp, \beta'_B)[z'_\perp + \infty p_1, z_\perp + \infty p_1] \mathcal{F}^{ai}(z_\perp, \beta_B) \end{aligned}$$

is gauge invariant

However, the gauge link $[z'_\perp + \infty p_1, z_\perp + \infty p_1]$ does not contribute at least at the one-loop level (γ_{cusp} and self-energy diagrams vanish)

Rapidity evolution: one loop

We study the evolution with respect to rapidity cutoff η

$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

Matrix element of $\tilde{\mathcal{F}}_i^a(k'_\perp, \beta'_B) \mathcal{F}^{ai}(k_\perp, \beta_B)$ at one-loop accuracy:
diagrams in the “external field” of gluons with rapidity $< \eta$.

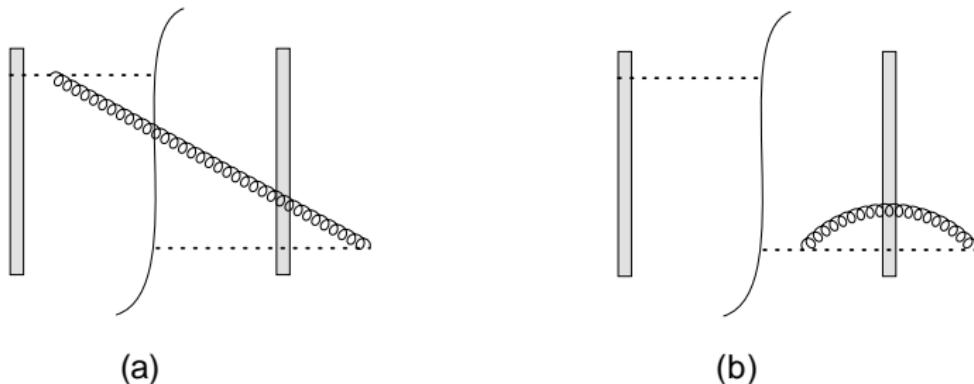


Figure : Typical diagrams for real (a) and virtual (b) contributions to the evolution kernel.

Real corrections: square of “Lipatov vergex”

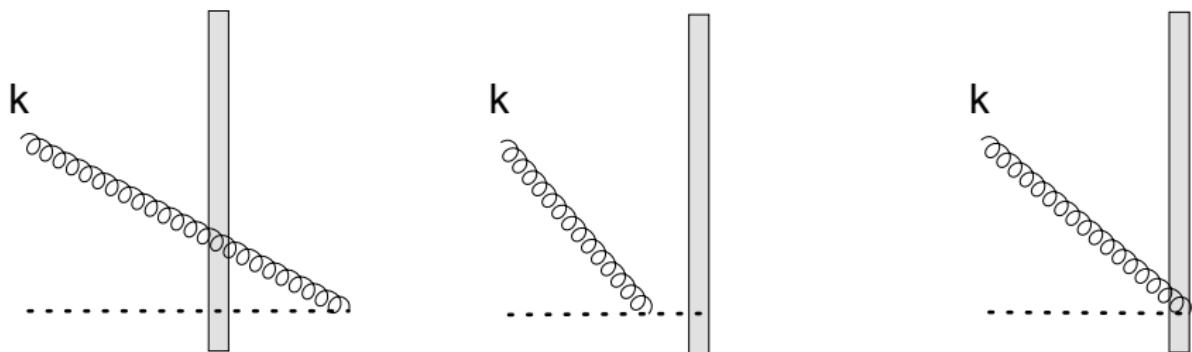


Figure : Lipatov vertex of gluon emission.

Result of calculation (in the $p_1^\mu A_\mu = 0$ gauge) A. Tarasov and I.B.

$$L_{\mu \perp}^{ab}(k, z_\perp) = (k_\perp | U \frac{p_\perp^2 g_{\mu i}^\perp + 2 p_i p_\mu^\perp}{p_\perp^2 + \alpha \beta_B s} U^\dagger - \frac{p_\perp^2 g_{\mu i}^\perp + 2 p_i p_\mu^\perp}{p_\perp^2 + \alpha \beta_B s} + 2 \frac{p_\mu^\perp}{p_\perp^2} \mathcal{F}_i(\beta_B) | z_\perp)^{ab}$$

Schwinger's notations $(x_\perp | \mathcal{O}(\hat{p}_\perp, \hat{X}_\perp) | y_\perp) \equiv \int d^2 p \mathcal{O}(p_\perp, x_\perp) e^{-i(p, x-y)_\perp}$

Virtual corrections

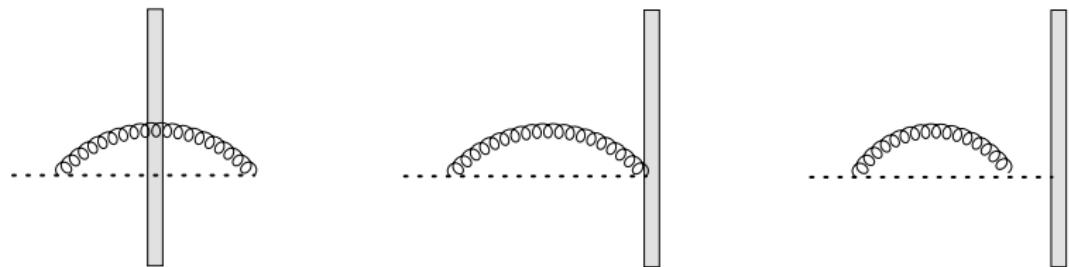


Figure : Virtual gluon corrections.

Result of the calculation (in light-like and background-Feynman gauges)

$$\begin{aligned} \langle \mathcal{F}_i^a(\beta_B, z_\perp) \rangle &= -i \frac{g^2}{2\pi} f^{abc} \\ &\times \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} (z_\perp | \frac{1}{p_\perp^2} \mathcal{F}_i(\beta_B) U \frac{\alpha \beta_B s}{\alpha \beta_B s + p_\perp^2} U^\dagger + \frac{1}{p_\perp^2} \partial_\perp^2 U \frac{p_i}{\alpha \beta_B s + p_\perp^2} U^\dagger | z_\perp)^{bc} \end{aligned}$$

NB: with $\alpha < \sigma$ cut off there is no UV divergence.

One-loop (leading log) result

$$\begin{aligned}
& \langle \tilde{\mathcal{F}}^{ai}(z'_\perp, \beta_B) \mathcal{F}_i^a(z_\perp, \beta_B) \rangle \\
&= \frac{g^2}{4\pi} \text{Tr} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \left\{ - (z'_\perp | \left[\tilde{U} \frac{p_\perp^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} \tilde{U}^\dagger - (\tilde{U} \rightarrow 1) \right] \right. \\
&\quad \times \left. \left[U \frac{p_\perp^2 g^{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} U^\dagger - (U \rightarrow 1) \right] | z_\perp \right) \\
&+ 2(z'_\perp | \left[\tilde{U} \frac{p^i}{p_\perp^2 + \alpha \beta_B s} \tilde{U}^\dagger - (\tilde{U} \rightarrow 1) \right] \mathcal{F}_i(\beta_B) \\
&\quad + \tilde{\mathcal{F}}^i(\beta_B) \left[U \frac{p_i}{p_\perp^2 + \alpha \beta_B s} U^\dagger - (U \rightarrow 1) \right] | z_\perp) \\
&+ 2\tilde{\mathcal{F}}^i(z'_\perp, \beta_B) \left[(z'_\perp | \frac{1}{p_\perp^2} \partial_\perp^2 U \frac{p_i}{\alpha \beta_B s + p_\perp^2} U^\dagger | z_\perp) - (z_\perp | \text{same} | z_\perp) \right] \\
&+ 2 \left[(z'_\perp | \tilde{U} \frac{p_i}{\alpha \beta_B s + p_\perp^2} \partial_\perp^2 \tilde{U}^\dagger \frac{1}{p_\perp^2} | z_\perp) - (z'_\perp | \text{same} | z'_\perp) \right] \mathcal{F}^i(z_\perp, \beta_B) \\
&+ 2\tilde{\mathcal{F}}^i(\beta_B, z'_\perp) \left[(z'_\perp | \frac{1}{p_\perp^2} \mathcal{F}_i(\beta_B) U \frac{\alpha \beta_B s}{\alpha \beta_B s + p_\perp^2} U^\dagger | z_\perp) - (z_\perp | \text{same} | z_\perp) \right] \\
&+ 2 \left[(z'_\perp | \tilde{U} \frac{\alpha \beta_B s}{\alpha \beta_B s + p_\perp^2} \tilde{U}^\dagger \tilde{\mathcal{F}}^i(\beta_B) \frac{1}{p_\perp^2} | z_\perp) - (z'_\perp | \text{same} | z'_\perp) \right] \mathcal{F}_i(\beta_B, z_\perp) \Big\}
\end{aligned}$$

This expression is UV and IR convergent.

Small $\beta_B = x_B$

At $\beta_B \sim \frac{m^2}{s}$ we can set $\beta_B = 0$ and get for $\mathcal{F}_i^a(z_\perp, \beta_B) = U_i(z_\perp) \equiv i\partial_i U U^\dagger$

$$\begin{aligned} & \langle \tilde{U}^{ai}(z'_\perp) U_i^a(z_\perp) \rangle \\ &= \frac{g^2}{4\pi} \text{Tr} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \left\{ - (z'_\perp | \left[2\tilde{U} \frac{p_i p_j}{p_\perp^2} \tilde{U}^\dagger - (\tilde{U} \rightarrow 1) \right] \left[2U \frac{p_i^j p^j}{p_\perp^2} U^\dagger - (U \rightarrow 1) \right] | z_\perp) \right. \\ &+ 2(z'_\perp | \left[\tilde{U} \frac{p^i}{p_\perp^2} \tilde{U}^\dagger - (\tilde{U} \rightarrow 1) \right] U_i + \tilde{U}^i(\beta_B) \left[U \frac{p_i}{p_\perp^2} U^\dagger - (U \rightarrow 1) \right] | z_\perp) \\ &+ 2\tilde{U}^i(z'_\perp, \beta_B) \left[(z'_\perp | \frac{1}{p_\perp^2} \partial_\perp^2 U \frac{p_i}{p_\perp^2} U^\dagger | z_\perp) - (z_\perp | \text{same} | z_\perp) \right] \\ &+ 2 \left[(z'_\perp | \tilde{U} \frac{p_i}{p_\perp^2} \partial_\perp^2 \tilde{U}^\dagger \frac{1}{p_\perp^2} | z_\perp) - (z'_\perp | \text{same} | z'_\perp) \right] U^i(z_\perp, \beta_B) \Big\} \end{aligned}$$

This evolution equation for WW TMD can be rewritten in terms of the BK kernel

$$\begin{aligned} & \frac{d}{d\eta} \tilde{U}_i^a(z_2) U_i^a(z_1) \\ &= - \frac{g^2}{8\pi^3} \text{Tr} (-i\partial_i^{z_2} + \tilde{U}_i^{z_2}) \left[\int d^2 z_3 (\tilde{U}_{z_2} \tilde{U}_{z_3}^\dagger - 1) \frac{z_{12}^2}{z_{13}^2 z_{23}^2} (U_{z_3} U_{z_1}^\dagger - 1) \right] (i \overset{\leftarrow}{\partial}_i^{z_1} + U_i^{z_1}) \end{aligned}$$

Moderate $\beta_B = x_B \sim 1$

At moderate $\beta_B \sim 1$ the kernel is logarithmic.

In the double-log approximation $\alpha\beta_B s \gg p_\perp^2$ we get

$$\begin{aligned} & \langle \tilde{\mathcal{F}}^{ai}(z'_\perp, \beta_B) \mathcal{F}_i^a(z_\perp, \beta_B) \rangle \\ &= -\frac{g^2 N_c}{\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \int \frac{d^2 p}{p^2} \theta(\alpha \beta_H s - p_\perp^2) [1 - e^{i(p, z' - z)_\perp}] \langle \tilde{\mathcal{F}}^{ai}(z'_\perp, \beta_B) \mathcal{F}_i^a(z_\perp, \beta_B) \rangle \end{aligned}$$

which can be rewritten as a linear equation

$$\frac{d}{d\eta} \mathcal{D}(x_B, k_\perp, \eta) = -\frac{\alpha_s N_c}{\pi^2} \mathcal{D}(x_B, k_\perp, \eta) [\eta + \log \frac{\beta_H s}{k_\perp^2}]$$

with the solution

$$\mathcal{D}(x_B, k_\perp, \eta) = \mathcal{D}(x_B, k_\perp, \eta) \exp\left\{-\frac{\alpha_s N_c}{2\pi^2} [\eta + \log \frac{\beta_H s}{k_\perp^2}]^2\right\}$$

One can also get single-log terms from the general one-loop equation (in works)

For $\frac{m^2}{s} \ll \beta_B = x_B \ll 1$ there will be a double-log linear evolution from $\alpha \sim 1$ to $\alpha \sim \frac{m^2}{\beta_B}$ and non-linear evolution from $\alpha \sim \frac{m^2}{\beta_B}$ to $\alpha \sim \frac{m^2}{s}$. These two evolutions should be matched at $\alpha \sim \frac{m^2}{\beta_B}$ using our one-loop general formula (work in progress)

Conclusions

- So far, so good...

Outlook

- TMD for SIDIS (with Wilson lines to $-\infty$)
- TMD and k_T -factorization.