Off-axis beam dynamics and optimisation strategies

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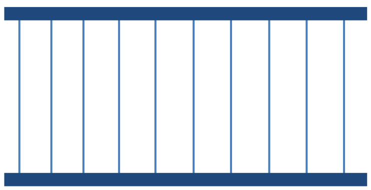
Overview

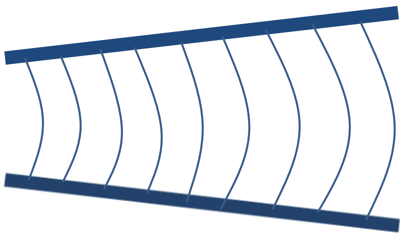
- Off-axis beam dynamics
  - Combined function magnets
  - Off-axis beams through linear optics
  - Nonlinear extension
- Optimisations strategies
  - Local vs. global optimisation
  - Examples for complex system

Combined function magnets

Normal dipole magnet

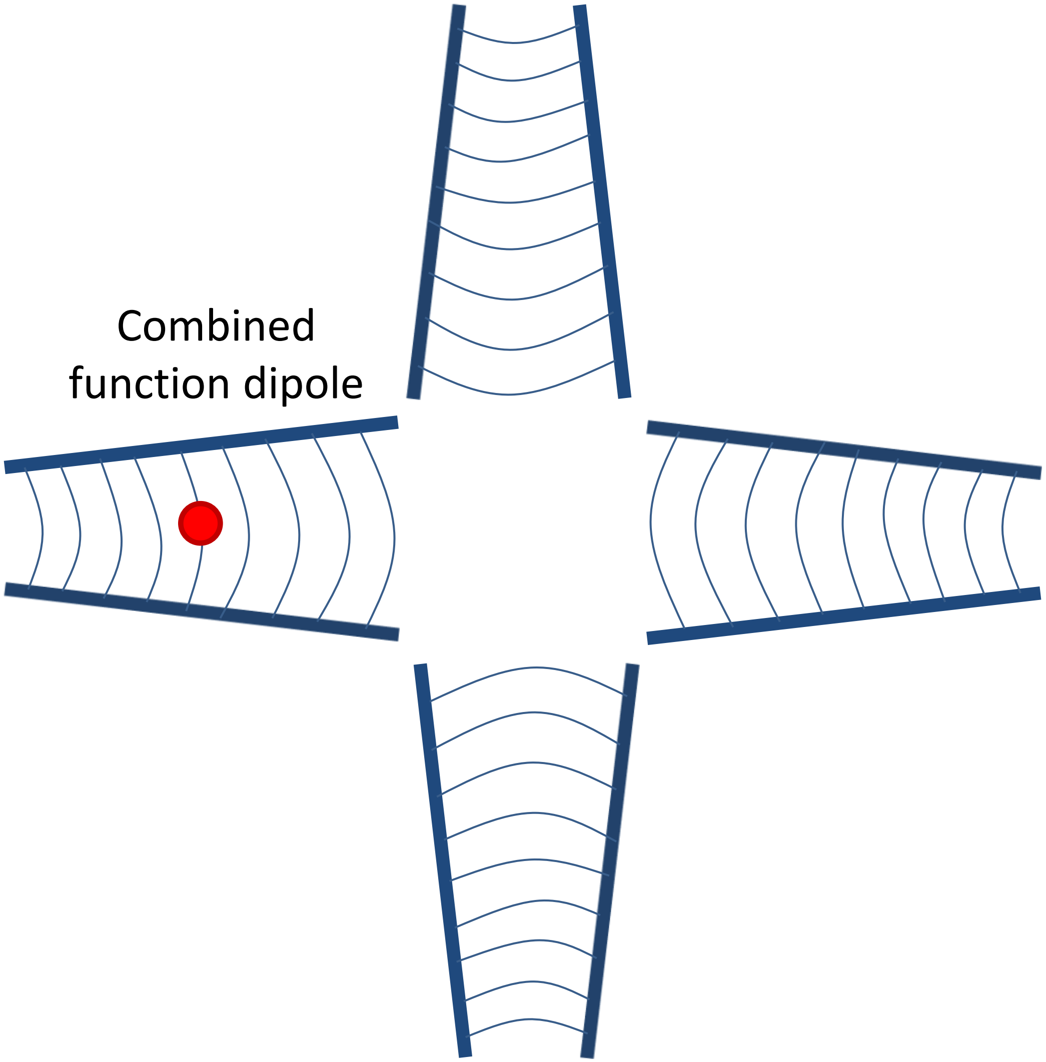


Combined function dipole



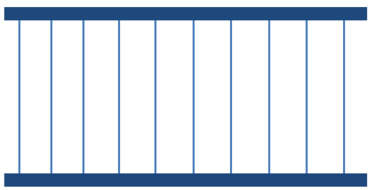
- Parallel faces of dipole yoke causes uniform magnetic field
  - Only dipole component

Combined function magnets

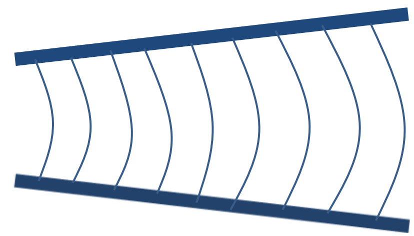


Combined function magnets

Normal dipole magnet



Combined function dipole



- Parallel faces of dipole yoke causes uniform magnetic field
  - Only dipole component

- Non-parallel faces changes the magnetic field
  - Dipole component
  - **AND** quadrupole component

Hill's Equation

$x'' = Kx$

$K = \frac{1}{\rho^2}$ (Dipole)
$K = k_1$ (Quadrupole)
$K = \frac{1}{\rho^2} + k_1$ (Combined function magnet)

Transfer matrix:

$$\begin{pmatrix} \cos\sqrt{K}s & \frac{1}{\sqrt{K}}\sin\sqrt{K}s & 0 & 0 & 0 & \frac{2}{\rho K}\sin^2\frac{\sqrt{K}s}{2} \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s & 0 & 0 & 0 & \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s \\ 0 & 0 & \cosh\sqrt{k_1}s & \frac{1}{\sqrt{k_1}}\sinh\sqrt{k_1}s & 0 & 0 \\ 0 & 0 & \sqrt{k_1}\sinh\sqrt{k_1}s & \cosh\sqrt{k_1}s & 0 & 0 \\ -\frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s & -\frac{2}{\rho K}\sin^2\sqrt{K}s & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\alpha = \frac{1}{\rho^2 K}\left(\frac{1}{\sqrt{K}}\sin\sqrt{K}s - s\right)$

Simplified transfer matrix

- Only consider horizontal plane
  - Vertical plane acts like a normal quadrupole
  - Ignore longitudinal dynamics for this lecture!

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{K}s & \frac{1}{\sqrt{K}}\sin\sqrt{K}s \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

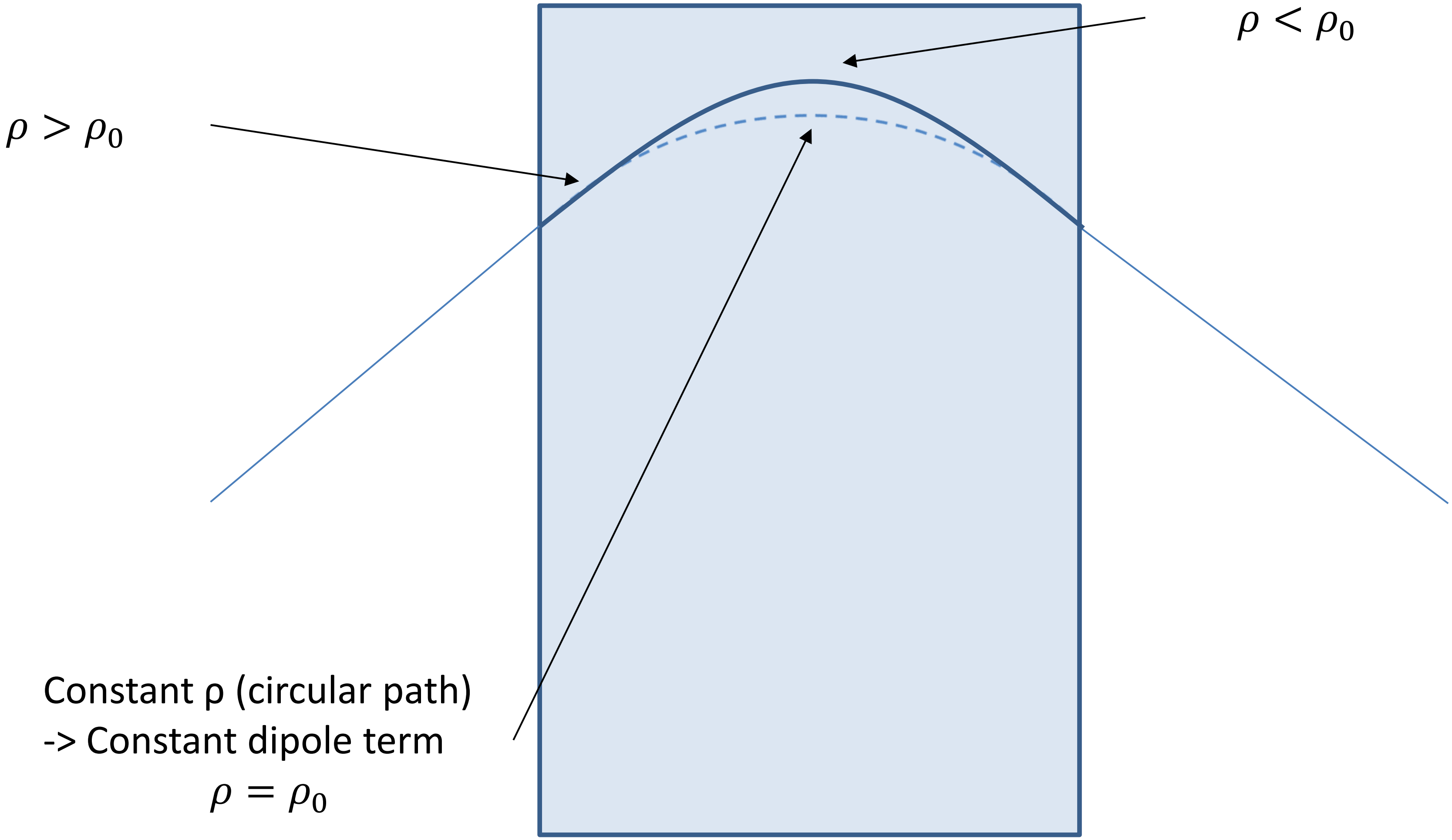
$$\begin{pmatrix} D_{x,1} \\ D'_{x,1} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{K}s & \frac{1}{\sqrt{K}}\sin\sqrt{K}s & \frac{2}{\rho K}\sin^2\sqrt{K}s \\ \sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s & \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_{x,0} \\ D'_{x,0} \\ 1 \end{pmatrix}$$

The 3rd row of the matrix is useful for matrix calculations

Off-axis beams through linear optics

- A combined function magnet is NOT equivalent to travelling off-axis through a quadrupole!

Trajectory through quadrupole



Trajectory through quadrupole

Beam trajectory through a quadrupole is well known solution from Hill's equation:

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{k_1}s & \frac{1}{\sqrt{k_1}}\sin\sqrt{k_1}s \\ -\sqrt{k_1}\sin\sqrt{k_1}s & \cos\sqrt{k_1}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \qquad \text{(focusing quad)}$$

Or $$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{k_1}s & \frac{1}{\sqrt{k_1}}\sinh\sqrt{k_1}s \\ \sqrt{k_1}\sinh\sqrt{k_1}s & \cosh\sqrt{k_1}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \qquad \text{(defocusing quad)}$$

But what about the dispersion?

As ρ varies, we cannot use the same equations as for a combined function magnet…

For simplicity, we will define the dispersion function as:

$$\begin{pmatrix} D_{x,1} \\ D'_{x,1} \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & D_q \\ M_{21} & M_{22} & D'_q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_{x,0} \\ D'_{x,0} \\ 1 \end{pmatrix}$$

Where M_{ij} are the quadrupole transfer matrix elements

D_q and D'_q are the dispersive contributions of the quadrupole, which we shall determine…

Off-axis dispersion

Recall that dispersion can be defined as:

$$D_x = M_{12}(l)\int_0^l \frac{\widetilde{M}_{11}(s)}{\rho}ds - M_{11}(l)\int_0^l \frac{\widetilde{M}_{12}(s)}{\rho}ds$$

$$D'_x = M_{22}(l)\int_0^l \frac{\widetilde{M}_{11}(s)}{\rho}ds - M_{21}(l)\int_0^l \frac{\widetilde{M}_{12}(s)}{\rho}ds$$

But we know that ρ is not constant, so we need to find an expression for this…

$$\rho(s) = \frac{dL}{d\theta} = \frac{(1 + x'^2)^{\frac{3}{2}}}{x''}$$

But the Hill's equation, $x'' = -k_1 x$, can be used to simplify this equation:

$$\rho(s) = -\frac{(1 + x'^2)^{\frac{3}{2}}}{k_1 x}$$

If $k_1 > 0$ we obtain the solution for a focusing quadrupole
If $k_1 < 0$ we obtain the solution for a defocusing quadrupole

Off-axis dispersion

Since we know that:

$$x(s) = \widetilde{M}_{11}(s)x_0 + \widetilde{M}_{12}(s){x'}_0$$
$$x'(s) = \widetilde{M}_{21}(s)x_0 + \widetilde{M}_{22}(s){x'}_0$$

Then:

$$D_q = k_1 \int_0^{l_q} \frac{\big(M_{12}(l)\widetilde{M}_{11}(s) - M_{11}(l)\widetilde{M}_{12}(s)\big)x}{(1+x'^2)^{\frac{3}{2}}} ds$$

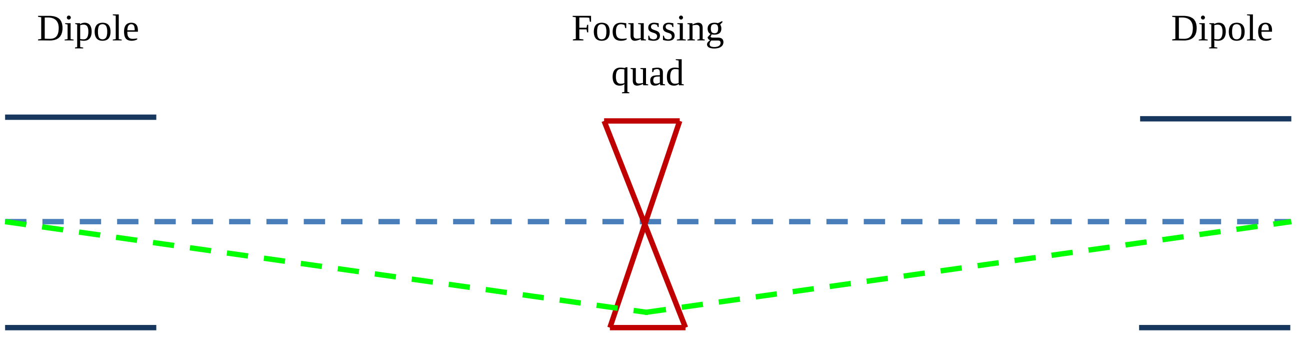
$$= k_1 \int_0^{l_q} \frac{\big(M_{12}(l)\widetilde{M}_{11}(s) - M_{11}(l)\widetilde{M}_{12}(s)\big)\big(x_0\widetilde{M}_{11} + {x'}_0\widetilde{M}_{12}\big)}{\left(1+\big(x_0\widetilde{M}_{21} + {x'}_0\widetilde{M}_{22}\big)^2\right)^{\frac{3}{2}}} ds$$

$$D'_q = k_1 \int_0^{l_q} \frac{\big(M_{22}(l)\widetilde{M}_{11}(s) - M_{21}(l)\widetilde{M}_{12}(s)\big)x}{(1+x'^2)^{\frac{3}{2}}} ds$$

$$= k_1 \int_0^{l_q} \frac{\big(M_{22}(l)\widetilde{M}_{11}(s) - M_{21}(l)\widetilde{M}_{12}(s)\big)\big(x_0\widetilde{M}_{11} + {x'}_0\widetilde{M}_{12}\big)}{\left(1+\big(x_0\widetilde{M}_{21} + {x'}_0\widetilde{M}_{22}\big)^2\right)^{\frac{3}{2}}} ds$$

So we now have expressions for the dispersive contribution

Example case:
1-quad local orbit bump



Dipole:

$$\begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\dfrac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Drift:

$$\begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Close the orbit bump

Determine x and x' just after the 1st dipole:

$$\begin{pmatrix} x_{d1} \\ x'_{d1} \end{pmatrix} = \begin{pmatrix} \rho(1-\cos\theta) \\ \tan\theta \end{pmatrix}$$

Determine x and x' just before the quadrupole:

$$\begin{pmatrix} x_{q1} \\ x'_{q1} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(1-\cos\theta) \\ \tan\theta \end{pmatrix} = \begin{pmatrix} \rho - \rho\cos\theta + L\tan\theta \\ \tan\theta \end{pmatrix}$$

Required x and x' at the end of the quadrupole to close the orbit:

$$\begin{pmatrix} x_{q2} \\ x'_{q2} \end{pmatrix} = \begin{pmatrix} M_{11}x_{q1} + M_{12}x'_{q1} \\ M_{21}x_{q1} + M_{22}x'_{q1} \end{pmatrix} = \begin{pmatrix} x_{q1} \\ -x'_{q1} \end{pmatrix}$$

Solving these simultaneous equations, we get:

$$\frac{x_{q1}}{x'_{q1}} = \frac{M_{12}}{1 - M_{11}} = -\frac{1 + M_{21}}{M_{22}}$$

Now we have closed the orbit bump, but we can use this to help with the dispersion integrals

Dispersion through the cell

To keep the maths easier, let's just look at the dispersion through the quadrupole:

$$\begin{pmatrix} D_{q2} \\ D'_{q2} \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & D_q \\ M_{21} & M_{22} & D'_q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_{q1} \\ D'_{q1} \\ 1 \end{pmatrix}$$

If we first assume that the beam travels on-axis through the quadrupole ($D_q = D'_q = 0$) then if $\frac{D_{q1}}{D'_{q1}} = \frac{x_{q1}}{x'_{q1}}$ we know that this cell would be dispersion free at the ends…

Dispersion after dipole:

$$\begin{pmatrix} D_{x1} \\ D'_{x1} \end{pmatrix} = \begin{pmatrix} \rho(1 - \cos\theta) \\ \sin\theta \end{pmatrix}$$

Dispersion at the start of the quad:

$$\begin{pmatrix} D_{q1} \\ D'_{q1} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(1 - \cos\theta) \\ \sin\theta \\ 1 \end{pmatrix} = \begin{pmatrix} \rho(1 - \cos\theta) + L\sin\theta \\ \sin\theta \\ 1 \end{pmatrix}$$

Dispersion through the cell

So:

$$\frac{D_{q1}}{D'_{q1}} = \frac{\rho(1-\cos\theta)}{\sin\theta} + L$$

$$\frac{x_{q1}}{x'_{q1}} = \frac{\rho(1-\cos\theta)}{\sin\theta}\cos\theta + L$$

If we assume that $\theta \ll 1$ (small angle approximation) then $\cos\theta \approx 1$ and the ratios are equal.

So in this case, the dispersion for an on-axis beam will be zero at the end of the cell. What about an off-axis beam?

Assuming the small angle approximation, the transfer matrix for a dipole is:

$$\begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\dfrac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & \rho\theta & \dfrac{\rho\theta^2}{2} \\ -\dfrac{\theta}{\rho} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion through the cell

Since we know that all terms that don't depend on D_q and D'_q cancel through the cell, we can simply consider the propagation of D_q and D'_q through the second part of the cell.

After the 2nd drift:

$$\begin{pmatrix} D_{x2} \\ D'_{x2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_q + \cdots \\ D'_q + \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} D_q + LD'_q + \cdots \\ D'_q + \cdots \\ 1 \end{pmatrix}$$

At the end of the cell:

$$\begin{pmatrix} 1 & \rho\theta & \frac{\rho\theta^2}{2} \\ -\frac{\theta}{\rho} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_q + LD'_q + \cdots \\ D'_q + \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} D_q + (L + \rho\theta)D'_q \\ -\frac{\theta}{\rho} D_q + \left(1 - \frac{\theta L}{\rho}\right) D'_q \\ 1 \end{pmatrix}$$

Dispersion through the cell

This is often known as **RESIDUAL DISPERSION**

In low emittance machines, this can be a limitation on performance:

- Difficult to locate sources of residual dispersion, so difficult to correct it
  - Beam based alignment, such as **DISPERSION FREE STEERING**, will reduce this
- Depends on beam position jitter, so often varies pulse to pulse
  - Can also depend on charge jitter if wakefields are a problem.

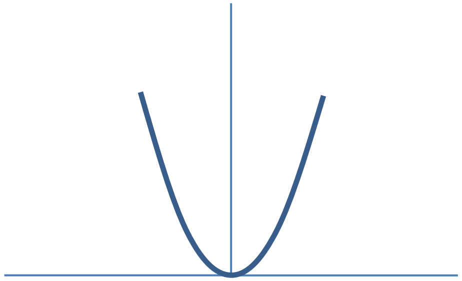
It is beyond the scope of this lecture, but it can be shown that it is not possible to have zero residual dispersion through any optical system when the beam travels off-axis.

Nonlinear extension

Consider a beam travelling through a higher order multipole, like a sextupole:

On-axis:

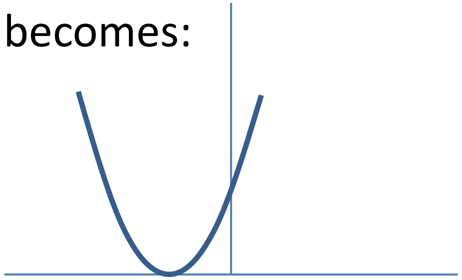
$$B_y = \frac{p}{c} k_n x^n$$



For example, sextupole field: $B_y = \frac{p}{c} k_2 x^2$

If the beam travels off-axis by a distance δx, then the field becomes:

$$B_y = \frac{p}{c} k_n (x + \delta x)^n = \sum \frac{p}{c} k_n \frac{n!}{k!(n-k)!} x^{n-k} \delta x^k$$



For sextupole: $B_y = \frac{p}{c} k_2 x^2 + 2 \frac{p}{c} k_2 x \delta x + \frac{p}{c} k_2 \delta x^2$

Sextupole term

Quadrupole term

Dipole term

Travelling off-axis through a multipole introduces lower order terms

Nonlinear extension

We will only consider the dipole and quadrupole terms:

$$B_y = \frac{p}{c} k_n \delta x^n + n \frac{p}{c} k_n \delta x^{n-1} x + o(x^2)$$

$$Kx = (n k_n \delta x^{n-1}) \frac{\delta x}{n} + (n k_n \delta x^{n-1}) x$$

So the Hill's equation is:

$$x'' + \left(\left(n k_n \delta x^{n-1}(x, x', s) \right) \frac{\delta x(x, x', s)}{n x} + \left(n k_n \delta x^{n-1}(x, x', s) \right) \right) x = 0$$

But for $n > 1$ (i.e. sextupole or higher), this is nonlinear and cannot be solved analytically. If we consider splitting the multipole into a large number of thin slices, then δx can be considered constant and each slice can be treated as a quadrupole and dipole term.

Nonlinear extension

If we consider the Hill's equation for an off-axis quadrupole:

$$Kx = k_1\delta x + k_1 x$$

Comparing this to the Hill's equation for an off-axis multipole:

$$Kx = (nk_n\delta x^{n-1})\frac{\delta x}{n} + (nk_n\delta x^{n-1})x$$

Then we can compare terms:

$$\tilde{k}_1 = nk_n\delta x^{n-1}$$

And the off-axis multipole term becomes:

$$Kx = \tilde{k}_1\frac{\delta x}{n} + \tilde{k}_1 x$$

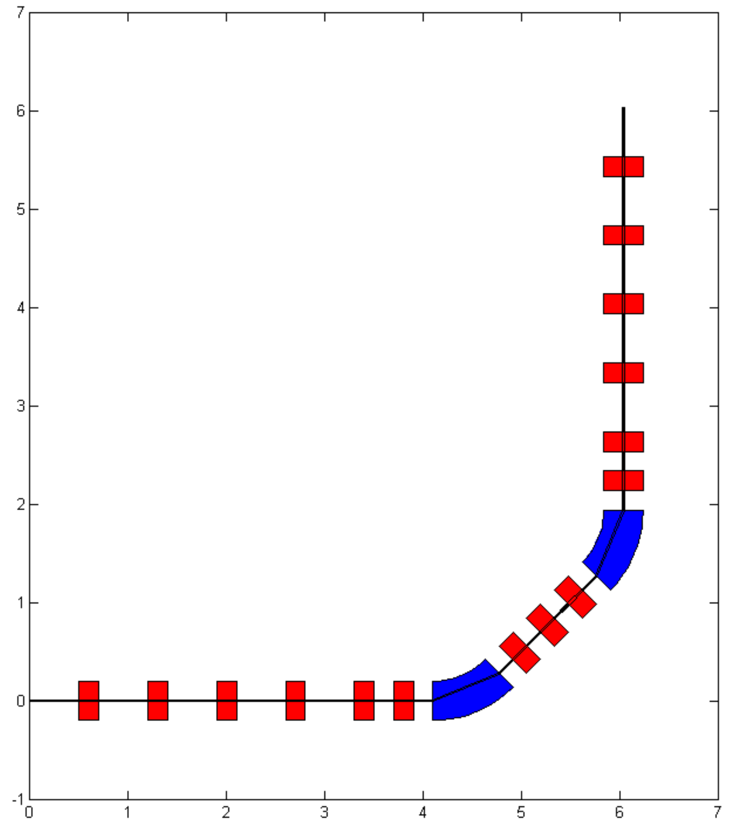
Therefore travelling off-axis through a multipole a distance δx is equivalent to travelling off-axis through a quadrupole by a distance $\frac{\delta x}{n}$.

- This means that the residual dispersion can be reduced, but still not removed.
- The strong dependence of magnetic field to radial position means that the sensitivity to beam jitter increases by a factor n; so nonlinear optics can cause more problems.

Optimisation strategies

Consider a short beamline consisting of 3 sections:

1) A quadrupole matching section
2) An achromatic arc section
3) Another quadrupole matching section



Local optimisation

Local optimisation:
- Match initial beam parameters into arc cell
- Match optics and dispersion in arc cell
- Match arc cell beam parameters into final optics

Advantages:
- Easy to implement in simulations such as MADX, PLACET, ELEGANT…
- Modular: can modify each section independently
- Good for linear transverse optics
    - Matching Twiss parameters
    - Dispersion

Disadvantages
- Not good for complex problems:
    - Chromatic corrections
    - Nonlinear optics
    - Longitudinal optics

Global optimisation

MATCH EVERYTHING TOGETHER!

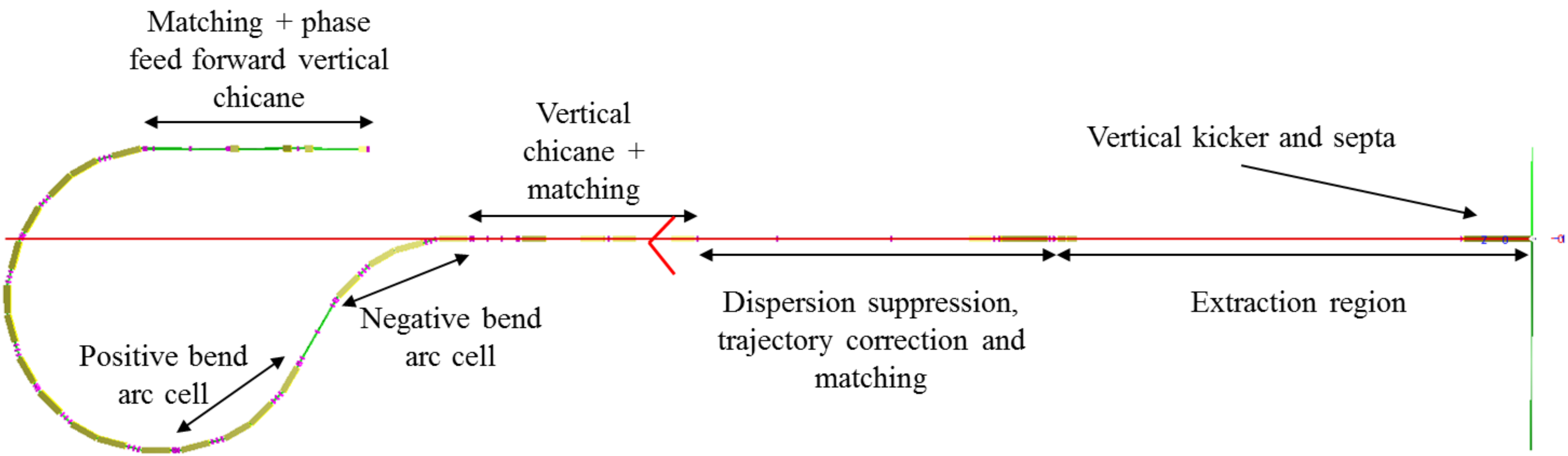
Advantage

- Very good at optimising complex problems

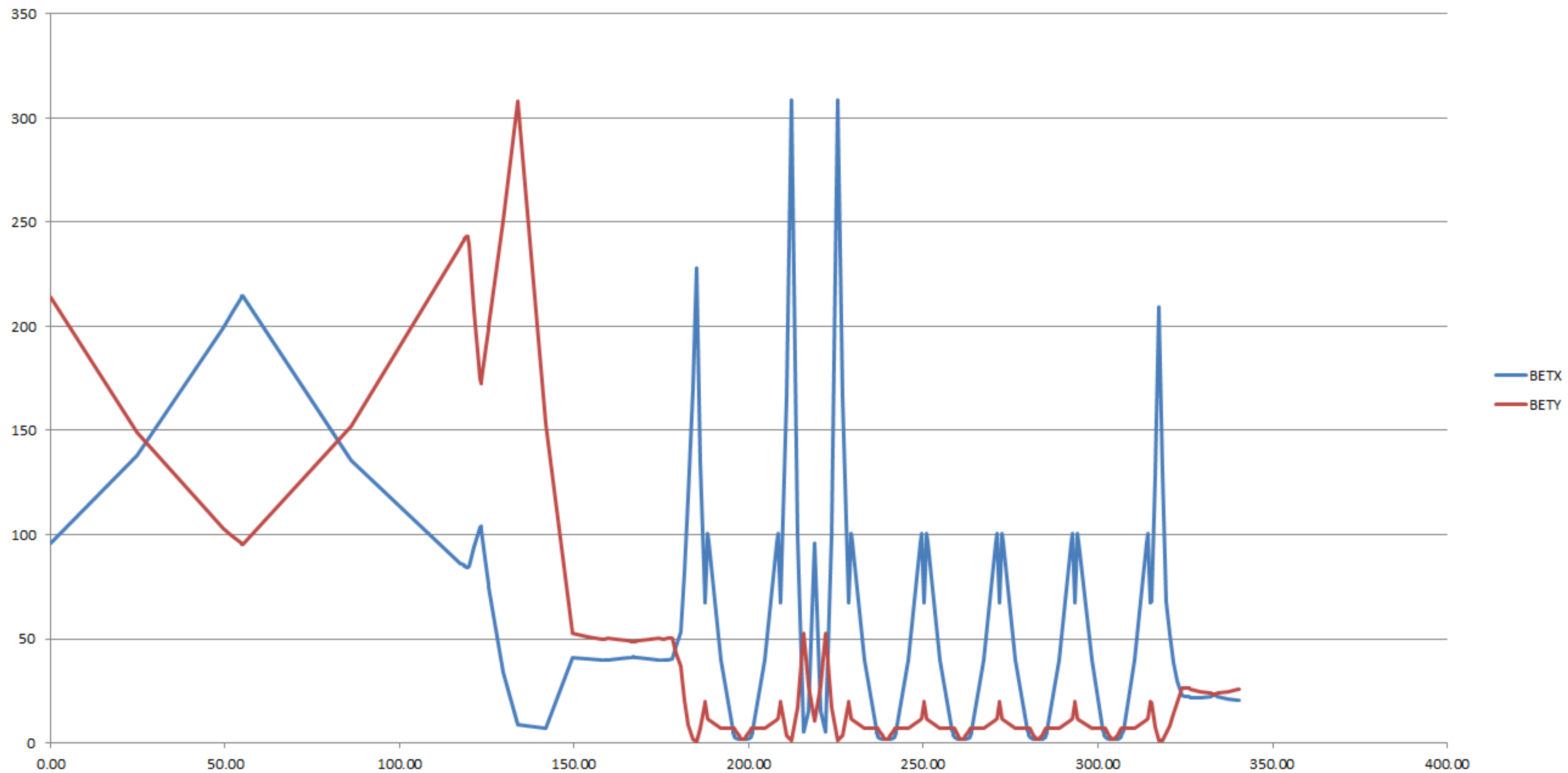
Disadvantages

- More difficult to implement
- Large number of parameters to optimise: optimal solution can be difficult to find
    - Often good to use local optimisation before global optimisation.

Example: CLIC drive beam TAL



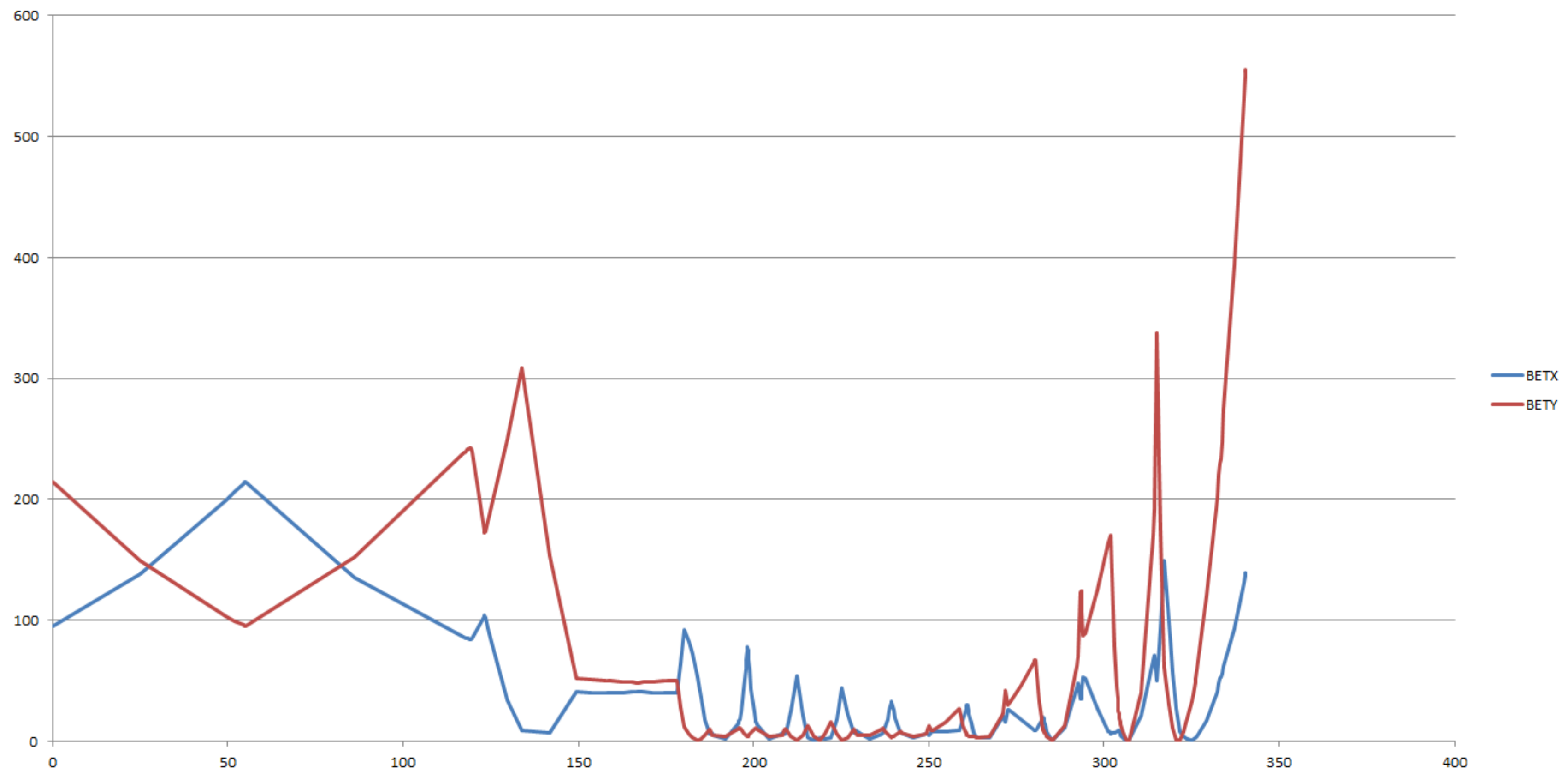
Locally optimised solution



Emittance growth:

Horizontal:	138 µmRad
Vertical:	2.2 µmRad
Longitudinal	1.8 µmGeV

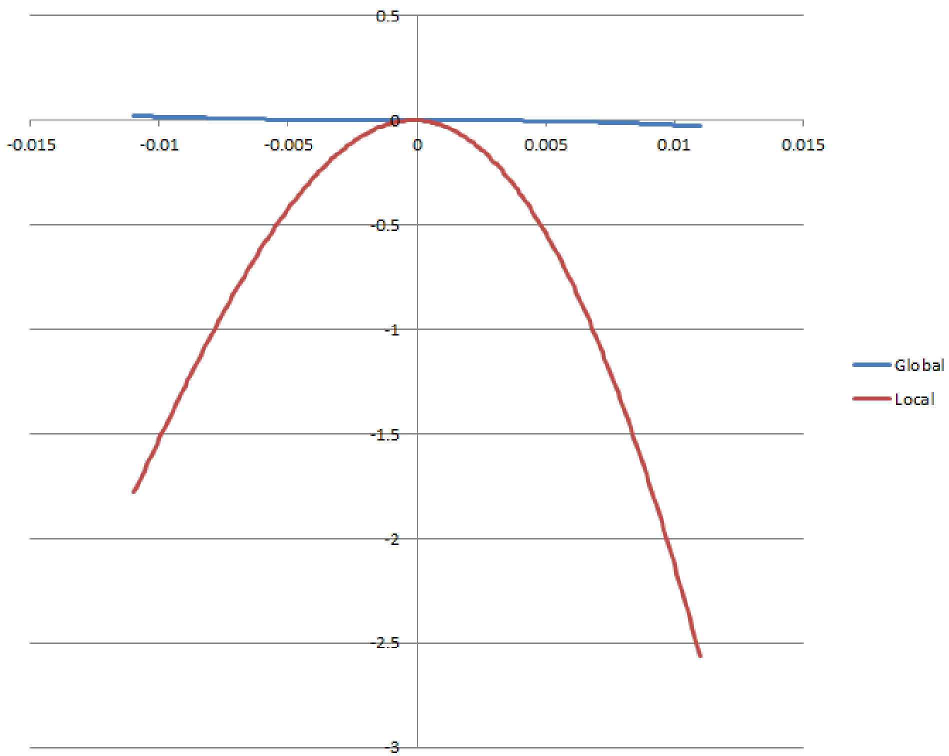
Globally optimised solution

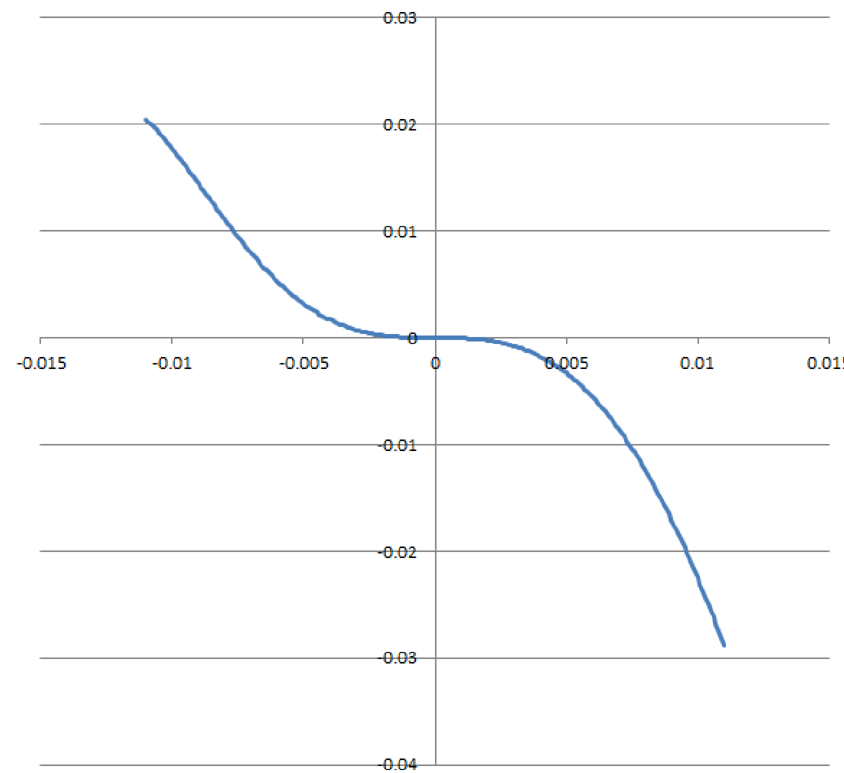


Emittance growth:

Horizontal:	3.0 µmRad
Vertical:	2.0 µmRad
Longitudinal	-0.0012 µmGeV

Dispersion energy dependence





R56 energy dependence