

Greedy Algorithm Fails in Compact Vector Summation

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Abstract. *Scheduling Theory has found many applications in various fields of sciences and business. Workflow scheduling algorithms play an important role in Grid-systems as well. Greedy algorithms, which in particular are applied e.g. in Network Planning problems, in general seem to be effective due to their simplicity and constructiveness. For the computational purposes they are of importance for they give approximate solution in a reasonable time. On the other hand greedy strategy rarely gives an optimal solution. Moreover, for the Compact Vector Summation (CVS) (one of the scheduling problems) we show that greedy-based solution is “unboundedly far” from the optimal solution.*

X – Linear normed space

$x_1, x_2, \dots, x_n, n \geq 1$ – collection of vectors of X .

$\pi: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ – permutation.

Consider the number

$$\varphi(\pi) = \max_{1 \leq k \leq n} \|a_{\pi(1)} + a_{\pi(2)} + \dots + a_{\pi(k)}\|$$

$\pi_{optimal}$ – $\varphi(\pi)$ attains its minimum.

Applications – e.g. in the problems of **Scheduling Theory**.

Estimation of $\pi_{optimal}$ is frequently called the **Problem of (dynamical) Compact Vector summation (CVS)**.

Greedy Algorithm (greedy permutation) to approach $\pi_{optimal}$:

Step I. Choose $x_{n_1} = \min_{1 \leq k \leq n} \{\|x_k\|\}$;

Step II. Choose x_{n_2} so that $\|x_{n_1} + x_{n_2}\|$ is minimal, etc.

Computation runs in **polynomial time**.

Is it of the same order as the optimal one?

Question: Is there a constant C , dependent only on the space X such that

$$\varphi(\pi_{greedy}) \leq C\varphi(\pi_{optimal})?$$

In the one-dimensional case the constant $C = 2$.

J. Wojtaszczyk – in l_∞^2 C is unbounded.

$$\overbrace{(1,1), (2,-3), (-3,2), \dots, (1,1), (2,-3), (-3,2)}^{3n}$$

$$\varphi(\pi_{greedy}) \geq n; \varphi(\pi_{optimal}) = 3$$

Theorem: For any two-dimensional normed space X (real or complex) and any prescribed number M there exists a collection $x_1, x_2, \dots, x_n, n \geq 1$ of vectors of X such that

$$\frac{\varphi(\pi_{greedy})}{\varphi(\pi_{optimal})} > M.$$

Proof.

H – linear inner product space $\dim H = 2$;

$$T: X \rightarrow H, \ker T = \{0\}, \|T\| = 1, a \in X, \|a\| = \|Ta\| = 1$$

Construction:

$$h_1 \equiv Ta, \text{ choose } h_2 \perp h_1, \|h_2\| = 2\sqrt{n}, b \equiv T^{-1}h_2;$$

$$x_1 \equiv -\frac{1}{\sqrt{n}}a, \quad x_2 \equiv \frac{1}{\sqrt{n}}(a + b), \quad x_3 \equiv \frac{1}{\sqrt{n}}(a - b)$$

Collection: $n \geq 1$

$$\overbrace{x_1, x_2, x_3, x_1, \dots, x_1, x_2, x_3, x_1}^{4n}$$

$$x_1 + x_2 + x_3 + x_1 = 0$$

Calculations

$$\|x_1\| = \frac{1}{\sqrt{n}}, \|x_2\| > 2, \|x_3\| > 2,$$

$$\|(k-1)x_1 + x_2\| > \|kx_1\| = \frac{k}{\sqrt{n}}, k = 1, \dots, n.$$

$$\|(k-1)x_1 + x_3\| > \|kx_1\| = \frac{k}{\sqrt{n}}, k = 1, \dots, n.$$

imply the greedy algorithm:

Step I. Choose x_1 ;

Step II. Choose x_1 ;

⋮

Step n. Choose x_1 ;

⋮

We get

$$\overbrace{x_1, \dots, x_1}^n, \dots$$

Thus

$$\varphi(\pi_{greedy}) \geq \|nx_1\| = \sqrt{n}.$$

While for the initial arrangement

$$\overbrace{x_1, x_2, x_3, x_1, \dots, x_1, x_2, x_3, x_1}^{4n}$$

$$\varphi(\pi) \leq 2\|T^{-1}\| \Rightarrow \varphi(\pi_{optimal}) \leq 2\|T^{-1}\|.$$

Choose

$$n > 4M^2\|T^{-1}\|^2.$$

We get

$$\frac{\varphi(\pi_{greedy})}{\varphi(\pi_{optimal})} > M.$$