Scattering Amplitudes

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THEORETICAL PHYSICS
LSA MICHIGAN CENTER FOR THEORETICAL PHYSICS UNIVERSITY OF MICHIGAN

Gluon scattering amplitudes

Using Feynman diagrams, the tree-level gluon amplitude requires

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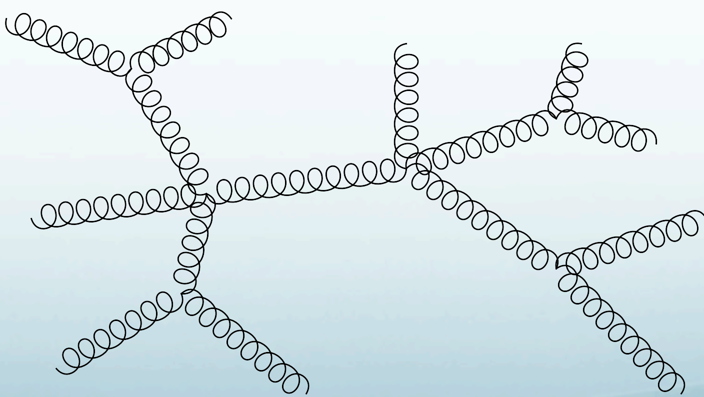
Each diagram gets increasingly complicated as n grows



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However, the result for the amplitudes can be written much more compactly, e.g.

$$A_n[1^+ \ldots i^- \ldots j^- \ldots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \,.$$

Here + and – indicate helicity of the gluons (all outgoing) and

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This is a major simplification!

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Scattering Amplitudes

Why are the results so simple?

Is there a better way to calculate amplitudes?

That is what this talk is about!

What makes gauge theory amplitudes difficult?

Feynman rules (and hence the diagrams) depend on

- Choice of gauge
- Field redefinitions

because the Lagrangian is off-shell.

But the on-shell amplitudes are *independent* of both.

If we can work only with on-shell invariant input, then these complications are avoided.

There is a way to do this: *on-shell recursion relations*

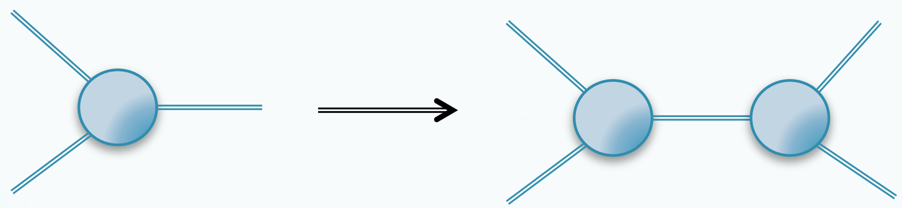
[Britto, Cachazo, Feng, Witten (2005)]

Better ways to calculate

On-shell recursion relations

Idea: build up n-point amplitudes from lower-point *on-shell* amplitudes

3-particle amplitude 4-particle amplitude



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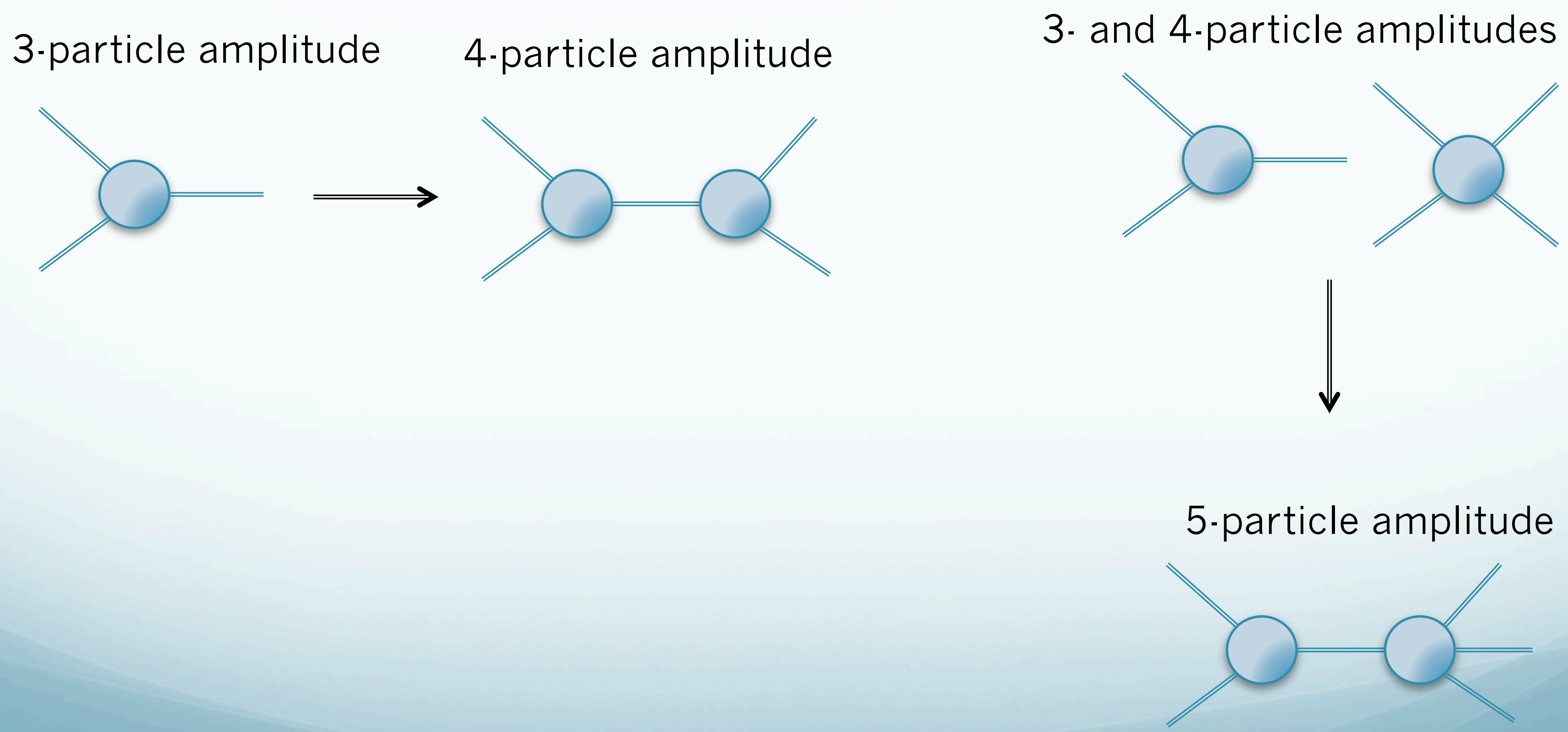

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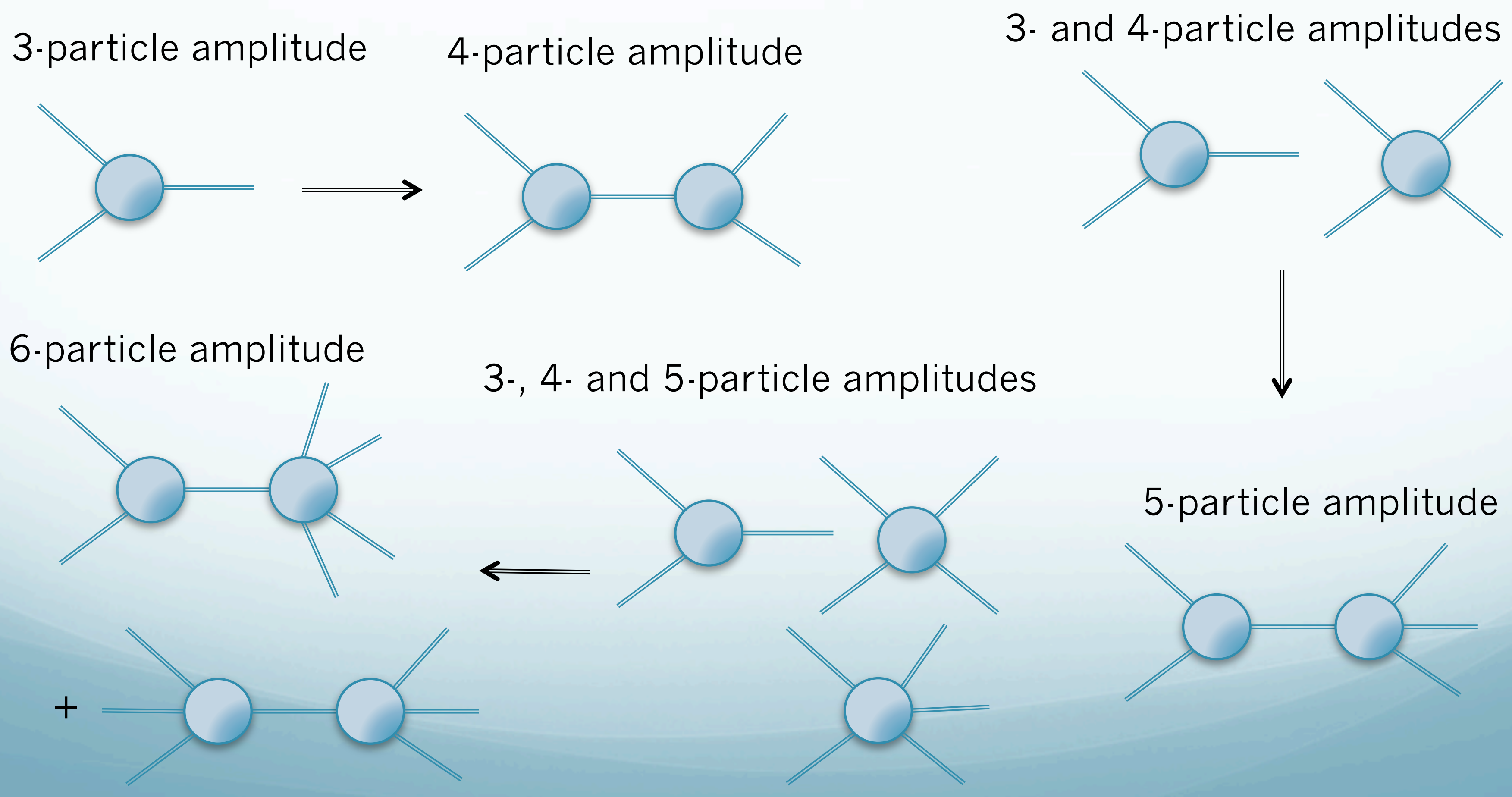

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Mathematically

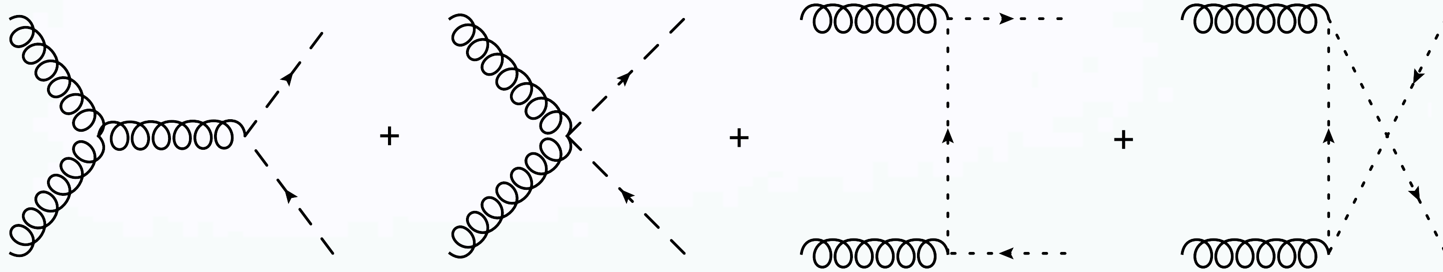
"hat" indicates momentum shift

$$A_n = \sum_{\text{diagrams } I} \hat{A}_{\mathrm{L}}(z_I)\,\frac{1}{P_I^2}\,\hat{A}_{\mathrm{R}}(z_I)$$

lower-point

The *derivation* exploits complex analysis and Cauchy's theorem in a very simple way: uses knowledge of physical poles in amplitudes and factorization.

Example 1 $\text{gluon} + \text{gluon} \rightarrow \text{squark} + \text{squark}$

$A_4 =$ 

As if four diagrams weren't bad enough.... having to square this $|A_4|^2$ and sum over colors & helicities to get the cross-section just makes it worse.

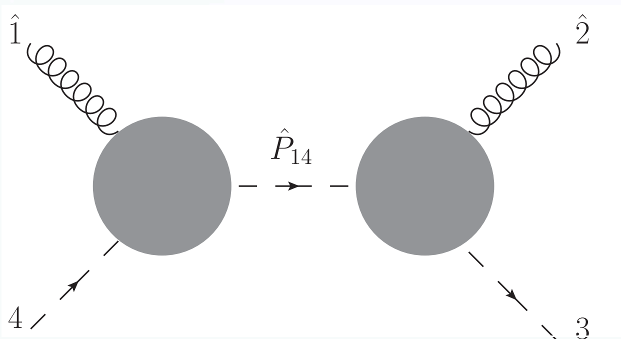
Recursion relations makes this much simpler!

I'll illustrate it first in the limit of $m_{squark} = 0$

Example 1 $\text{gluon} + \text{gluon} \to \text{squark} + \text{squark}$

$m_{squark} = 0$

Recursion relations: only one diagram



$$A_4[g_1^\pm g_2^\pm \tilde{q}_3 \tilde{q}_4^*] = 0$$

$$A_4[g_1^- g_2^+ \tilde{q}_3 \tilde{q}_4^*] = \frac{[23]^2[24]^2}{[12][23][34][41]}\,.$$

$$A_4[g_1^+ g_2^- \tilde{q}_3 \tilde{q}_4^*] = ([ij] \leftrightarrow \langle ij \rangle)$$

$$\langle ij \rangle = \sqrt{2p_i.p_j}\, e^{i\text{phase}}$$
$$[ij] = \sqrt{2p_i.p_j}\, e^{-i\text{phase}}$$

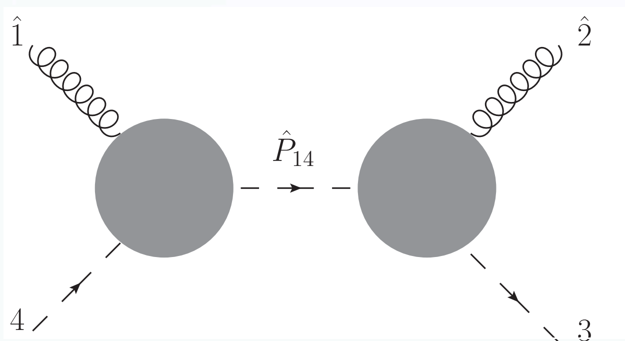
$$\langle 12 \rangle [12] = (p_1 + p_2)^2 = -s, \qquad \text{etc}$$

$$\left| A_4[g_1^- g_2^+ \tilde{q}_3 \tilde{q}_4^*] \right|^2 = \frac{s_{23}^2 s_{24}^2}{s_{12} s_{23} s_{34} s_{41}} = \frac{t^2}{s^2}$$

Example 1 gluon + gluon → squark + squark

Recursion relations: only one diagram

m_{squark} nonzero



$$A_4[g_1^- g_2^- \tilde{q}_3 \tilde{q}_4^*] = \frac{1}{[12]^2}\frac{s m_{\tilde{q}}^2}{u_1}$$

$$A_4[g_1^- g_2^+ \tilde{q}_3 \tilde{q}_4^*] = -\frac{1}{u_1 s}\langle 1|4|2][2|3|1\rangle,$$

$$u_1 = u - m_{\tilde{q}}^2$$

$$t_1 = t - m_{\tilde{q}}^2$$

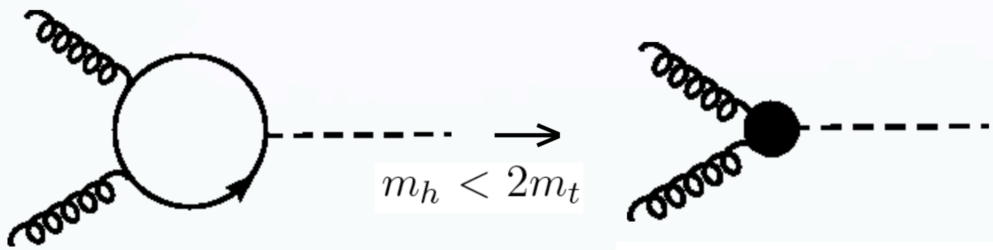
$$\sum \left|A_4(g_1 g_2 \tilde{q}_3 \tilde{q}_4^*)\right|^2 = 2g^4 \left\{ N(N^2-1)\left(1 - 2\frac{u_1 t_1}{s^2}\right) - \frac{N^2-1}{N} \right\} \left[1 - 2\frac{m_{\tilde{q}}^2 s}{u_1 t_1}\left(1 - \frac{m_{\tilde{q}}^2 s}{u_1 t_1}\right)\right]$$

This is the well-known result from the gluon to squark cross-section in the literature. Here derived via recursion relations by a Michigan undergrad, Filipe Rudriguez.

Example 2: Gluon fusion

Gluon-Higgs fusion: g g → Higgs

[Dixon, Glover, Khoze' 04; Berger, Del Duca, Dixon' 06]



$$h\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}$$

dim-5 operator

Applied
on-shell recursion
to gluon-Higgs fusion

⇒ Higgs+*n* gluon
amplitudes

1-loop
⇒ tree-level

2-loop
⇒ 1-loop

etc

Proof of validity
[Cohen, H.E., Kiermaier (2010)]

Study of Scattering Amplitudes

Two pillars

Practical application
to phenomenologically
relevant processes

Uncover the
mathematical structure
of the amplitudes

TRUTH

BEAUTY

Three Major Research Directions

1) Try to push loop calc as far as possible in a very controlled simple theory: "(planar) N=4 Super Yang Mills Theory" (SYM) (gluons, gluinos, scalars - all massless)

Goals: `Solve' theory at all loop order.
Compact expressions? Understand why!

2) Adapt lessons from N=4 SYM to phenomenologically relevant theories to find new computational methods.

Goals: application to analysis of data from
LHC and future particle experiments. New physics insights?

3) Use new methods to explore perturbative quantum gravity.

Goals: Point-particle quantum gravity perturbatively sensible?
Gravity as (gauge theory)^2
Structure of string theory amplitudes

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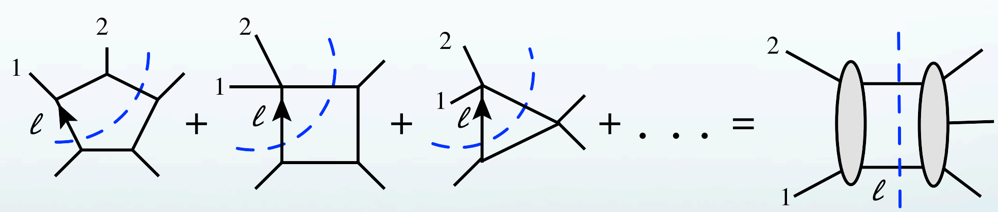
Tree-level: all tree amplitudes in N=4 SYM solved via recursion.

Loops:

Generalized Unitarity [Bern, Dixon, Kosower,...]



tree x tree

Loop *integrand* recursion relations [Arkani-Hamed, Bourjaily, Cachazo, Trnka]

Symmetry/integrability [Vieira, Sever, Basso,...]

Planar N=4 Super Yang Mills Theory and GEOMETRY

It turns out that amplitudes have a geometric interpretation.

Recursion relations => amplitude = sum of terms

Each of these terms can be interpreted as the volume of a 4-simplex

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\begin{aligned}
&\text{0-simplex} = \text{a point}\\
&\text{1-simplex} = \text{line segment}\\
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&\text{3-simplex} = \text{tetrahedron.}
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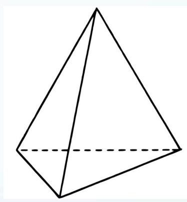

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Just like a polygon can be triangulated in different ways for computation of its area, a polytope can be triangulated into simplices.

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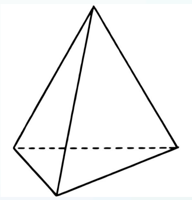
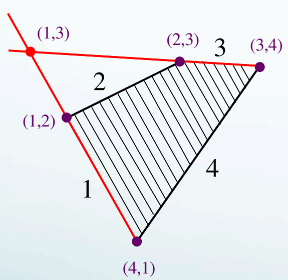
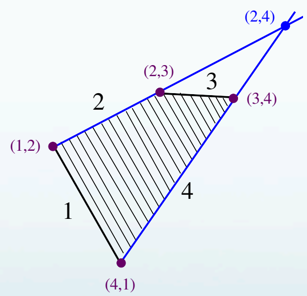
1-simplex = line segment
2-simplex = triangle
3-simplex = tetrahedron.







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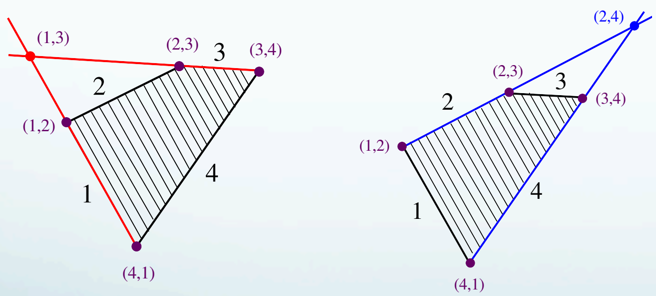

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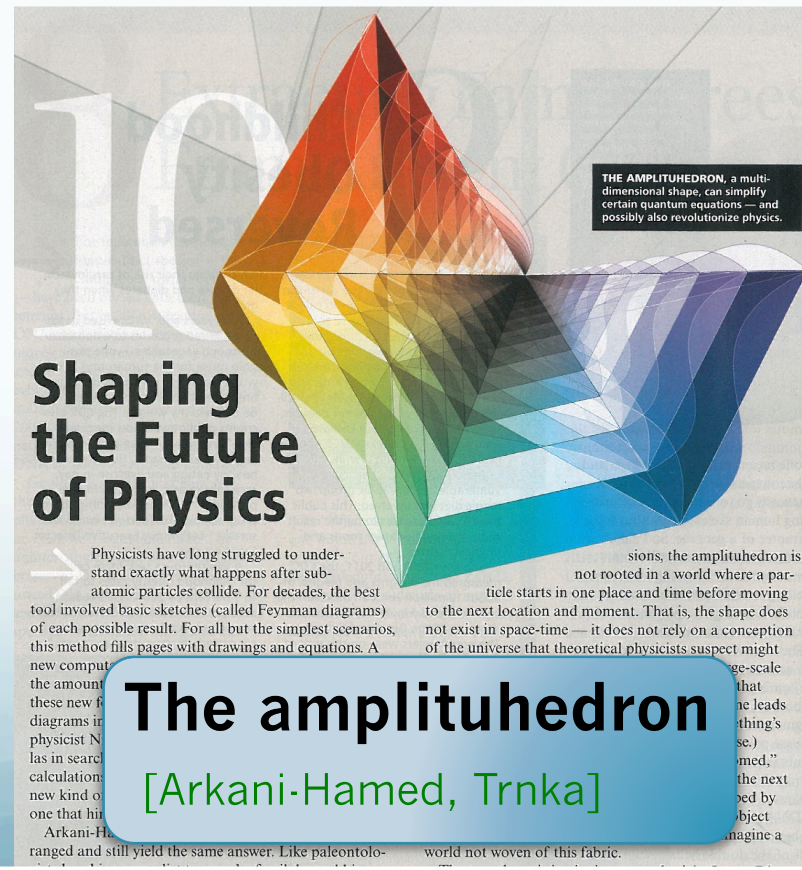

The amplituhedron

[Arkani-Hamed, Trnka]

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Is $\mathcal{N} = 8$ Supergravity Ultraviolet Finite?

Z. Bern[a], L. J. Dixon[b], R. Roiban[c] (2007)

N=8 supergravity in 4d:

< 2007: 4-graviton 1,2,3-loop finite
2009: 4-graviton 4-loop finite

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)+(2009)]

Done using *generalized unitarity* AND ***gravity as (gauge theory)^2***

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2010: all *n*-graviton amplitudes finite L<7
guaranteed by symmetries.

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)+(2009)]

Used amplitude techniques to asses possible UV counterterms. No loops needed.

[Freedman, Kiermaier, HE; Kiermaier, HE; Beisert, Kiermaier, Freedman, Morales, Stieberger (2010)]

Perturbative finiteness still an open question.

Summary

Modern on-shell methods for scattering amplitudes are incredibly powerful.

Applications in a wide range of problems:

- Pheno amplitudes
- Formal developments (mathematical structure, geometry)
- Quantum gravity
- Studies of formal aspects of QFT
- Non-renormalization theorems

and much more!

Textbook

Published by
Cambridge University Press
(2015)

(preview: arXiv:1308.1697)

Includes introductions to:

Spinor helicity formalism
for Feynman diagrams.

On-shell recursion relations.

Supersymmetry applications.

Unitarity cuts and loops.

Advanced subjects
(Grassmannians, on-shell diagrams, polytopes, supergravity,
supergravity = $(SYM)^2$...)