Iterative Schemes for Coupled Multiphysical Problems in Electrical Engineering

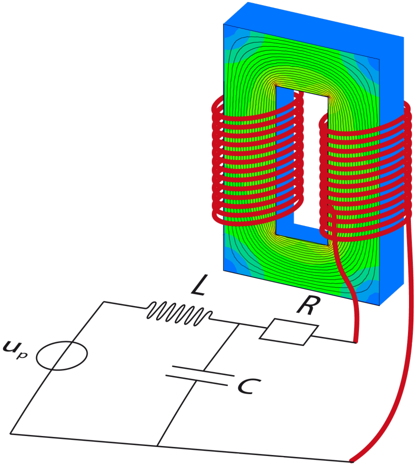
Sebastian Schöps[1,2]

[1]Institut für Theorie Elektromagnetischer Felder and
[2]Graduate School CE, TU Darmstadt

TECHNISCHE UNIVERSITÄT DARMSTADT

Outline of the Talk

TECHNISCHE UNIVERSITÄT DARMSTADT



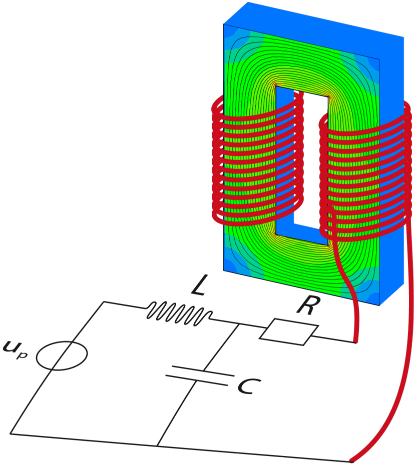
Outline of the Talk

1 Introduction
- Motivation
- Modular systems
- Pendulum
- DAE Index

2 Coupled systems in time domain

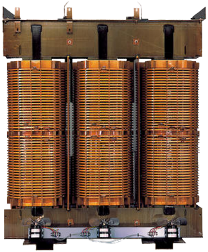
3 Low-Frequency field-circuit coupling

4 Conclusions

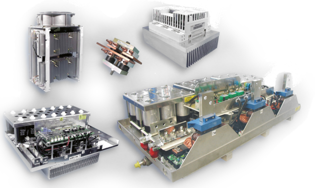


Motivation: Coupling

- **multiphysics: ’EM + X’**
  - electromagnetics
  - solid-body motion
  - thermodynamics
  - fluid dynamics
- **multiscale modeling: ’EM + EM’**
  - effects on multiple scales (m to nm)
  - elevated frequencies in some devices
  - focus is on particular device or domain
- **hybrid formulations**
  - formulation (e.g. variables)
  - discretization (e.g. FEM & FDM)
  - models (e.g. network vs. spatially resolved)



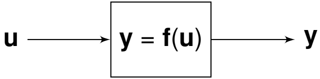
VPI dry type transformer, ABB



smart grid power semiconductors, Infineon

Coupled systems I

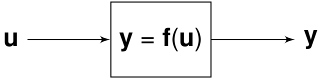
- Monolithic solution of a single system



with inputs u and e.g. two outputs $\mathbf{y} = [y_1, y_2]$

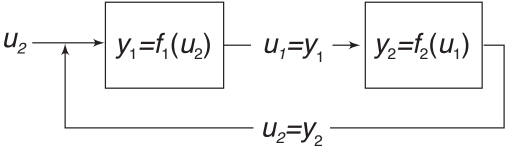
Coupled systems I

- Monolithic solution of a single system



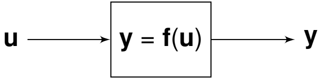
with inputs u and e.g. two outputs y = $[y_1, y_2]$

- Two coupled (sub-)systems



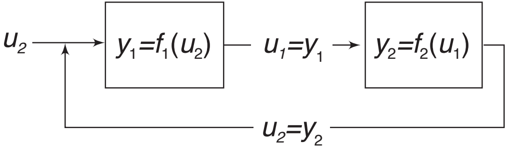
Coupled systems I

- Monolithic solution of a single system



with inputs u and e.g. two outputs y = $[y_1, y_2]$

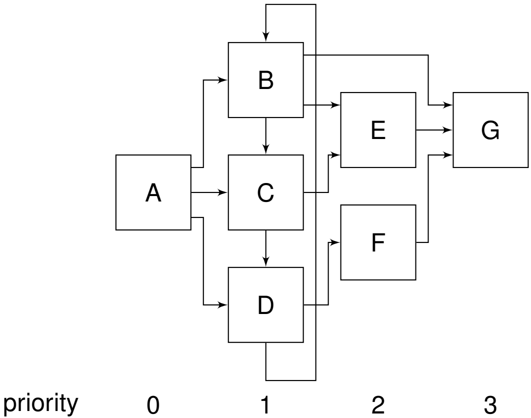
- Two coupled (sub-)systems



- Inputs and outputs can depedent on space and time, i.e.,

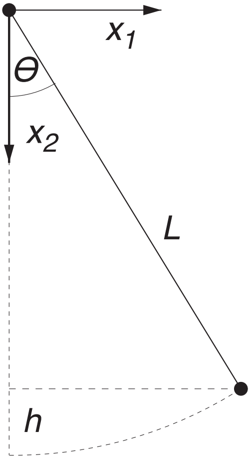
$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t) \qquad \text{and} \qquad \mathbf{y} = \mathbf{y}(\mathbf{r}, t)$$

Coupled systems II



J. Bastian, C. Clauß, S. Wolf, and P. Schneider. Master for co-simulation using FMI. In C. Clauß, editor, *Proceedings of the 8th International Modelica Conference*, Dresden, Germany, 2011.

Example: Pendulum as one Block



Each pendulum as Ordinary Differential Equation (ODE)

$$\ddot{\theta} = -\frac{q}{L} \cdot \sin(\theta) + \ldots$$

with coordinates

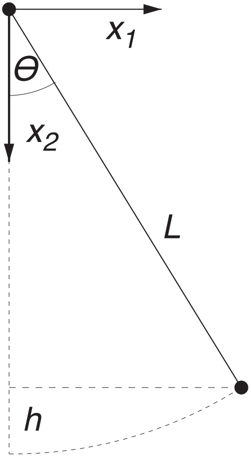
$$x_1 = L \cdot \cos(\theta)$$
$$x_2 = L \cdot \sin(\theta)$$

more general: Differential-Algebraic Equations (DAEs)

$$m \cdot \ddot{x}_1 + 2\lambda \cdot x_1 = \ldots,$$
$$m \cdot \ddot{x}_2 + 2\lambda \cdot x_2 = \ldots,$$
$$x_1^2 + x_2^2 = L^2$$

Example: Cartesian Pendulum in Modelica

TECHNISCHE UNIVERSITÄT DARMSTADT




```
model PendulumIndexXX "PendulumIndexXX.mo"

emph/*emph emphConstantsemph emphandemph emphParametersemph emph*/
constant Modelica.SIunits.Acceleration g=9.81;
parameter Modelica.SIunits.Length l(min=0)=1;
parameter Modelica.SIunits.Mass m=1;

emph/*emph emphVariablesemph emph*/
Modelica.SIunits.Position x1(start=0) ;
Modelica.SIunits.Position x2(start=-l) ;
Modelica.SIunits.Velocity v(start=1) ;
Modelica.SIunits.Velocity w(start=0) ;
Modelica.SIunits.Length lambda(start=(m*(1+l*g))/(2*l*l));

emph/*emph emphInputemph emphandemph emphOutputemph emphModelicaemph
      emphinterfacesemph emph*/
output Modelica.Blocks.Interfaces.RealOutput xValue;
output Modelica.Blocks.Interfaces.RealOutput yValue;

equation

emph/*emph emphRESTRICTIONemph emphEQUATIONemph emphHEREemph emph*/
der(x1) = v;
der(x2) = w;
m * der(v) = -2 * x1 * lambda;
m * der(w) = -m * g - 2 * x2 * lambda;
xValue = x1;
yValue = x2;
```

DAEs and the Index

- many problems are not characterized by ODEs but differenital algebraic equations (DAEs), i.e.,

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, \mathbf{z})$$

with an additional algebraic constraint

$$0 = \mathbf{g}(t, \mathbf{y}, \mathbf{z})$$

with unknown $\mathbf{x} = [\mathbf{y}, \mathbf{z}]$.

DAEs and the Index

- many problems are not characterized by ODEs but differenital algebraic equations (DAEs), i.e.,

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, \mathbf{z})$$

  with an additional algebraic constraint

$$0 = \mathbf{g}(t, \mathbf{y}, \mathbf{z})$$

  with unknown $\mathbf{x} = [\mathbf{y}, \mathbf{z}]$.

- the **DAE index** is the number of derivatives with respect to time that are needed to obtain an ODE from an DAE.

DAEs and the Index

TECHNISCHE UNIVERSITÄT DARMSTADT

- many problems are not characterized by ODEs but differenital algebraic equations (DAEs), i.e.,

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, \mathbf{z})$$

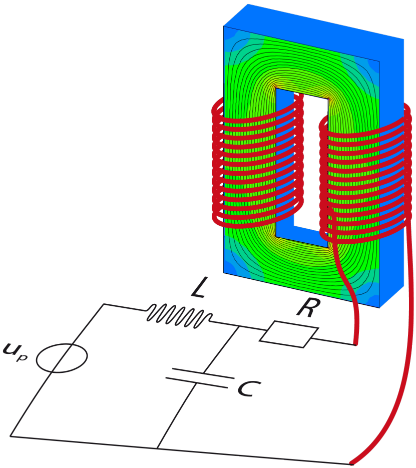
  with an additional algebraic constraint

$$0 = \mathbf{g}(t, \mathbf{y}, \mathbf{z})$$

  with unknown $\mathbf{x} = [\mathbf{y}, \mathbf{z}]$.

- the **DAE index** is the number of derivatives with respect to time that are needed to obtain an ODE from an DAE.
- high index problems are **difficult to solve** because errors are amplified (integration and differentiation).

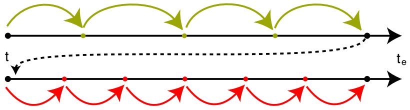
Outline of the Talk



Coupled systems in time domain

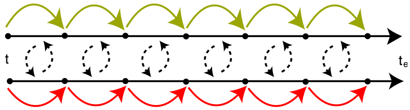
TECHNISCHE UNIVERSITÄT DARMSTADT

One-way Coupling



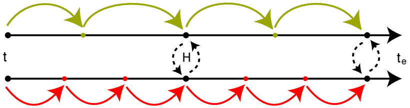
(parameter extraction)

Strong Coupling



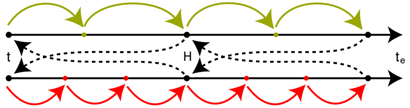
(synchronization in linear solver or multilevel Newton)

Weak Coupling



(requires synchronization points)

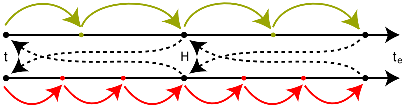
Dynamic iteration / waveform relaxation



(requires synchronization points, inter and extrapolation!)

Coupled systems in time domain

Dynamic iteration / waveform relaxation



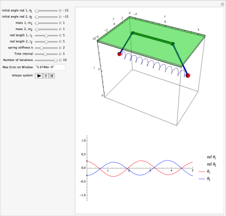
(requires synchronization points, inter and extrapolation!)

Simple Example: Double Pendulum (ODEs)

System of 2nd order Ordinary Differential Equations:

$$\ddot{\theta}_1 = \frac{k l_2 \sin(\theta_2) - k l_1 \sin(\theta_1) + l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) + g m_1 \tan(\theta_1)}{l_1 m_1 \cos(\theta_1)}$$

$$\ddot{\theta}_2 = \frac{k l_1 \sin(\theta_1) + l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2) - \tan(\theta_2)\left(g m_2 + k l_2 \cos(\theta_2)\right)}{l_2 m_2 \cos(\theta_2)}$$



- **unknowns**: angles $\theta_\star$
- **parameters**: masses $m_\star$, length $l_\star$ and gravitation g.
- **initial values**: $\theta_\star$ and $\dot{\theta}_\star$

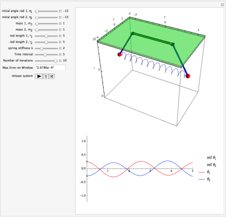
Simple Example: Double Pendulum (ODEs)

TECHNISCHE UNIVERSITÄT DARMSTADT

System of 2nd order Ordinary Differential Equations:

$$\ddot{\theta}_1{}^{(k)} = \frac{k l_2 \sin\left(\theta_2{}^{(k-1)}(t)\right) - k l_1 \sin(\theta_1) + l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) + g m_1 \tan(\theta_1)}{l_1 m_1 \cos(\theta_1)}$$

$$\ddot{\theta}_2{}^{(k)} = \frac{k l_1 \sin\left(\theta_1{}^{(k)}(t)\right) + l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2) - \tan(\theta_2)\left(g m_2 + k l_2 \cos(\theta_2)\right)}{l_2 m_2 \cos(\theta_2)}$$



- **unknowns**: angles $\theta_\star$
- **parameters**: masses $m_\star$, length $l_\star$ and gravitation g.
- **initial values**: $\theta_\star$ and $\dot{\theta}_\star$
- **dynamic iteration:** Gauss-Seidel scheme with $k = 1, ..., N$ sweeps for a single time window

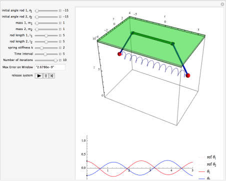
Simple Example: Double Pendulum (ODEs)

TECHNISCHE UNIVERSITÄT DARMSTADT

System of 2nd order Ordinary Differential Equations:

$$\ddot{\theta}_1 = \frac{kl_2 \sin(\theta_2) - kl_1 \sin(\theta_1) + l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) + gm_1 \tan(\theta_1)}{l_1 m_1 \cos(\theta_1)}$$

$$\ddot{\theta}_2 = \frac{kl_1 \sin(\theta_1) + l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2) - \tan(\theta_2)\left(gm_2 + kl_2 \cos(\theta_2)\right)}{l_2 m_2 \cos(\theta_2)}$$



- **unknowns**: angles $\theta_\star$
- **parameters**: masses $m_\star$, length $l_\star$ and gravitation g.
- **initial values**: $\theta_\star$ and $\dot{\theta}_\star$
- **dynamic iteration:** Gauss-Seidel

Convergence

Super-linear convergence for ODEs (bounded right-hand-sides f).

Waveform Relaxation: Playing with the loops

TECHNISCHE UNIVERSITÄT DARMSTADT

possible application to CUDI (Arjan Verweij's talk):

- Loop for parametric sweep
  - Iterate with different h_{ext} to calculate the Minimum Quench Energy
    - Loop through all time steps
      - ~~Iterate until the time step is small enough~~
        - Iterate until the electrical module has converged
          - Iterate until the thermal module has converged
            - Loop over all the bands of the cable
              - Loop over all the strand sections of a band
                - Iterate until the temperatures of the section and the adjacent helium have converged

each problem gets it's own time step

move time Loop here

decouple

Dynamic Iteration: Fixed Point Analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

Index-1 systems

$$\dot{\mathbf{y}}_i = \mathbf{f}_i(\mathbf{y}, \mathbf{z}), \quad 0 = \mathbf{g}_i(\mathbf{y}, \mathbf{z}) \text{ with } \det \partial_{\mathbf{z}_i} \mathbf{g}_i \neq 0, \qquad i = 1, \dots, n$$

Dynamic Iteration: Fixed Point Analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

Index-1 systems

$$\dot{\mathbf{y}}_i^{(k+1)} = \mathbf{F}_i(\mathbf{y}^{(k+1)}, \mathbf{z}^{(k+1)}, \mathbf{y}^{(k)}, \mathbf{z}^{(k)}), \quad 0 = \mathbf{G}_i(\mathbf{y}^{(k+1)}, \mathbf{z}^{(k+1)}, \mathbf{y}^{(k)}, \mathbf{z}^{(k)})$$

Errors

- differential splitting error $\delta_y^{(k)}$
- algebraic splitting error $\delta_z^{(k)}$

Dynamic Iteration: Fixed Point Analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

Index-1 systems

$$\dot{\mathbf{y}}_i^{(k+1)} = \mathbf{F}_i(\mathbf{y}^{(k+1)}, \mathbf{z}^{(k+1)}, \mathbf{y}^{(k)}, \mathbf{z}^{(k)}), \quad 0 = \mathbf{G}_i(\mathbf{y}^{(k+1)}, \mathbf{z}^{(k+1)}, \mathbf{y}^{(k)}, \mathbf{z}^{(k)})$$

Errors

- differential splitting error $\delta_y^{(k)}$
- algebraic splitting error $\delta_z^{(k)}$

Fixed point iteration

$$\begin{bmatrix} \delta_y^{(k+1)} \\ \delta_z^{(k+1)} \end{bmatrix} \leq \begin{bmatrix} \mathcal{O}(H) & \mathcal{O}(H) \\ \beta + \mathcal{O}(H) & \alpha + \mathcal{O}(H) \end{bmatrix} \begin{bmatrix} \delta_y^{(k)} \\ \delta_z^{(k)} \end{bmatrix} + \text{"error in initial values"}$$

Dynamic Iteration: Fixed Point Analysis

TECHNISCHE UNIVERSITÄT DARMSTADT

Index-1 systems

$$\dot{\mathbf{y}}_i^{(k+1)} = \mathbf{F}_i(\mathbf{y}^{(k+1)}, \mathbf{z}^{(k+1)}, \mathbf{y}^{(k)}, \mathbf{z}^{(k)}), \quad 0 = \mathbf{G}_i(\mathbf{y}^{(k+1)}, \mathbf{z}^{(k+1)}, \mathbf{y}^{(k)}, \mathbf{z}^{(k)})$$

Errors

- differential splitting error $\delta_y^{(k)}$
- algebraic splitting error $\delta_z^{(k)}$

Fixed point iteration

$$\begin{bmatrix} \delta_y^{(k+1)} \\ \delta_z^{(k+1)} \end{bmatrix} \leq \begin{bmatrix} \mathcal{O}(H) & \mathcal{O}(H) \\ \beta + \mathcal{O}(H) & \alpha + \mathcal{O}(H) \end{bmatrix} \begin{bmatrix} \delta_y^{(k)} \\ \delta_z^{(k)} \end{bmatrix} + \text{"error in initial values"}$$

Theorem (Contraction)

The iteration converges if the coupling between old and new algebraic variables is weak: $\alpha = \left\| \left(\frac{\partial \mathbf{G}}{\partial \mathbf{z}^{(k+1)}} \right)^{-1} \frac{\partial \mathbf{G}}{\partial \mathbf{z}^{(k)}} \right\| < 1$ *because „convergence rate" = „spectral radius"* $r < 1$

Optimal Time Discretization Order

TECHNISCHE UNIVERSITÄT DARMSTADT

The splitting error on the *i*-th time window is

$$\delta^k_{\mathbf{y},i} = \mathcal{O}(H^{pk})$$

where p is:

(a) for a coupled system of 2 DAEs with only differential coupling $p = 2$,
(b) for a DAE with no algebraic-to-algebraic coupling $p = 1$,
(c) for a general DAE $p = 0$.

Thus the *optimal order* for the numerical time integration is:

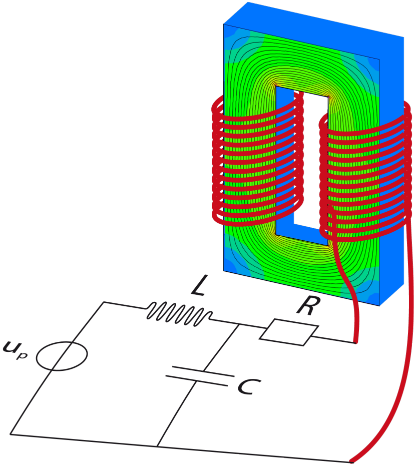
$$q \geq p \cdot k.$$

with the total error (splitting + time discretization)

$$\left\|\tilde{\mathbf{x}}^{(k)} - \mathbf{x}\right\| = \mathcal{O}(H^{pk}) + \mathcal{O}(m \cdot h^q) = \mathcal{O}(H^{pk}).$$

where $m < \infty$ is the number of (local) time steps.

Outline of the Talk

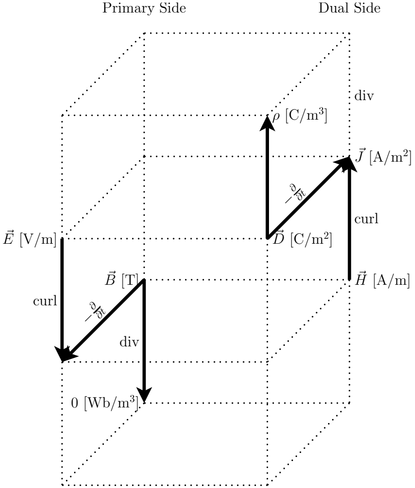


Maxwell's Equations

TECHNISCHE UNIVERSITÄT DARMSTADT

- differential formulation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,,$$
$$\nabla \cdot \mathbf{B} = 0 \,,$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \,,$$
$$\nabla \cdot \mathbf{D} = \rho \,.$$



[F. Freschi, M. Repetto, SCEE2010]

Maxwell's Equations

TECHNISCHE UNIVERSITÄT DARMSTADT

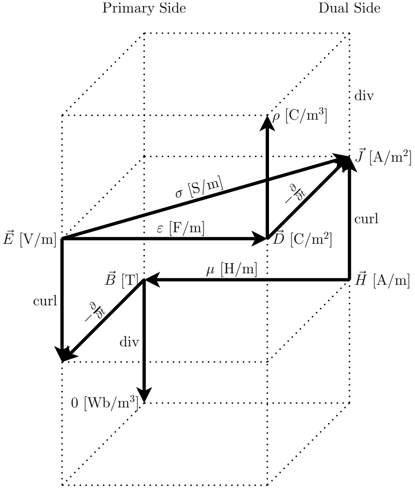
- differential formulation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\ ,$$
$$\nabla \cdot \mathbf{B} = 0\ ,$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\ ,$$
$$\nabla \cdot \mathbf{D} = \rho\ .$$

- with material laws

$$\mathbf{J} = \sigma \mathbf{E}\ , \quad \mathbf{D} = \varepsilon \mathbf{E}\ , \quad \mathbf{H} = \mu^{-1} \mathbf{B}$$

where $\mu^{-1} = \nu(|\mathbf{B}|)$ is nonlinear.



[F. Freschi, M. Repetto, SCEE2010]

Maxwell's Equations

TECHNISCHE UNIVERSITÄT DARMSTADT

- differential formulation

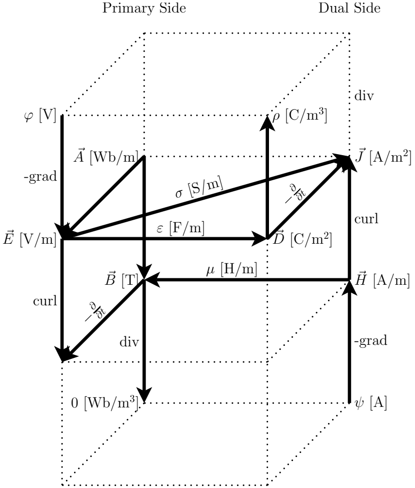
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,,$$
$$\nabla \cdot \mathbf{B} = 0\,,$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\,,$$
$$\nabla \cdot \mathbf{D} = \rho\,.$$

- with material laws

$$\mathbf{J} = \sigma \mathbf{E}\,, \quad \mathbf{D} = \varepsilon \mathbf{E}\,, \quad \mathbf{H} = \mu^{-1} \mathbf{B}$$

where $\mu^{-1} = \nu(|\mathbf{B}|)$ is nonlinear.

- introduction of potentials, e.g. magnetic vector potential $\mathbf{B} = \nabla \times \mathbf{A}$



[F. Freschi, M. Repetto, SCEE2010]

Maxwell's Equations

TECHNISCHE UNIVERSITÄT DARMSTADT

- differential formulation

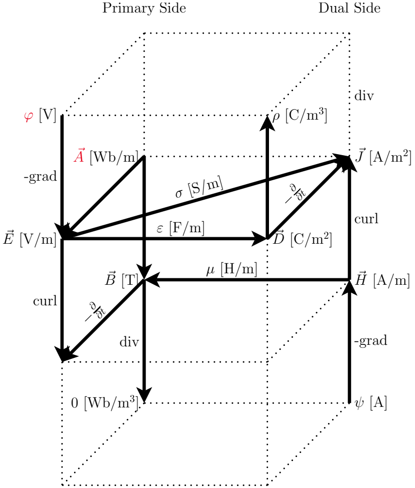
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,,$$
$$\nabla \cdot \mathbf{B} = 0 \,,$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \,,$$
$$\nabla \cdot \mathbf{D} = \rho \,.$$

- with material laws

$$\mathbf{J} = \sigma \mathbf{E} \,, \quad \mathbf{D} = \varepsilon \mathbf{E} \,, \quad \mathbf{H} = \mu^{-1} \mathbf{B}$$

where $\mu^{-1} = \nu(|\mathbf{B}|)$ is nonlinear.

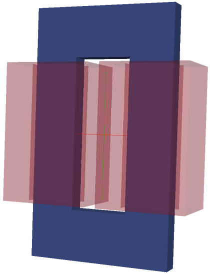
- introduction of potentials, e.g. magnetic vector potential $\mathbf{B} = \nabla \times \mathbf{A}$



[F. Freschi, M. Repetto, SCEE2010]

Maxwell: Magnetoquasistatic Approximation

TECHNISCHE UNIVERSITÄT DARMSTADT



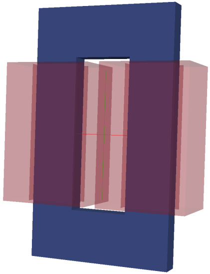
- reformulate Maxwell in terms of **magnetic vector potential A** with $\mathbf{B} = \nabla \times \mathbf{A}$

- **hyperbolic curl-curl equation on Ω**

$$\varepsilon \frac{\partial^2}{\partial t^2}\mathbf{A} + \sigma \frac{\partial}{\partial t}\mathbf{A} + \nabla \times \left(\nu \left(|\nabla \times \mathbf{A}| \right) \nabla \times \mathbf{A} \right) = -\sigma \nabla \varphi \ - \ \varepsilon \frac{\partial}{\partial t} \nabla \varphi$$

Maxwell: Magnetoquasistatic Approximation

TECHNISCHE UNIVERSITÄT DARMSTADT



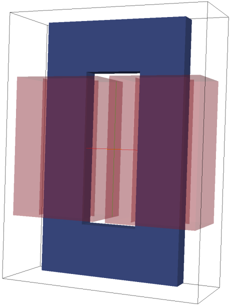
- reformulate Maxwell in terms of **magnetic vector potential A** with $\mathbf{B} = \nabla \times \mathbf{A}$
- low frequency: $\max \left|\frac{\partial}{\partial t}\mathbf{D}\right| \ll \max |\mathbf{J}|$
  $\rightarrow$ disregard displacement currents

- **parabolic / elliptic curl-curl equation on Ω**

$$\varepsilon \frac{\partial^2}{\partial t^2}\mathbf{A} + \sigma \frac{\partial}{\partial t}\mathbf{A} + \nabla \times \left(\nu\left(|\nabla \times \mathbf{A}|\right) \nabla \times \mathbf{A}\right) = -\sigma \nabla \varphi - \varepsilon \frac{\partial}{\partial t}\nabla\varphi$$

Maxwell: Magnetoquasistatic Approximation

TECHNISCHE UNIVERSITÄT DARMSTADT



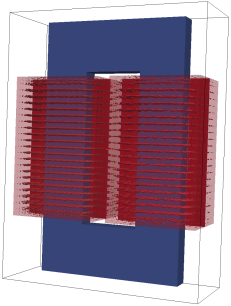
- reformulate Maxwell in terms of **magnetic vector potential A** with $\mathbf{B} = \nabla \times \mathbf{A}$
- low frequency: $\max |\frac{\partial}{\partial t}\mathbf{D}| \ll \max |\mathbf{J}|$
  $\rightarrow$ disregard displacement currents
- homogeneous **Dirichlet** BC on Γ

- **parabolic / elliptic curl-curl equation on Ω**

$$\sigma \frac{\partial}{\partial t}\mathbf{A} + \nabla\times\left(\nu\left(|\nabla\times\mathbf{A}|\right)\nabla\times\mathbf{A}\right) = -\sigma\nabla\varphi \qquad \mathbf{A}_t|_\Gamma = 0$$

Maxwell: Magnetoquasistatic Approximation

TECHNISCHE UNIVERSITÄT DARMSTADT

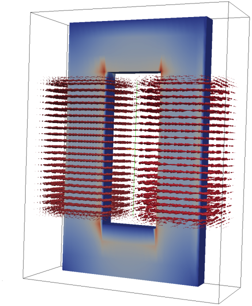


- reformulate Maxwell in terms of **magnetic vector potential A** with $\mathbf{B} = \nabla \times \mathbf{A}$
- low frequency: $\max |\frac{\partial}{\partial t}\mathbf{D}| \ll \max |\mathbf{J}|$
  $\rightarrow$ disregard displacement currents
- homogeneous **Dirichlet** BC on Γ
- **excitation** by spatially distributed current density $\mathbf{J}_{\mathrm{src}} := -\sigma\nabla\varphi$

- **parabolic / elliptic curl-curl equation on Ω**

$$\sigma\frac{\partial}{\partial t}\mathbf{A} + \nabla\times\left(\nu\left(|\nabla\times\mathbf{A}|\right)\nabla\times\mathbf{A}\right) = \mathbf{J}_{\mathrm{src}} \qquad \mathbf{A}_t|_{\Gamma} = 0$$

Maxwell: Magnetoquasistatic Approximation



- reformulate Maxwell in terms of **magnetic vector potential A** with $\mathbf{B} = \nabla \times \mathbf{A}$
- low frequency: $\max |\frac{\partial}{\partial t}\mathbf{D}| \ll \max |\mathbf{J}|$
  $\rightarrow$ disregard displacement currents
- homogeneous **Dirichlet** BC on Γ
- **excitation** by spatially distributed current density $\mathbf{J}_{\text{src}} := -\sigma\nabla\varphi$
- exhibits **eddy currents** where $\sigma > 0$

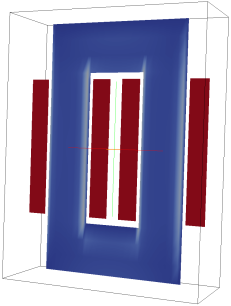
- **parabolic / elliptic curl-curl equation on Ω**

$$\sigma\frac{\partial}{\partial t}\mathbf{A} + \nabla\times\left(\nu\left(|\nabla \times \mathbf{A}|\right)\nabla \times \mathbf{A}\right) = \mathbf{J}_{\text{src}} \qquad \mathbf{A}_t|_{\Gamma} = 0$$

the system is elliptic in subdomains where $\sigma = 0$

Maxwell: Magnetoquasistatic Approximation

TECHNISCHE UNIVERSITÄT DARMSTADT



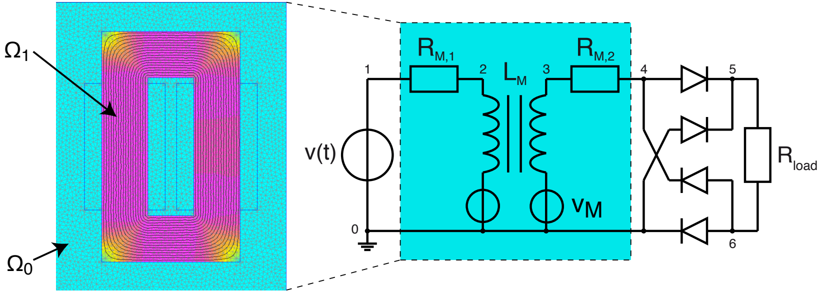
- reformulate Maxwell in terms of **magnetic vector potential A** with $\mathbf{B} = \nabla \times \mathbf{A}$
- low frequency: $\max |\frac{\partial}{\partial t}\mathbf{D}| \ll \max |\mathbf{J}|$
  $\rightarrow$ disregard displacement currents
- homogeneous **Dirichlet** BC on Γ
- **excitation** by spatially distributed current density $\mathbf{J}_{\mathrm{src}} := -\sigma\nabla\varphi$
- exhibits **eddy currents** where $\sigma > 0$
- 2D reduction: $\mathbf{A} = [0, 0, u(x_1, x_2)]^\top$

- **parabolic / elliptic curl-curl equation on Ω**

$$\sigma \frac{\partial}{\partial t} u - \nabla\cdot\left(\nu\left(|\nabla u|\right)\nabla u\right) = J_z \qquad u|_\Gamma = 0$$

the system is elliptic in subdomains where $\sigma = 0$

Field-Circuit Coupling



(a) 2D transformer model (b) rectifier circuit with embeded PDE model

Weak formulation: find $u(\mathbf{x}) \in H_0^1(\Omega)$ with $u_i = u|_{\Omega_i}$ such that $\forall v \in H_0^1(\Omega)$

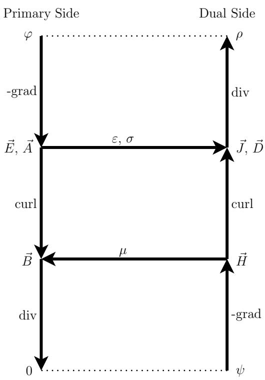
$$\int_{\Omega_1} \left(\sigma_1 \frac{\partial u_1}{\partial t} v + \nu_1 \nabla u_1 \cdot \nabla v \right) \mathrm{d}\mathbf{x} + \int_{\Omega_0} \nu_0 \nabla u_0 \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_{\Omega_0} \chi^\top \mathbf{i}_\mathrm{M} v \, \mathrm{d}\mathbf{x},$$

Interpretation of Coupling Interface

[De Gersem, S., 2013]

- Interface equation in matrix-form

$$\mathbf{L}_\mathrm{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{i}_\mathrm{M} + \mathbf{R}_\mathrm{M}\mathbf{i}_\mathrm{M} + \mathbf{v}_\mathrm{M}(u_1) = \mathbf{v}_\mathrm{M}$$



Introduction to electric circuits from differenial forms perspective

P. Bamberg and S. Sternberg, *A course in mathematics for students of physics*, Volume 2.

Interpretation of Coupling Interface

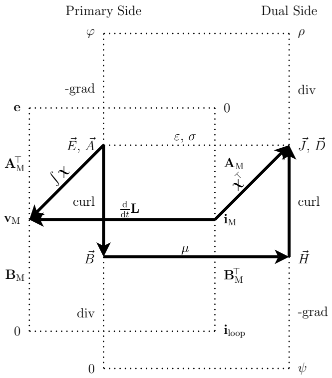
[De Gersem, S., 2013]

- Interface equation in matrix-form

$$\mathbf{L}_\mathrm{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{i}_\mathrm{M} + \mathbf{R}_\mathrm{M}\mathbf{i}_\mathrm{M} + \mathbf{v}_\mathrm{M}(u_1) = \mathbf{v}_\mathrm{M}$$

- with constant inductance (s.p.d.)

$$\mathbf{L}_\mathrm{M} := \int_{\Omega_0} \chi\,(\nabla\cdot\nu\nabla)^{-1}\,\chi\ \mathrm{d}\mathbf{x}$$



Interpretation of Coupling Interface

[De Gersem, S., 2013]

- Interface equation in matrix-form

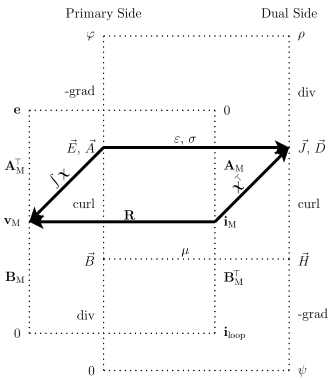
$$\mathbf{L}_\mathrm{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{i}_\mathrm{M} + \mathbf{R}_\mathrm{M}\mathbf{i}_\mathrm{M} + \mathbf{v}_\mathrm{M}(u_1) = \mathbf{v}_\mathrm{M}$$

- with constant inductance (s.p.d.)

$$\mathbf{L}_\mathrm{M} := \int_{\Omega_0} \chi\,(\nabla\cdot\nu\nabla)^{-1}\chi\;\;\mathrm{d}\mathbf{x}$$

- constant resistance (diag.)

$$\mathbf{R}_\mathrm{M} := \int_{\Omega_0} \chi\sigma^{-1}\chi^\top\;\mathrm{d}\mathbf{x},$$



Interpretation of Coupling Interface

TECHNISCHE UNIVERSITÄT DARMSTADT

- Interface equation in matrix-form

$$\mathbf{L}_\mathrm{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{i}_\mathrm{M} + \mathbf{R}_\mathrm{M}\mathbf{i}_\mathrm{M} + \mathbf{v}_\mathrm{M}(u_1) = \mathbf{v}_\mathrm{M}$$

- with constant inductance (s.p.d.)

$$\mathbf{L}_\mathrm{M} := \int_{\Omega_0} \chi\,(\nabla\cdot\nu\nabla)^{-1}\chi\;\mathrm{d}\mathbf{x}$$

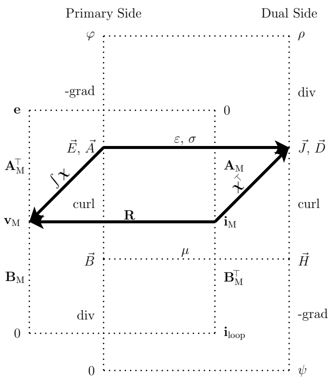
- constant resistance (diag.)

$$\mathbf{R}_\mathrm{M} := \int_{\Omega_0} \chi\sigma^{-1}\chi^\top\;\mathrm{d}\mathbf{x},$$

- and voltage source

$$\mathbf{v}_\mathrm{M}(u_1) := \int_{\Omega_0} \chi\,(\nabla\cdot\nu_0\nabla)^{-1}\left(-\sigma^{-1}\nabla\cdot(\nu_1(|\nabla u_1|)\nabla u_1)\right)\mathrm{d}\mathbf{x}.$$

[De Gersem, S., 2013]

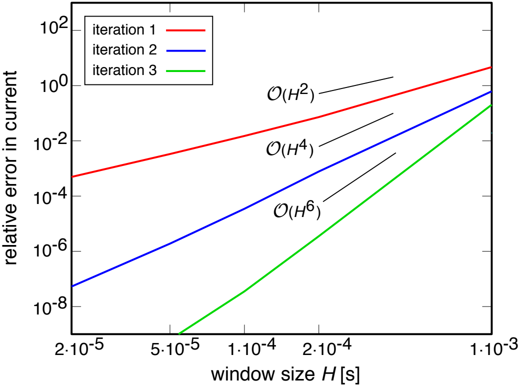


Numerical Example – Rectifier

TECHNISCHE UNIVERSITÄT DARMSTADT

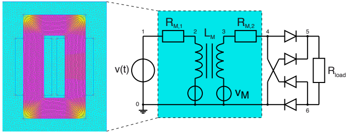
Global Convergence rate $\mathcal{O}(H^2)$ can be achieved:

[Bartel, Brunk, Günther, S., 2013]



reference: monolithic solution with very high accuracy (RADAU5)

Example:



(a) 2D transformer model (b) rectifier circuit with embeded PDE model

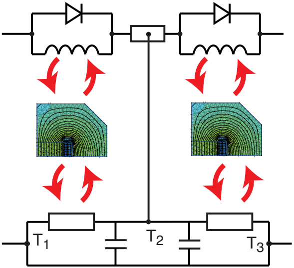
Parameter:

- window size H
- sweeps ($k = 1, 2, 3$)
- time integrator RADAU5
- interpolation: splines

Conclusions

TECHNISCHE UNIVERSITÄT DARMSTADT

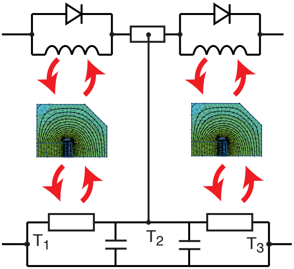
- dynamic iteration is the justification of many ad-hoc approaches
- Iterative coupling schemes ensure stability and are essential for high accuracy ($\rightarrow$ splitting error)
- You can play with the time stepping loop (sometimes it can be skipped...)
- Implementation effort of iterative coupling is only one additional loop
- Convergence cannot be taken for granted
- Mathematical verification by fixed point analysis
- Clever modeling can overcome divergence (similar to sophisticated interface conditions in Domain Decomposition)



Conclusions

TECHNISCHE UNIVERSITÄT DARMSTADT

- dynamic iteration is the justification of many ad-hoc approaches
- Iterative coupling schemes ensure stability and are essential for high accuracy ($\rightarrow$ splitting error)
- You can play with the time stepping loop (sometimes it can be skipped...)
- Implementation effort of iterative coupling is only one additional loop
- Convergence cannot be taken for granted
- Mathematical verification by fixed point analysis



Final question

When to update which model?

Example: Index Reduction for Pendulum

TECHNISCHE UNIVERSITÄT DARMSTADT

[any DAE textbook, Mattsson 1993]

„Physical" model

DAE index 3

$$
\begin{aligned}
x_1^2 + x_2^2 &= l^2 \\
m\ddot{x}_1 &= -(\lambda/L)x_1 \\
m\ddot{x}_2 &= -(\lambda/L)x_2\breve{\ }mg
\end{aligned}
$$

- g force of gravity, λ is a Lagrange multiplier; $(\lambda/L)x_i$ represent the force which holds the solution onto the constraint

Example: Index Reduction for Pendulum

TECHNISCHE UNIVERSITÄT DARMSTADT

[any DAE textbook, Mattsson 1993]

„Physical" model

DAE index 3

$$
\begin{aligned}
x_1^2 + x_2^2 &= l^2 \\
m\dot{x}_3 &= -(\lambda/L)x_1 \\
m\dot{x}_4 &= -(\lambda/L)x_2 \breve{\ } mg \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned}
$$

- g force of gravity, λ is a Lagrange multiplier; $(\lambda/L)x_i$ represent the force which holds the solution onto the constraint

Example: Index Reduction for Pendulum

[any DAE textbook, Mattsson 1993]

„Physical" model

DAE index 3

$$
\begin{aligned}
x_1^2 + x_2^2 &= l^2 \\
m\dot{x}_3 &= -(\lambda/L)x_1 \\
m\dot{x}_4 &= -(\lambda/L)x_2\breve{\ }mg \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned}
$$

derviative w.r.t. time

Manual reduction

DAE index 2

$$
\begin{aligned}
2x_1 \cdot x_3 &= -2x_2 \cdot x_4 \\
m\dot{x}_3 &= -(\lambda/L)x_1 \\
m\dot{x}_4 &= -(\lambda/L)x_2\breve{\ }mg \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned}
$$

- g force of gravity, λ is a Lagrange multiplier; $(\lambda/L)x_i$ represent the force which holds the solution onto the constraint

Example: Index Reduction for Pendulum

TECHNISCHE UNIVERSITÄT DARMSTADT

[any DAE textbook, Mattsson 1993]

„Physical" model
DAE index 3

$$
\begin{aligned}
x_1^2 + x_2^2 &= l^2 \\
m\dot{x}_3 &= -(\lambda/L)x_1 \\
m\dot{x}_4 &= -(\lambda/L)x_2 \breve{\ } mg \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned}
$$

Manual reduction
DAE index 2

$$
\begin{aligned}
2x_1 \cdot x_3 &= -2x_2 \cdot x_4 \\
m\dot{x}_3 &= -(\lambda/L)x_1 \\
m\dot{x}_4 &= -(\lambda/L)x_2 \breve{\ } mg \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned}
$$

Manual reduction
ODE / DAE index 0

$$
\begin{aligned}
\dot{\lambda} &= -3(mg/L)x_4 \\
m\dot{x}_3 &= -(\lambda/L)x_1 \\
m\dot{x}_4 &= -(\lambda/L)x_2 - mg \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned}
$$

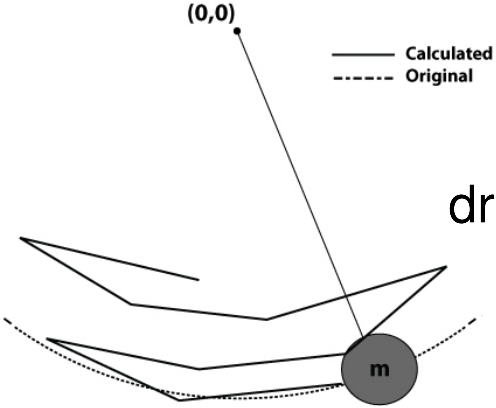
derviative w.r.t. time

2 derviatives w.r.t. time

- g force of gravity, λ is a Lagrange multiplier; $(\lambda/L)x_i$ represent the force which holds the solution onto the constraint

Example: Pendulum (Drift-Off)

[any DAE textbook, Mattsson 1993]



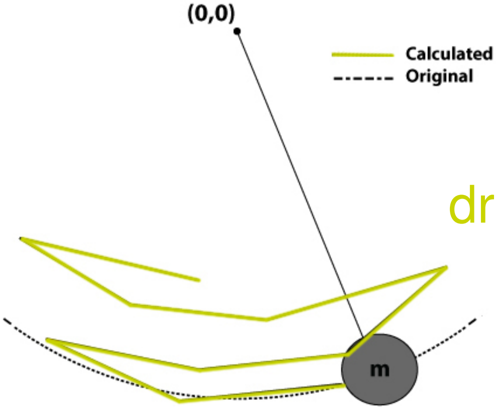
drift-off if constraint

$$x_1^2 + x_2^2 = L^2$$

is not enforced

Example: Pendulum (Drift-Off)

[any DAE textbook, Mattsson 1993]



drift-off if constraint

$$x_1^2 + x_2^2 = L^2$$

is not enforced

Formulation as Dynamic Iteration

TECHNISCHE UNIVERSITÄT DARMSTADT

Find magnetic vector potential **A** *and temperature* T $\qquad \forall w \in H_0(curl;\Omega)$

$$\int_\Omega \left(\left(\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} + \sigma(T)\frac{\partial \mathbf{A}}{\partial t}\right) \cdot \mathbf{w} + \nu \nabla \times \mathbf{A} \cdot \nabla \times \mathbf{w}\right) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{J}_{\mathrm{src}} \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} = 0$$

$\forall v \in H_0^1(\Omega)$

$$\int_\Omega \left(\rho c \frac{\partial T}{\partial t} v - k \nabla T \cdot \nabla v\right) \mathrm{d}\mathbf{x} - \int_\Omega Q\left(T, \frac{\partial \mathbf{A}}{\partial t}\right) v\, \mathrm{d}\mathbf{x} = 0$$

Formulation as Dynamic Iteration

TECHNISCHE UNIVERSITÄT DARMSTADT

Find magnetic vector potential **A** *and temperature* T $\forall w \in H_0(curl;\Omega)$

$$\int_\Omega \left(\left(\epsilon\frac{\partial^2 \mathbf{A}^{(k)}}{\partial t^2} + \sigma(T^{(k-1)})\frac{\partial \mathbf{A}^{(k)}}{\partial t}\right)\cdot \mathbf{w} + \nu\nabla \times \mathbf{A}^{(k)} \cdot \nabla \times \mathbf{w}\right) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{J}_{\mathrm{src}} \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} = 0$$

$\forall v \in H_0^1(\Omega)$

$$\int_\Omega \left(\rho c\frac{\partial T^{(k)}}{\partial t} v - k\nabla T^{(k)} \cdot \nabla v\right) \mathrm{d}\mathbf{x} - \int_\Omega Q\left(T^{(k-1)}, \frac{\partial \mathbf{A}^{(k)}}{\partial t}\right) v\, \mathrm{d}\mathbf{x} = 0$$

- apply dynamic iteration scheme

Formulation as Dynamic Iteration

TECHNISCHE UNIVERSITÄT DARMSTADT

Find magnetic vector potential **A** *and temperature* T $\forall w \in H_0(curl;\Omega)$

$$\int_\Omega \left(\left(\epsilon \frac{\partial^2 \mathbf{A}^{(k)}}{\partial t^2} + \sigma^{(k-1)} \frac{\partial \mathbf{A}^{(k)}}{\partial t}\right) \cdot \mathbf{w} + \nu \nabla \times \mathbf{A}^{(k)} \cdot \nabla \times \mathbf{w}\right) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{J}_{\mathrm{src}} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = 0$$

$\forall v \in H_0^1(\Omega)$

$$\int_\Omega \left(\rho c \frac{\partial T^{(k)}}{\partial t} v - k \nabla T^{(k)} \cdot \nabla v\right) \mathrm{d}\mathbf{x} - \int_\Omega Q\left(T^{(k-1)}, \frac{\partial \mathbf{A}^{(k)}}{\partial t}\right) v \, \mathrm{d}\mathbf{x} = 0$$

- apply dynamic iteration scheme
- average conductivity $\bar{\sigma}^{(k-1)} := \frac{1}{T} \int_T \sigma\left(T^{(k-1)}(t)\right) \mathrm{d}t$ for each sweep k

Formulation as Dynamic Iteration

TECHNISCHE UNIVERSITÄT DARMSTADT

Find magnetic vector potential $\mathbf{A}$ *and temperature* T $\qquad \forall w \in H_0(curl;\Omega)$

$$\int_\Omega \left(\left(\omega^2 \epsilon \frac{\partial^2 \mathbf{A}^{(k)}}{\partial t^2} \mathbf{A}^{(k)} + j\omega\sigma^{(k-1)} \mathbf{A}^{(k)} \right) \cdot \mathbf{w} + \nu \nabla \times \mathbf{A}^{(k)} \cdot \nabla \times \mathbf{w} \right) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{J}_{\mathrm{src}} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = 0$$

$$\forall v \in H_0^1(\Omega)$$

$$\int_\Omega \left(\rho c \frac{\partial T^{(k)}}{\partial t} v - k \nabla T^{(k)} \cdot \nabla v \right) \mathrm{d}\mathbf{x} - \int_\Omega Q\left(T^{(k-1)}, \mathbf{A}^{(k)} \right) v \, \mathrm{d}\mathbf{x} = 0$$

- apply dynamic iteration scheme
- average conductivity $\bar{\sigma}^{(k-1)} := \frac{1}{T} \int_T \sigma\left(T^{(k-1)}(t) \right) \mathrm{d}t$ for each sweep k
- assume sinusoidal current exication $\mathbf{J}_{\mathrm{src}} \sim \sin(\omega t) \rightarrow$ frequency domain

Formulation as Dynamic Iteration

TECHNISCHE UNIVERSITÄT DARMSTADT

Find magnetic vector potential $\mathbf{A}$ *and temperature* T $\forall w \in H_0(curl;\Omega)$

$$\int_\Omega \left(\left(\epsilon \mathbf{A}^{(k)} + j\omega\sigma^{(k-1)}\mathbf{A}^{(k)}\right) \cdot \mathbf{w} + \nu\nabla \times \mathbf{A}^{(k)} \cdot \nabla \times \mathbf{w}\right) \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{J}_{\mathrm{src}} \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} = 0$$

$\forall v \in H_0^1(\Omega)$

$$\int_\Omega \left(\rho c \frac{\partial T^{(k)}}{\partial t} v - k\nabla T^{(k)} \cdot \nabla v\right) \mathrm{d}\mathbf{x} - \int_\Omega Q\left(T^{(k-1)}, \mathbf{A}^{(k)}\right) v\, \mathrm{d}\mathbf{x} = 0$$

- apply dynamic iteration scheme
- average conductivity $\bar{\sigma}^{(k-1)} := \frac{1}{T}\int_T \sigma\left(T^{(k-1)}(t)\right)\mathrm{d}t$ for each sweep k
- assume sinusoidal current exication $\mathbf{J}_{\mathrm{src}} \sim \sin(\omega t) \rightarrow$ frequency domain
- space and time discretization (FIT, Cell Method, FEM; implicit Euler, RK, ...)

Formulation as Dynamic Iteration

TECHNISCHE UNIVERSITÄT DARMSTADT

Find magnetic vector potential **A** *and temperature* T $\qquad \forall w \in H_0(curl;\Omega)$

$$\int_\Omega \left(\left(\epsilon \mathbf{A}^{(k)} + j\omega\sigma^{(k-1)}\mathbf{A}^{(k)}\right)\cdot \mathbf{w} + \nu\nabla\times\mathbf{A}^{(k)}\cdot\nabla\times\mathbf{w}\right)\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{J}_{\mathrm{src}}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} = 0$$

$$\forall v \in H_0^1(\Omega)$$

$$\int_\Omega \left(\rho c\frac{\partial T^{(k)}}{\partial t}v - k\nabla T^{(k)}\cdot\nabla v\right)\mathrm{d}\mathbf{x} - \int_\Omega Q\left(T^{(k-1)}, \mathbf{A}^{(k)}\right)v\,\mathrm{d}\mathbf{x} = 0$$

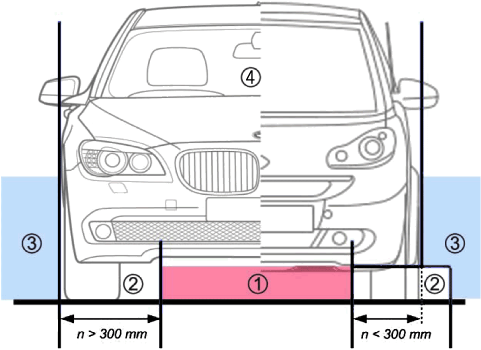
- apply dynamic iteration scheme
- average conductivity $\bar{\sigma}^{(k-1)} := \frac{1}{T}\int_T \sigma\left(T^{(k-1)}(t)\right)\mathrm{d}t$ for each sweep k
- assume sinusoidal current exication $\mathbf{J}_{\mathrm{src}} \sim \sin(\omega t) \longrightarrow$ frequency domain
- space and time discretization (FIT, Cell Method, FEM; implicit Euler, RK, ...)

P. Alotto, F. Freschi, M. Repetto. *Multiphysics Problems via the Cell Method: The Role of Tonti Diagrams*, 2010.

Motivation: Inductive Charging of Electric Cars

ICNIRP based recommendation

[Kaufmann et al. (with KOSTAL), 2014]



Source: Elektrische Ausrüstung von Elektro-Straßenfahrzeugen
Induktive Ladung von Elektrofahrzeugen - Teil 4-2: Niedriger Leistungsbereich. German. VDE-AR-E 2122-4-2. 2011.

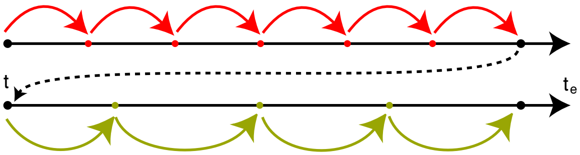
Public area: $|\mathbf{B}| \leq 6.25\,\mu$T

Functional area: test object must not heat up above 80°C.

Coupling in time domain (2 systems)

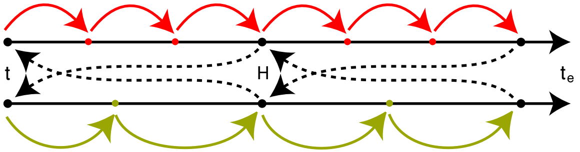
TECHNISCHE UNIVERSITÄT DARMSTADT

One-way Coupling



(parameter extraction)

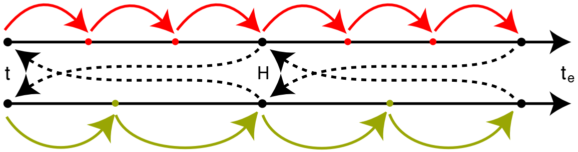
Dynamic iteration / waveform relaxation



(requires synchronization points, inter and extrapolation!)

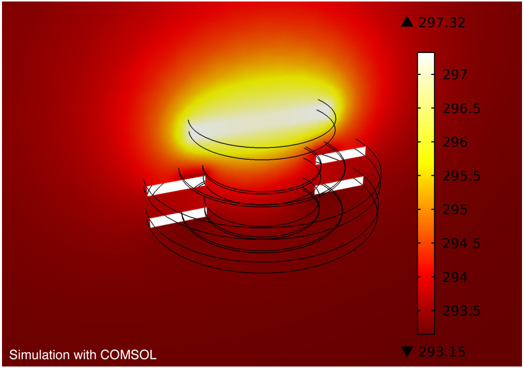
Coupling in time domain (2 systems)

Dynamic iteration / waveform relaxation



(requires synchronization points, inter and extrapolation!)

Numerical Example with Comsol



simulation time (20 min):

- frequency-transient model: 7 s
- purely transient model: ca. 2 years

time steps (for 20 min):

- frequency-transient model: 17
- purely transient model: ca. $250 \cdot 10^6$

- without window iteration: 290°C, i.e., $\leq$ 3% error.
- when using implicit Euler: one iteration is sufficient for $\mathcal{O}(H)$