Flow measurements with multiparticle azimuthal correlations

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Workshop on "Ions at the Future Circular Collider”

centre for particle physics
Discovery

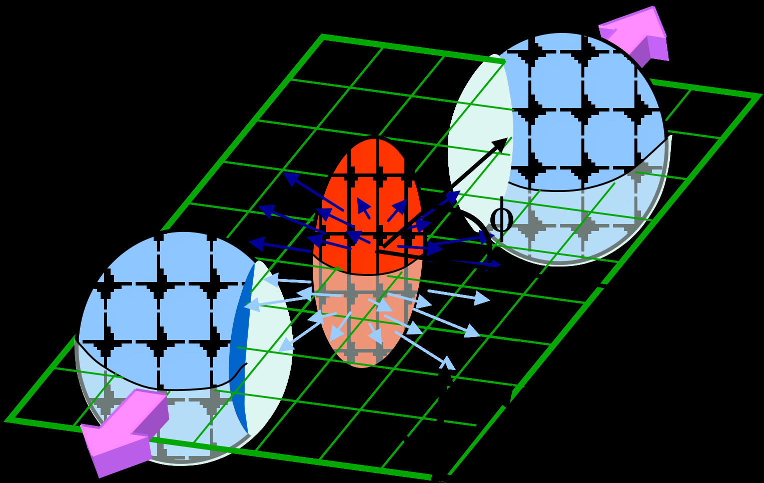
Disclaimer

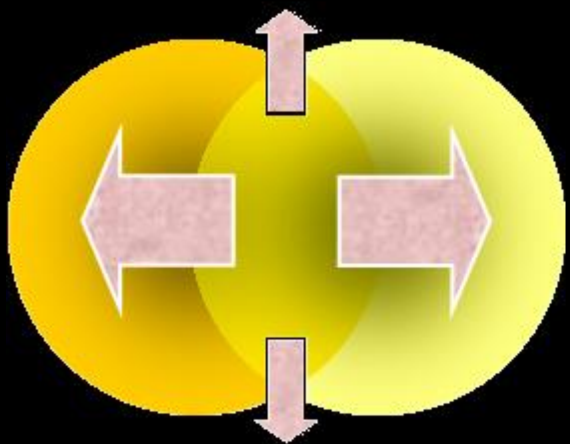
- The primary focus of this talk are recent developments in the multiparticle technology itself and the discussion of its future potential
- This is **not a review talk** on the experimental or theoretical flow results obtained with multiparticle correlations
  - There are plenty of such review/summary talks in the air at the moment focusing on flow results
  - On the other hand, the multiparticle technology itself wasn't addressed for ages
- What will be presented here is a brand new and mostly technical material from:
  - 'Generic framework for anisotropic flow analyses with multiparticle azimuthal correlations,' Phys.Rev. C89 (2014) 064904
  - 'Higher order moments of multiparticle azimuthal correlations,' arXiv:1409.5636 (prepared exclusively for this workshop)
- Therefore, follow this talk at your own peril...

Outline

- Introduction
- Recent progress on multiparticle correlations
  - Measurement
  - Systematical biases
  - Statistical properties
    - Sensitivity
- What's next?

Anisotropic flow (1/2)





- Anisotropies in momentum space (S. Voloshin and Y. Zhang (1996)):

$$E\frac{d^3N}{d^3\vec{p}} = \frac{1}{2\pi}\frac{d^2N}{p_T\,dp_T\,dy}\left(1+\sum_{n=1}^{\infty}2v_n\cos\left(n\left(\phi-\Psi_{\mathrm{RP}}\right)\right)\right)$$

$$v_n = \langle\cos(n(\phi-\Psi_{\mathrm{RP}}))\rangle$$

- Harmonics v_n quantify anisotropic flow
  - v_1 is directed flow, v_2 is elliptic flow, v_3 is triangular flow, etc.

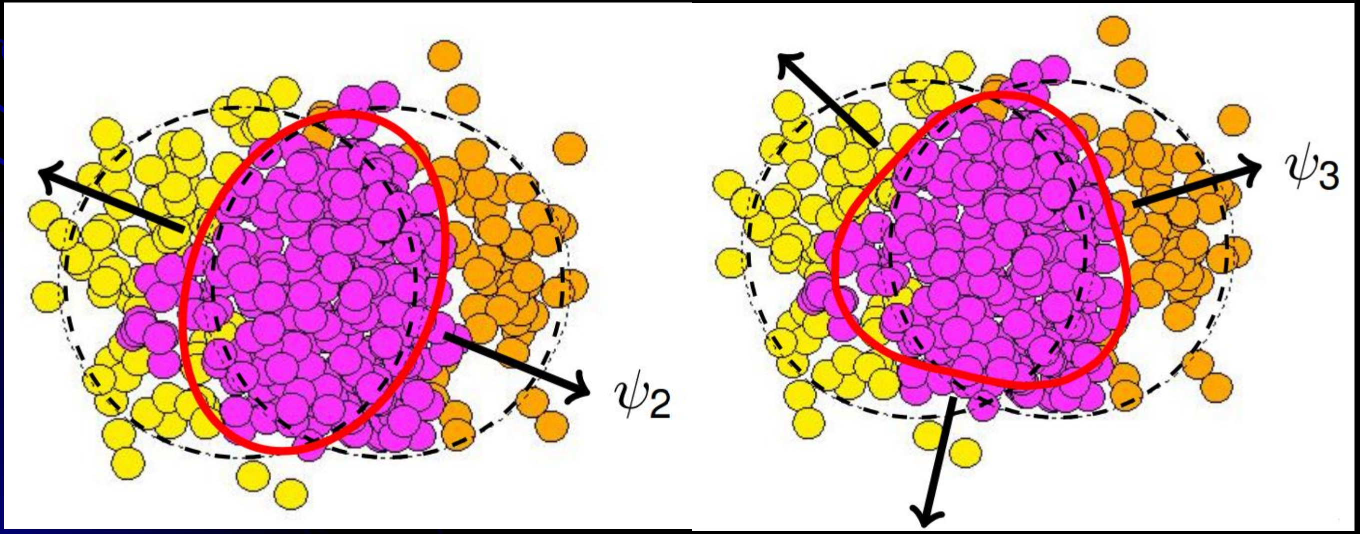
Anisotropic flow (2/2)

- We need full Fourier decompositon to also take into account effects of fluctuations, each harmonic has its own symmetry plane:

$$v_n = \langle \cos(n(\phi - \Psi_n)) \rangle$$

$$f(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1} v_n \cos[n(\varphi - \Psi_n)]\right]$$

$$f(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} (c_n \cos n\varphi + s_n \sin n\varphi)\right]$$



Why correlations?

- Originally, the idea was to estimate the amplitudes v_n without the explicit knowledge of symmetry planes

$$\begin{aligned}\left\langle\left\langle e^{in(\phi_1-\phi_2)}\right\rangle\right\rangle &= \left\langle\left\langle e^{in(\phi_1-\Psi_{\mathrm{RP}}-(\phi_2-\Psi_{\mathrm{RP}}))}\right\rangle\right\rangle \\ &= \left\langle\left\langle e^{in(\phi_1-\Psi_{\mathrm{RP}})}\right\rangle\left\langle e^{-in(\phi_2-\Psi_{\mathrm{RP}})}\right\rangle\right\rangle = \langle v_n^2\rangle\end{aligned}$$

- Autocorrelations: Must be removed by definition

$$\begin{aligned}\langle 2\rangle_{n,-n} &\equiv \left\langle e^{in(\varphi_1-\varphi_2)}\right\rangle = \langle\cos n(\varphi_1-\varphi_2)\rangle \\ &= \frac{1}{M(M-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{M}\cos n(\phi_i-\phi_j)\end{aligned}$$

- Higher order moments v_n^k
- **Key point:** Factorization of joint multivariate p.d.f.

$$f(\varphi_1,\ldots,\varphi_n) = f_{\varphi_1}(\varphi_1)\cdots f_{\varphi_n}(\varphi_n)$$

Historical account

- A great amount of effort been invested in the last ~15 years to calculate efficiently and exactly multiparticle azimuthal correlations free from trivial (yet dominant) contributions from autocorrelations
- ‘Brute force’ approach with nested loops fails already at the level of three-particle correlations in heavy-ion collisions at LHC
- Sophisticated and efficient formalism of generating functions

$$G_n(z) \equiv \prod_{j=1}^{M}\left(1 + \frac{z^* e^{in\phi_j} + z e^{-in\phi_j}}{M}\right)$$

$$\langle G_n(z)\rangle = \sum_{k=0}^{M/2} \frac{|z|^{2k}}{M^{2k}} \binom{M}{k} \binom{M-k}{k} \left\langle e^{in(\phi_1+\cdots+\phi_k-\phi_{k+1}-\cdots-\phi_{2k})}\right\rangle$$

N. Borghini, P. M. Dinh and J.-Y. Ollitrault, “Flow analysis from multiparticle azimuthal correlations,” PRC 64 (2001) 054901

- Non-exact, limited in scope to very specific correlators when all harmonics coincide
- *Q*-cumulants: exact, however also limited in scope

Generic case

- Generic definition of m-particle azimuthal correlation:

$$\langle m\rangle_{n_1,n_2,\ldots,n_m} = \frac{\sum\limits_{\substack{k_1,k_2,\ldots,k_m=1\\ k_1\neq k_2\neq\ldots\neq k_m}}^{M} w_{k_1} w_{k_2}\cdots w_{k_m}\, e^{i(n_1\varphi_{k_1}+n_2\varphi_{k_2}+\cdots+n_m\varphi_{k_m})}}{\sum\limits_{\substack{k_1,k_2,\ldots,k_m=1\\ k_1\neq k_2\neq\ldots\neq k_m}}^{M} w_{k_1} w_{k_2}\cdots w_{k_m}}$$

- We want all possible multiparticle correlations:
  - Measured exactly and efficiently free from autocorrelations
  - Corrected for various detector inefficiencies
- **New:** Generic framework for anisotropic flow analyses with multiparticle azimuthal correlations had been developed in the last year for this reason

Generic framework

Phys.Rev. C89 (2014) 064904

http://www.nbi.dk/~cholm/mcorrelations/

Analytic solutions

- Analytic standalone formulas in terms of weighted Q-vectors

$$Q_{n,p} \equiv \sum_{k=1}^{M} w_k^p \, e^{in\varphi_k}$$

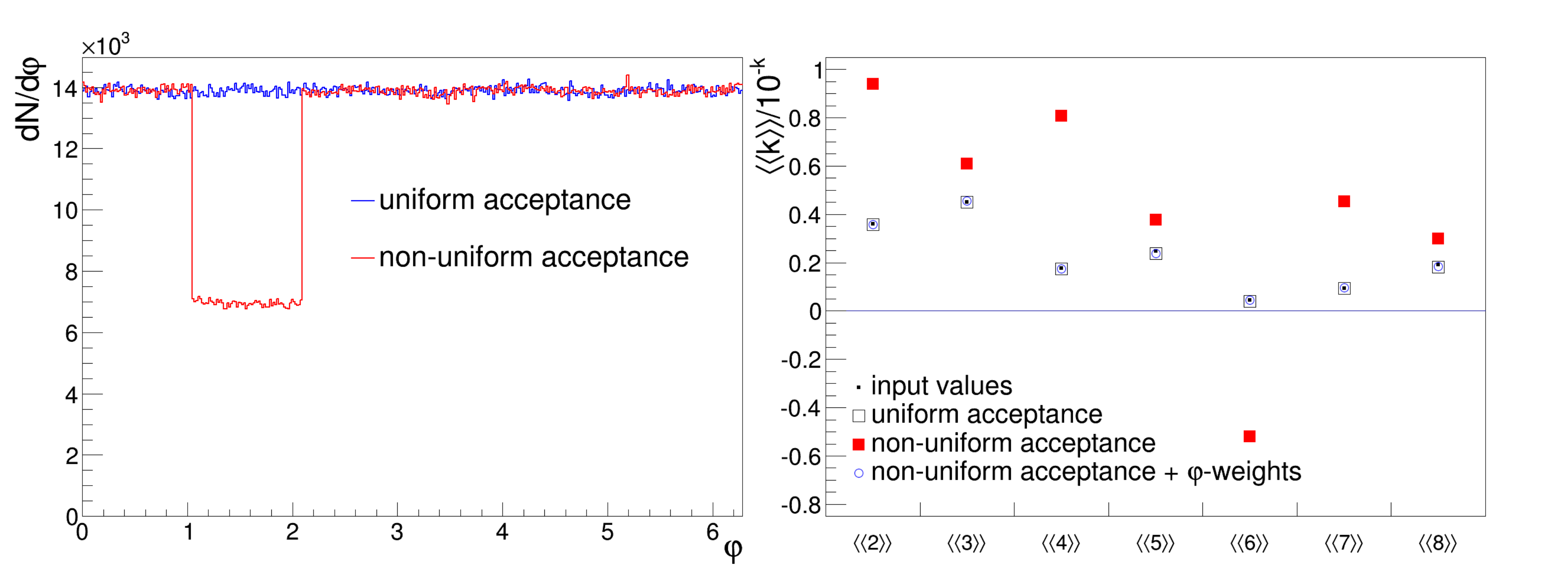
- Exact and efficient solutions for the problem of autocorrelations for the most general case
- Example: Generic solution for 4-particle azimuthal correlation

$$\begin{aligned}
\mathrm{Num}\langle 4\rangle_{n_1,n_2,n_3,n_4} \quad = \quad & Q_{n_1,1}Q_{n_2,1}Q_{n_3,1}Q_{n_4,1} - Q_{n_1+n_2,2}Q_{n_3,1}Q_{n_4,1} - Q_{n_2,1}Q_{n_1+n_3,2}Q_{n_4,1} \\
& - Q_{n_1,1}Q_{n_2+n_3,2}Q_{n_4,1} + 2Q_{n_1+n_2+n_3,3}Q_{n_4,1} - Q_{n_2,1}Q_{n_3,1}Q_{n_1+n_4,2} \\
& + Q_{n_2+n_3,2}Q_{n_1+n_4,2} - Q_{n_1,1}Q_{n_3,1}Q_{n_2+n_4,2} + Q_{n_1+n_3,2}Q_{n_2+n_4,2} \\
& + 2Q_{n_3,1}Q_{n_1+n_2+n_4,3} - Q_{n_1,1}Q_{n_2,1}Q_{n_3+n_4,2} + Q_{n_1+n_2,2}Q_{n_3+n_4,2} \\
& + 2Q_{n_2,1}Q_{n_1+n_3+n_4,3} + 2Q_{n_1,1}Q_{n_2+n_3+n_4,3} - 6Q_{n_1+n_2+n_3+n_4,4}
\end{aligned}$$

- The number of distinct terms per correlator form a well known **Bell sequence**: 1, 2, 5, 15, 52, 203, 877, 4140, 21147 …
- Recursive algorithms (credits to my colleagues @ NBI!)
  - Developed specifically for higher orders (>6) because such a huge expressions actually cannot be compiled!!

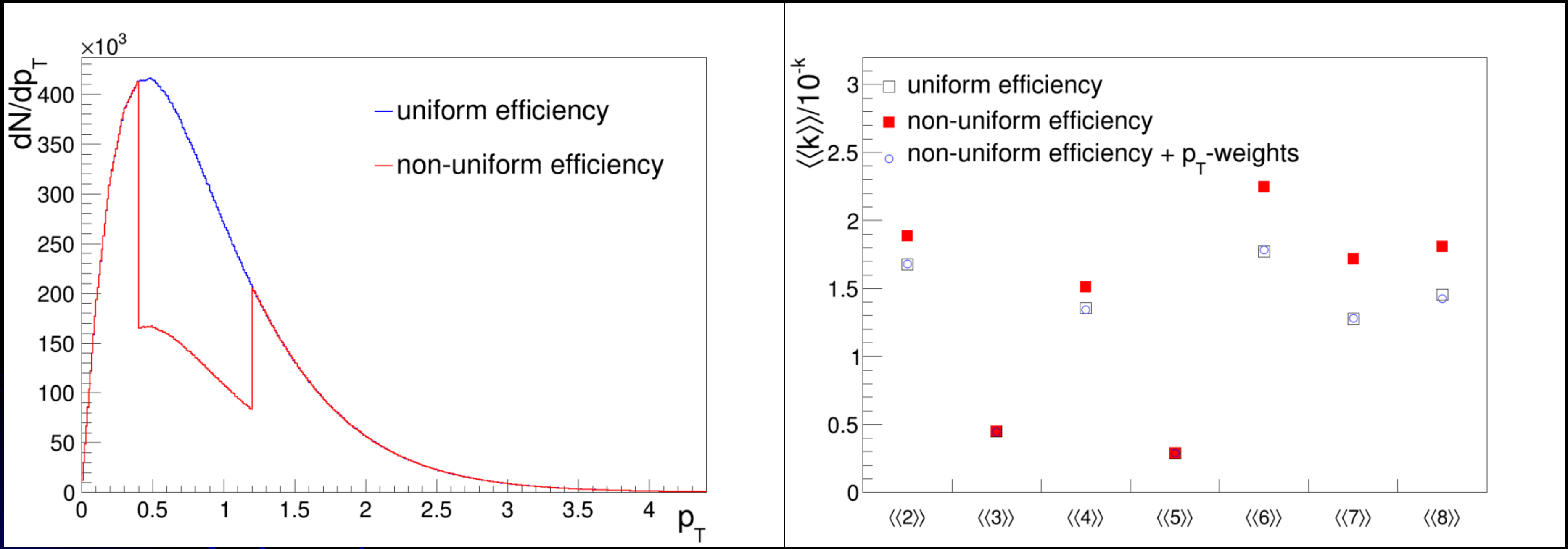
Weights (1/2)

- Full-scale efficiency framework with generic weights
- Example 1: Non-uniform acceptance



Weights (2/2)

- Example 2: p_T dependent reconstruction efficiency

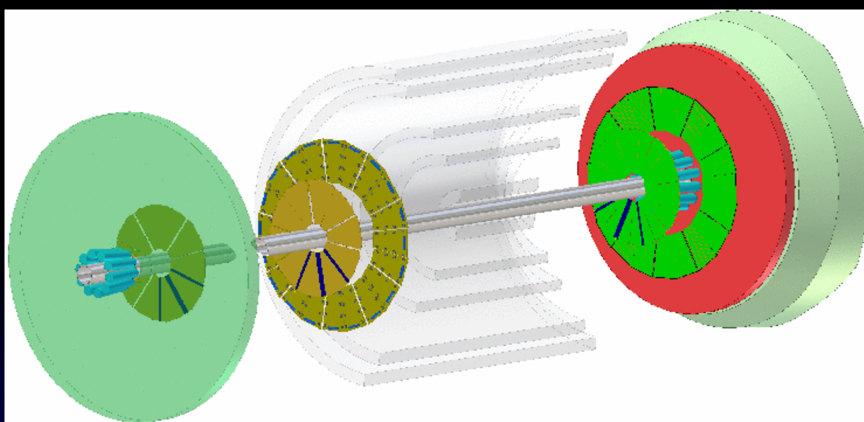


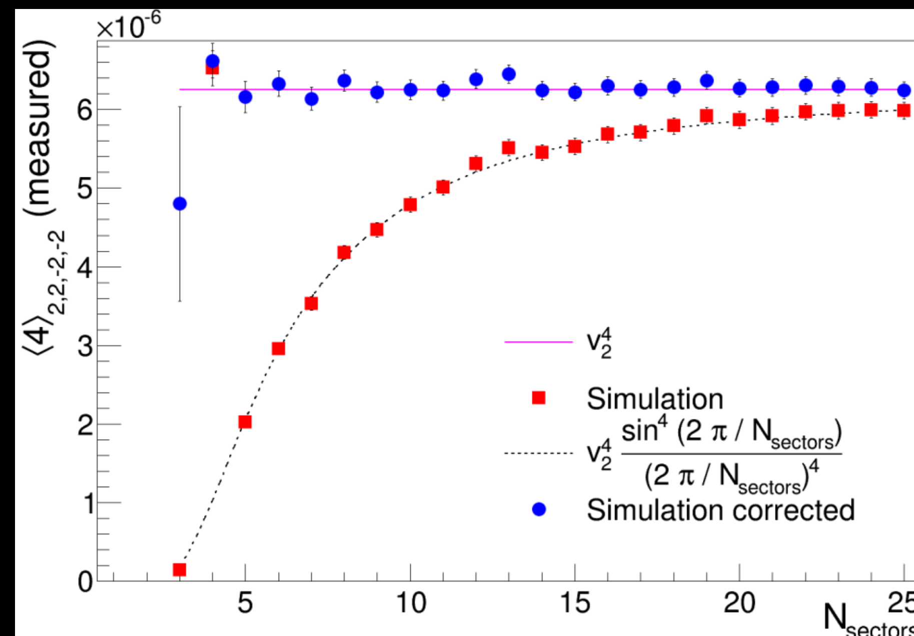
- Independent weights can also be combined within the framework

Finite granularity

- Design considerations for future detectors: Finite granularity introduces a systematical bias in anisotropic flow estimates with multiparticle azimuthal correlations

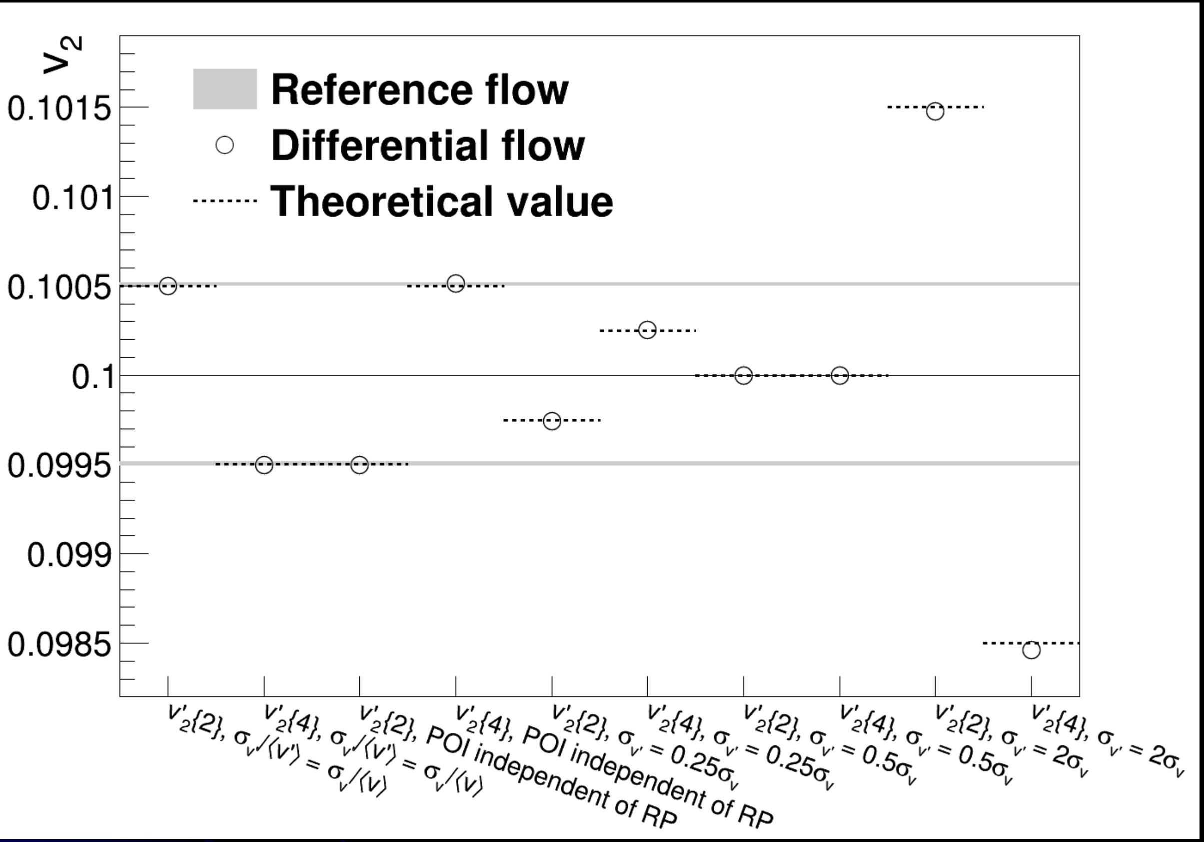
ALICE forward detectors:





Exotica

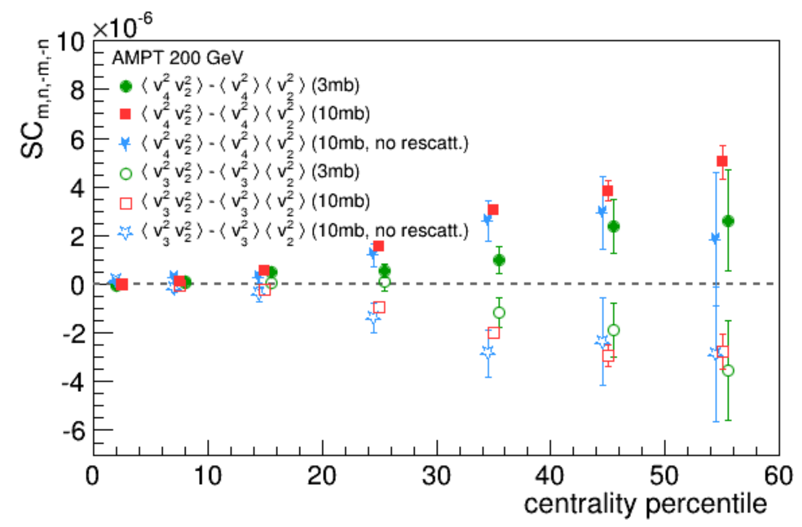
- We have found a peculiar bias which is present in the traditional differential flow analysis with correlation techniques, which stem solely from the particle selection criteria, and is present also in an ideal case when only flow correlations are present

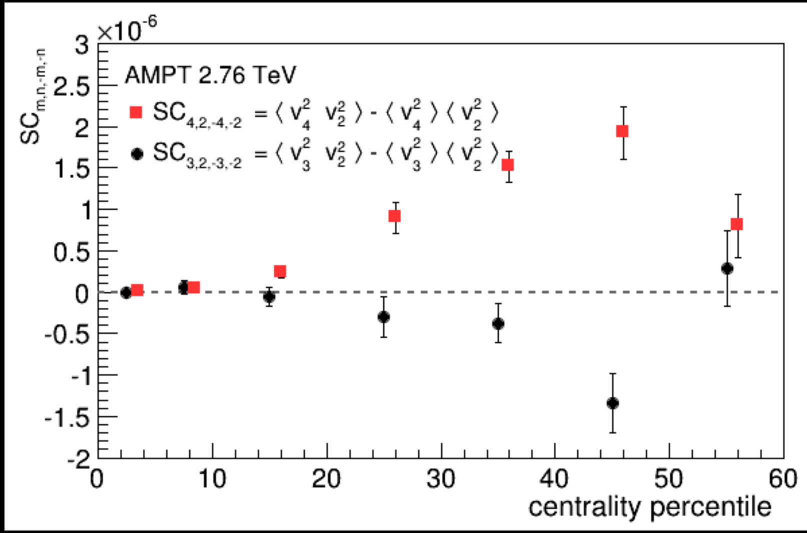


New observables

- Plethora of new anisotropic flow observables available
- Example: We can quantify correlations between event-by-event fluctuations of different harmonics

$$
\begin{aligned}
\langle\langle\cos(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)\rangle\rangle_c &= \langle\langle\cos(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)\rangle\rangle \\
&\quad -\langle\langle\cos[m(\varphi_1-\varphi_2)]\rangle\rangle\,\langle\langle\cos[n(\varphi_1-\varphi_2)]\rangle\rangle \\
&= \left\langle v_m^2 v_n^2\right\rangle - \left\langle v_m^2\right\rangle\left\langle v_n^2\right\rangle \\
&= 0
\end{aligned}
$$





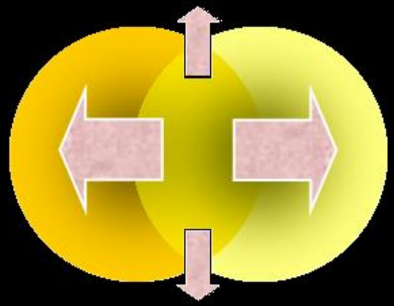
Quest for the Holy p.d.f.

arXiv:1409.5636

Tribute to Q-vectors

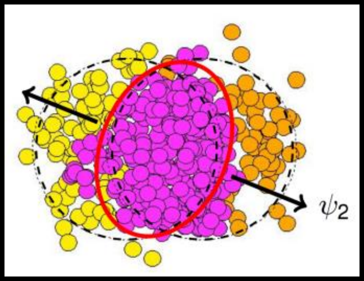
- M-particle Q-vector:

$$Q_n \equiv \sum_{k=1}^{M} e^{in\varphi_k}$$



- Single-particle Q-vector:

$$u_n = e^{in\varphi}$$

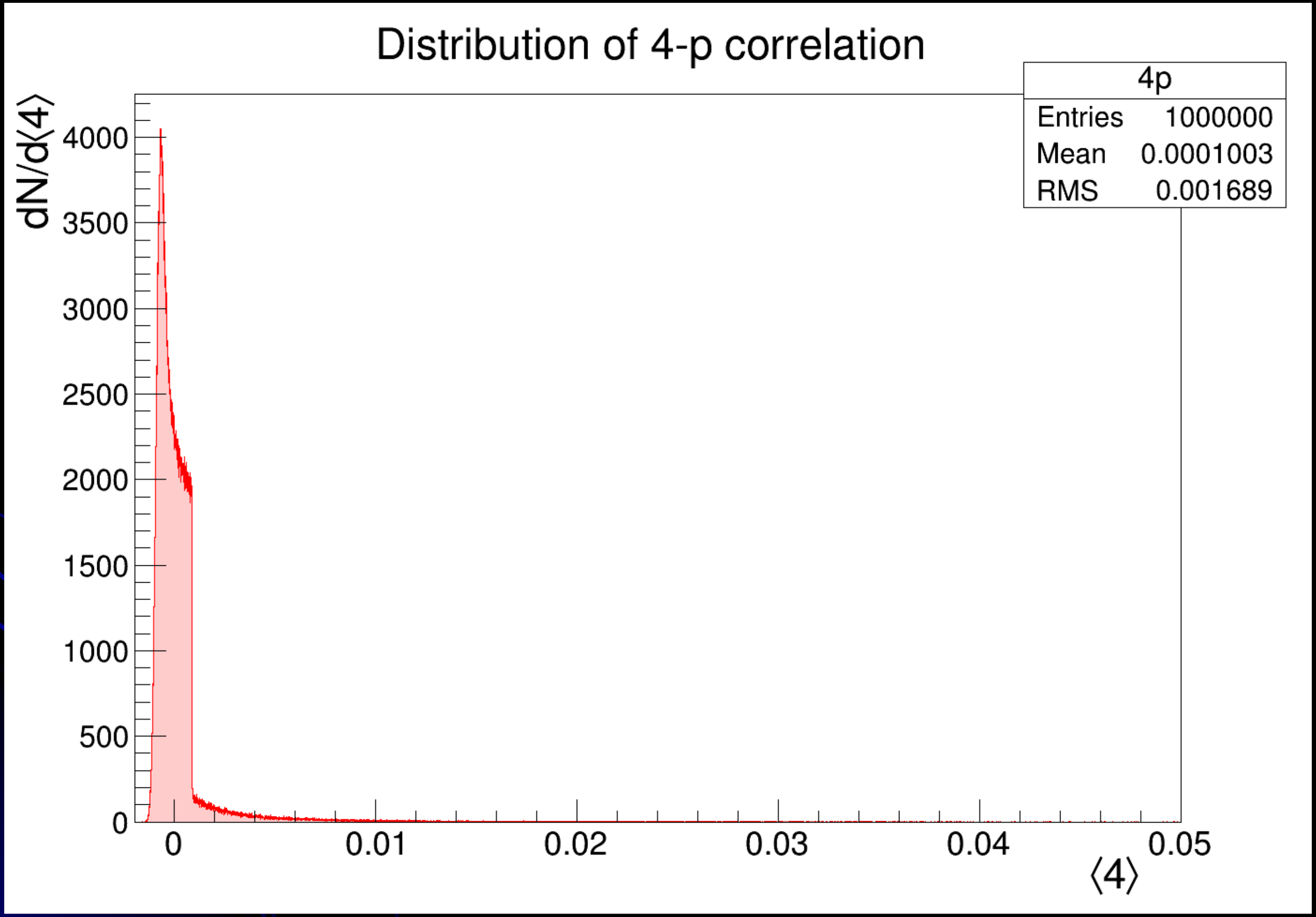


- Reduced Q-vector:

$$q_n = \frac{Q_n}{\sqrt{M}}$$

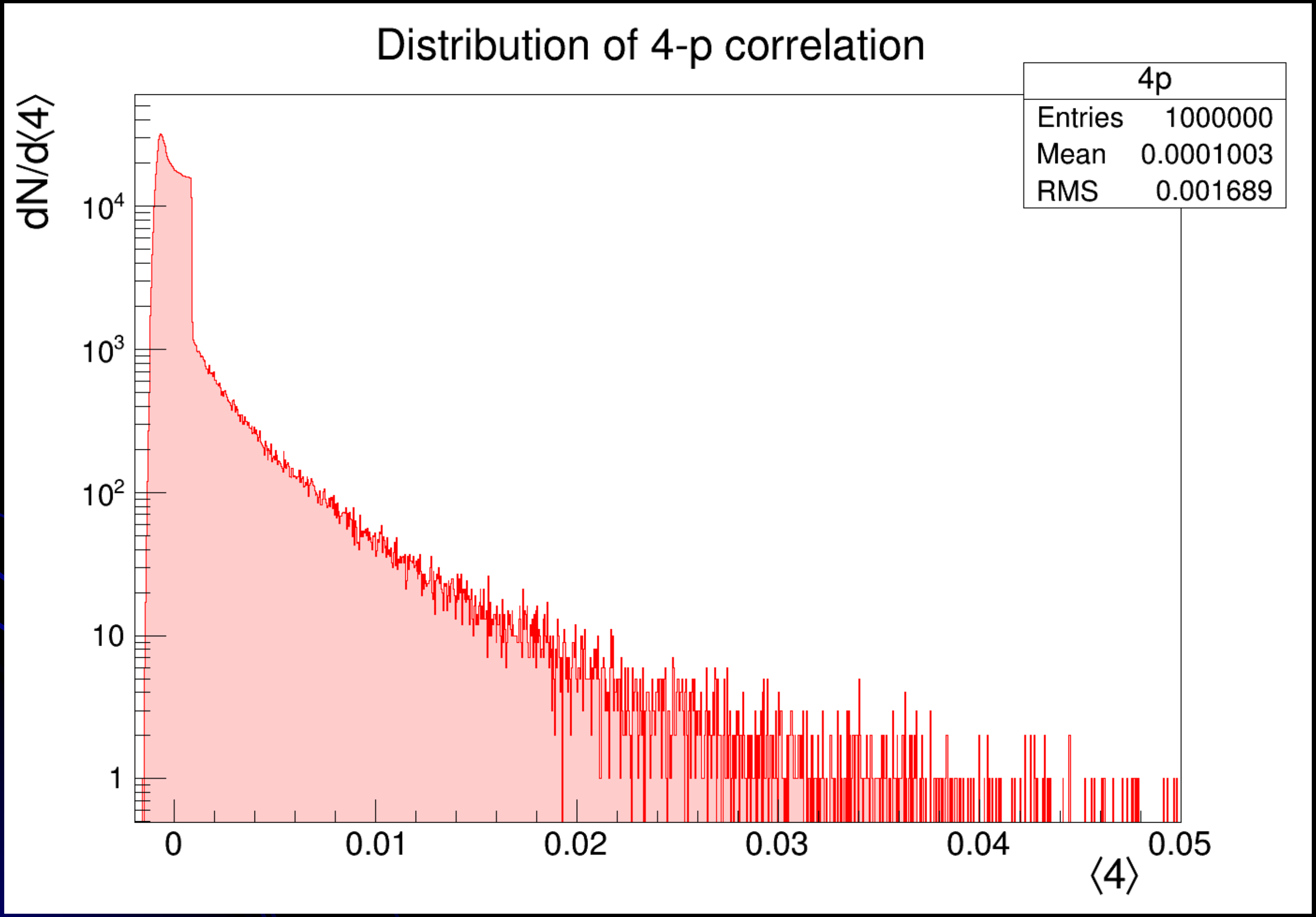
Invitation (1/4)

- The simplest possible toy Monte Carlo example: $M = 50$, $v_2 = 0.10$



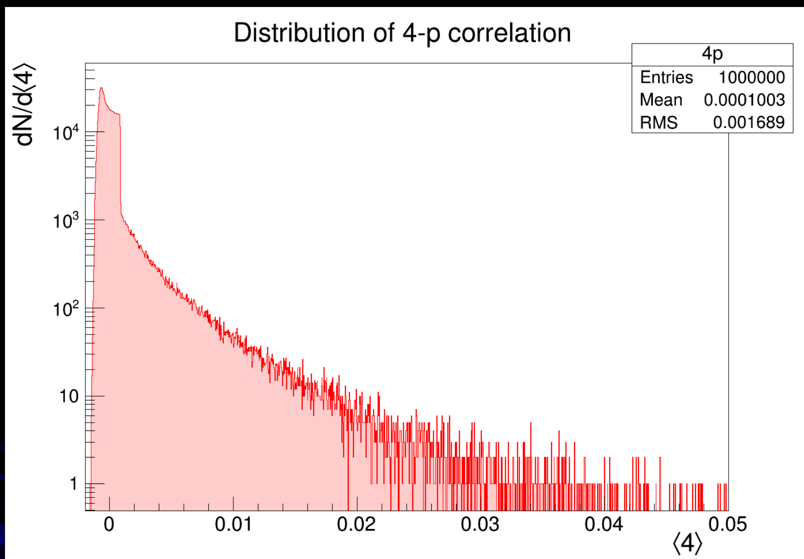
Invitation (2/4)

- The simplest possible toy Monte Carlo example: $M = 50$, $v_2 = 0.10$



Invitation (3/4)

- It is used widely, but what actually we know about mathematical and statistical properties of this ‘funny thing’ below?



- **1)** We know that its first algebraic moment (mean) is given by v^4 for the case when we can factorize the joint multivariate p.d.f. (a.k.a. ‘when only flow correlations are present’) into the product of single particle Fourier-like p.d.f.’s

$$f(\varphi_1,\ldots,\varphi_n) = f_{\varphi_1}(\varphi_1)\cdots f_{\varphi_n}(\varphi_n)$$

- **2)** We know how to express it analytically in terms of Q-vectors:

$$\langle 4\rangle = \frac{|Q_n|^4 + |Q_{2n}|^2 - 2\cdot\mathbf{Re}\left[Q_{2n}Q_n^*Q_n^*\right] - 4(M-2)\cdot|Q_n|^2}{M(M-1)(M-2)(M-3)} + \frac{2}{(M-1)(M-2)}$$

- **3)** ?

Invitation (4/4)

- By measuring only the first moment of such a complicated distribution, we are only **scratching the surface**
- **Long tails:** We cannot even in principle distinguish the healthy events from the outliers
- p.d.f. is not defined by one functional form, i.e. the p.d.f. is clearly a **piecewise-defined** function
- The **higher order moments** by definition carry an independent information about the underlying multivariate p.d.f. and have different dependence on the degrees of freedom of the initial single-particle p.d.f.'s

$$\begin{aligned}\langle 4\rangle &= \frac{|Q_n|^4+|Q_{2n}|^2-2\cdot\mathbf{Re}\left[Q_{2n}Q_n^*Q_n^*\right]-4(M-2)\cdot|Q_n|^2}{M(M-1)(M-2)(M-3)}\\ &+ \frac{2}{(M-1)(M-2)}\end{aligned}$$

Terminology

- Fourier series parametrizations, two conventions widely used:

$$f(\varphi) = \frac{1}{2\pi}\left[1+2\sum_{n=1} v_n \cos[n(\varphi-\Psi_n)]\right]$$

$$f(\varphi) = \frac{1}{2\pi}\left[1+2\sum_{n=1}^{\infty}(c_n \cos n\varphi + s_n \sin n\varphi)\right]$$

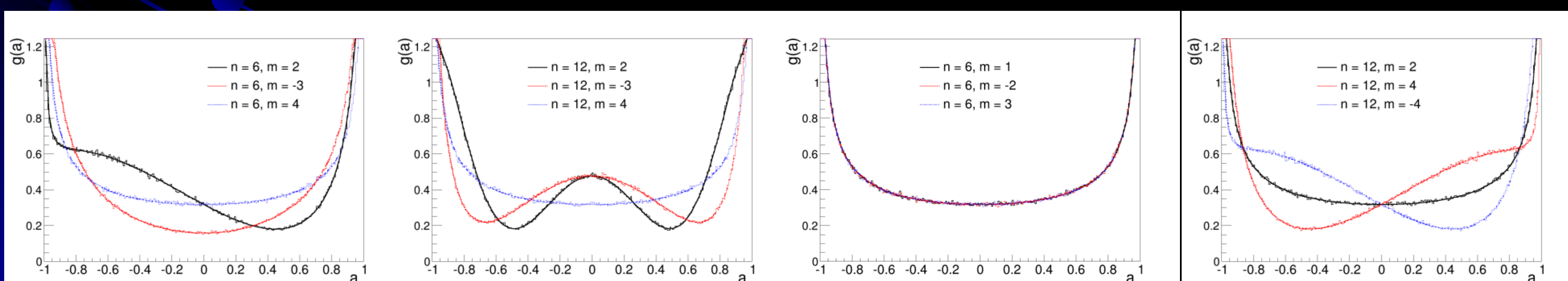
- **Random walk:** All harmonics are zero
- **Monochromatic flow:** Only one harmonic is non-zero
- **Multichromatic flow:** All harmonics are non-zero

p.d.f. of single-particle Q-vectors

- **Analytic solutions** for p.d.f.'s available
- p.d.f.'s of both of the real and imaginary parts of single-particle *Q*-vectors and for the most general case of multichromatic flow, are given solely in terms of **Chebyshev polynomials of the first kind:**

$$g(a) = \frac{1}{\pi\sqrt{1-a^2}}\left(1+\sum_{l=1}^{\infty} 2c_{l\cdot m}T_l(a)\right), \qquad a = \cos mx$$

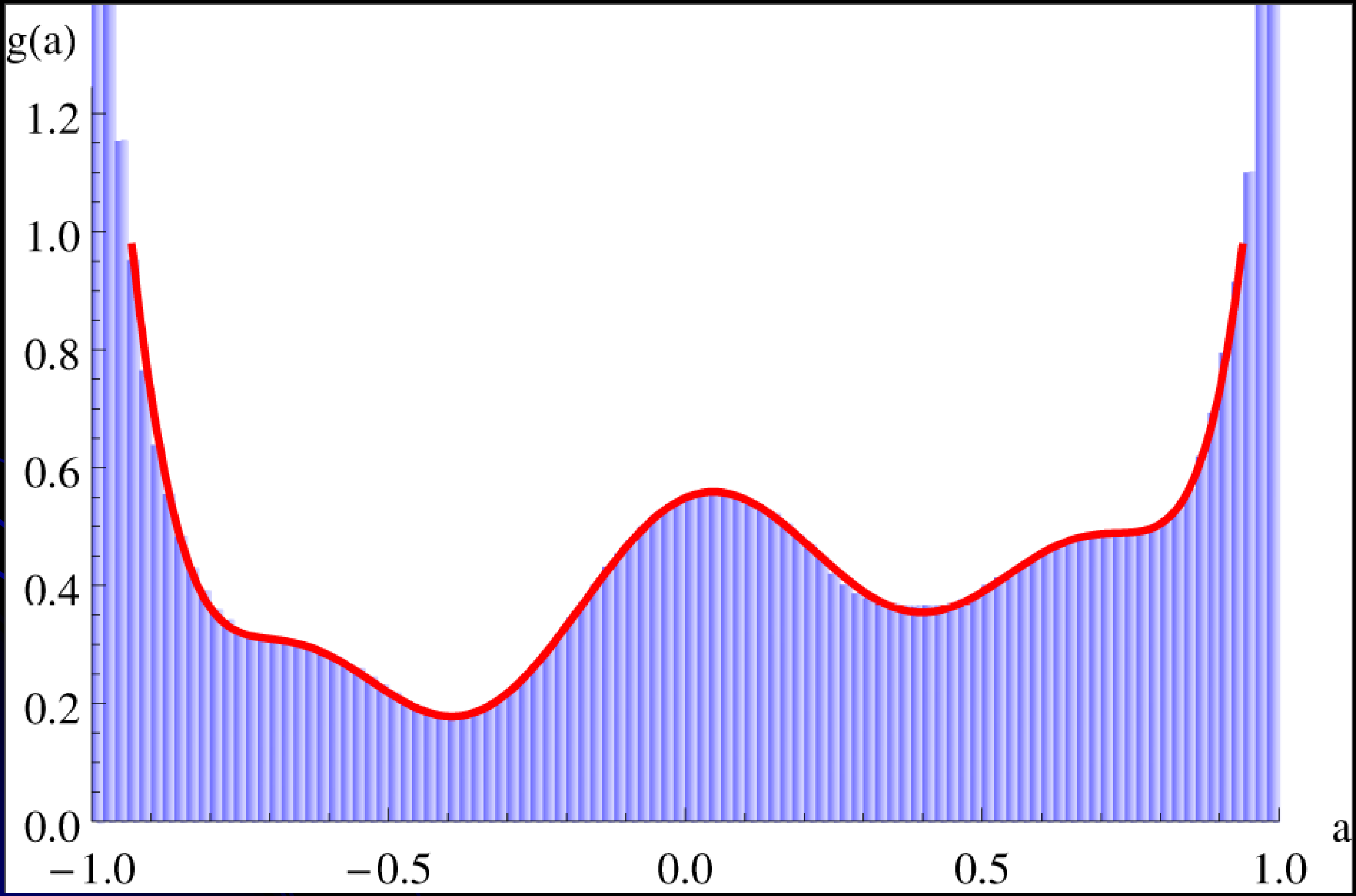
$$g(a) = \frac{1}{\pi\sqrt{1-a^2}}\left(1+\sum_{l=1}^{\infty} 2(-1)^l\left(c_{2l\cdot m}T_{2l}(a) - s_{(2l-1)\cdot m}T_{2l-1}(a)\right)\right), \qquad a = \sin mx$$



See Appendix B in arXiv:1409.5636

p.d.f. of single-particle Q-vectors

- Example: Distribution (blue) and theoretical p.d.f. (red) of imaginary part of single-particle Q-vector for the case of multichromatic flow



Characteristic functions

- Characteristic function $\phi_x(k)$ of random observable x is defined as the inverse Fourier transform of its p.d.f. $f(x)$
  - The characteristic function for a **sum** of independent random observables is given by the **product** of the individual characteristics functions
- Analytic solutions for characterisitc functions, both for real and imaginary parts of M-particle Q-vectors and for the most general case of multichromatic flow, are given in terms of **Bessel functions of the first kind:**

$$\phi_{\mathrm{Re}\,Q_m}(k) = \left[J_0(k)+2\sum_{p=1}^{\infty}(-1)^p\left[c_{2p\cdot m}J_{2p}(k)-ic_{(2p-1)\cdot m}J_{2p-1}(k)\right]\right]^M$$

$$\phi_{\mathrm{Im}\,Q_m}(k) = \left[J_0(k)+2\sum_{p=1}^{\infty}\left[c_{2p\cdot m}J_{2p}(k)+is_{(2p-1)\cdot m}J_{2p-1}(k)\right]\right]^M$$

Okay, so what?

- Well, once we have the characterisic functions we have all moments of underlying p.d.f. even without knowing explicitly the functional form of p.d.f.:

$$\mu'_{a,n} = i^{-n}\frac{d^n}{dk^n}\,\phi_a(k)\Big|_{k=0}$$

- Therefore, the problem of determining any expectation value of the type $\mathrm{E}[(\mathrm{Re}\,\mathrm{Q_m})^k]$ or $\mathrm{E}[(\mathrm{Im}\,\mathrm{Q_m})^k]$ is analytically solved now, for the most general case of multichromatic flow. First three moments:

$$\begin{aligned}
\mu'_{\mathrm{Re}\,Q_m,1} &= Mc_m\\
\mu'_{\mathrm{Im}\,Q_m,1} &= Ms_m\\
\mu'_{\mathrm{Re}\,Q_m,2} &= \frac{M}{2}\left[1+2(M-1)c_m^2+c_{2m}\right]\\
\mu'_{\mathrm{Im}\,Q_m,2} &= \frac{M}{2}\left[1+2(M-1)s_m^2-c_{2m}\right]\\
\mu'_{\mathrm{Re}\,Q_m,3} &= \frac{M}{4}\left[4(M-2)(M-1)c_m^3+c_m\left[-3+6M+6(M-1)c_{2m}\right]+c_{3m}\right]\\
\mu'_{\mathrm{Im}\,Q_m,3} &= \frac{M}{4}\left[\left[-3+6M-6(M-1)c_{2m}\right]s_m+4(M-2)(M-1)s_m^3-s_{3m}\right]
\end{aligned}$$

The key point

- The highly non-trivial consenquence of complete factorization of a joint multivariate p.d.f. is imprinted in the **strict functional forms** of previous moments, and their mutual relations
- On the other hand, the initial single particle distribution can be **always** described with a Fourier series, whether or not it is due to collective anisotropic flow
- Because of random fluctuations of impact parameter vector, which in an experiment can't be controlled, the previous moments are suitable only for the theorists
- Therefore, we are not done yet, our primary goal are higher order moments and p.d.f.'s of **isotropic** multiparticle correlations, which are not affected by random fluctuations of impact parameter vector
- Stay tuned!

Sensitive issues

Sensitivity (1/3)

- Typical question within heavy-ion collaborations: How many events we have to collect in order to supress statistical noise beyond n significant digits in our next data taking run?
- An easy, natural, important, unavoidable, etc. question to ask, but in fact it is very difficult to answer it in the context of multiparticle azimuthal correlations... Anyway, let's do what we can at the moment
- Formulation:

$$\frac{\text{statistical noise}}{\text{result}} < 10^{-n} \quad \Rightarrow \quad \frac{\sigma_{\langle 2\rangle}}{\mu_{\langle 2\rangle}\sqrt{N}} < 10^{-n}$$

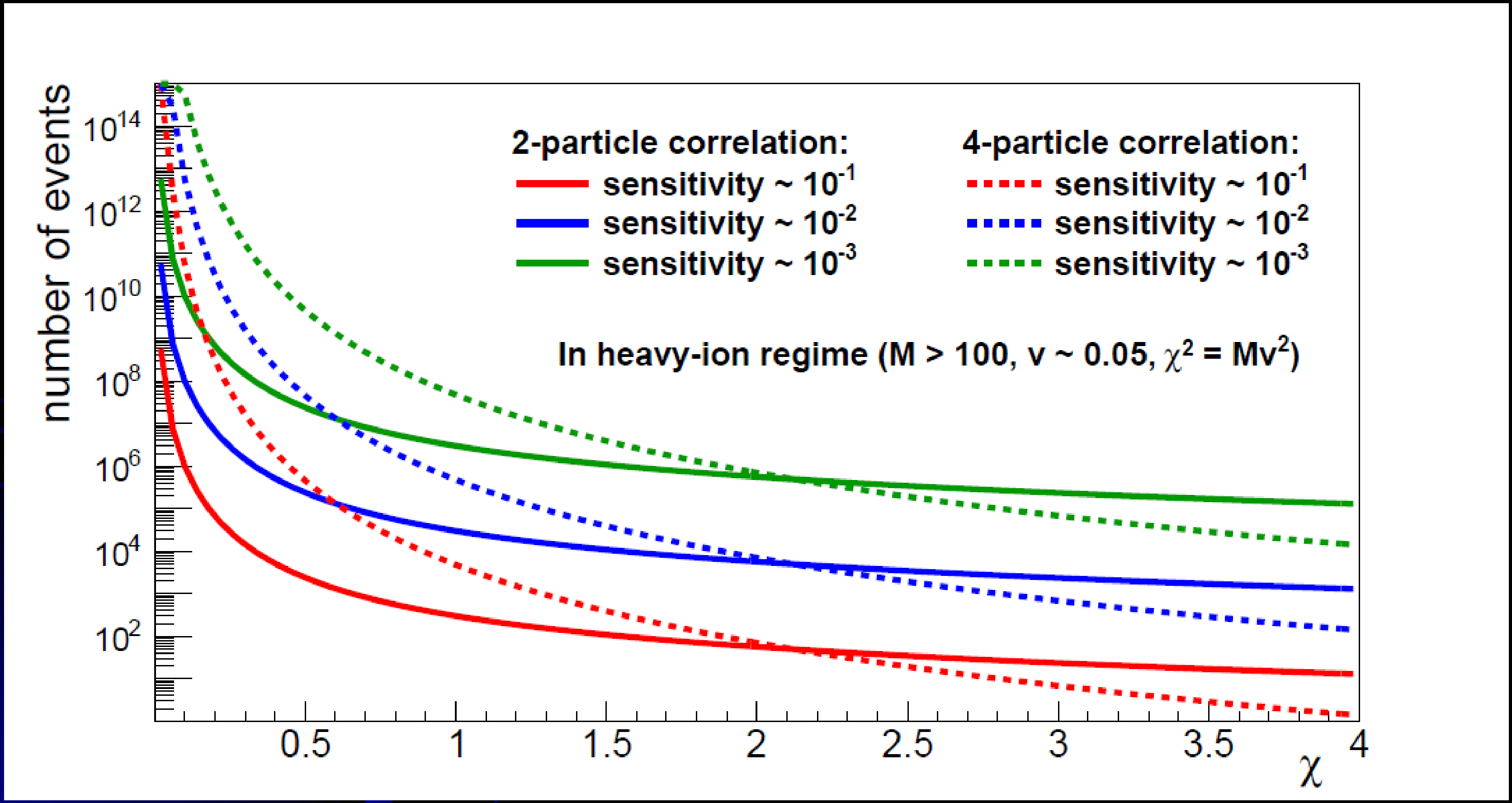
- Approximate results (accuracy better than 5%) for variances in 'heavy-ion regime', in terms of resolution parameter $\chi^2 = M v^2$:

$$\sigma^2_{\langle 2\rangle} \simeq \frac{1}{M^2}\left(1+2\chi^2\right)$$

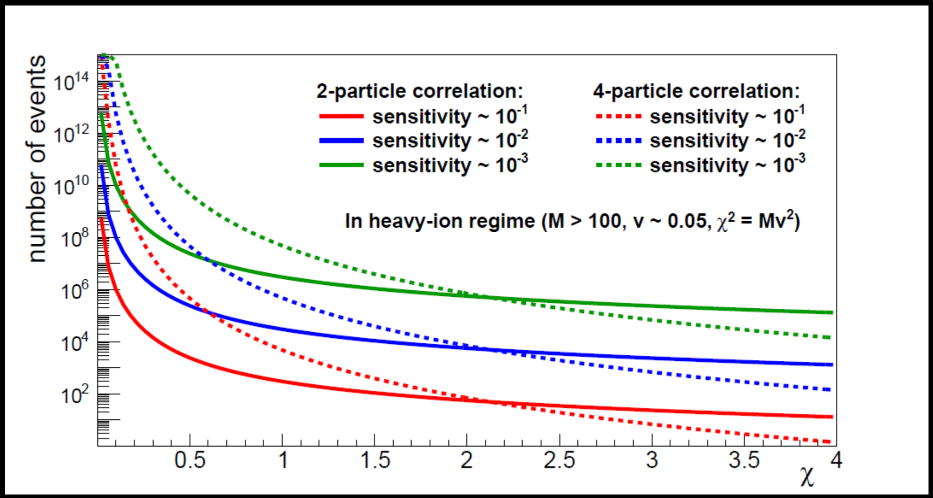
$$\sigma^2_{\langle 4\rangle} \simeq \frac{1}{M^4}\left(4+16\chi+20\chi^2+8\chi^3\right)$$

Sensitivity (2/3)

- **Warning:** Check carefully assumptions which led to this figure before spreading/selling it around



Sensitivity (3/3)



- Although derived under the specific assumptions of 'heavy-ion' regime, these results already show some interesting and non-trivial features

- Three key observations:
  - **Cross-over** at $\chi \sim 2$: With increased resolution the four-particle correlations become more sensitive than the two-particle correlations
  - **Flatteting** as χ increases: In their present form, it is impossible to use correlation tecniques to perform event-by-event flow measurements, some major conceptual breakthrough is required
  - **Rapid increase** at low χ

Thanks!