

ON A LINEAR GENERALIZED CENTRAL SPLINE ALGORITHM OF  
COMPUTERIZED TOMOGRAPHY

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*Absrtact. A linear generalized central spline algorithm for computerized  
tomography problem is constructed and studied.*

The main problem of computerized tomography is contained in the reconstruction of function by its integral over hyperplanes. This mathematical problem is encountered in a growing number of diverse settings in medicine, science, technology, and, in general, in the cases, when the inner structure of an object is investigated with the help of some emanations. An important well-known example of computerized tomography is its application in the roentgendiagnostic. The scheme of this process is the following.

Consider a roentgenbeams following a line  $L$  that passes through an object. Let  $f(t)$  be the absorption coefficient by biocloth of the rentgen beams in a point  $t$  and  $I(t)$  be the intensity of the beam at  $t$ . If the initial intensity of the beam  $L$  is  $I_0$  and its intensity after the passage of the body is  $I_1$ , then  $I_1/I_0 = \exp\{-\int_L f(t)dt\}$ . This means that after the scanning we get the linear integrals along to such line  $L$ . The problem is in the reconstruction of  $f$  with the help of such set of integrals. If we investigate a space body, then line integrals will be replace by integrals along hyperplanes. The map  $\mathfrak{R}$ , which corresponds to a defined on  $\mathbb{R}^n$  function  $f$  the integrals of  $f$  along all hyperplanes, is called the Radon transform.

Generally for the definition of Radon transform  $\mathfrak{R}$  in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , the standard parametrization of a hyperplane by a normal unit vector  $\omega$  and its distance  $s$  from the origin are used. The Radon transform  $\mathfrak{R}$  maps a density function  $u$  to its integrals over all hyperplanes and is defined by the formula

$$\mathfrak{R}u(\omega, s) = \int_{(t,\omega)=s} u(t)dt = \int_{\omega^\perp} u(s\omega + t)dt, \quad (1)$$

where  $\omega \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ .  $\mathfrak{R}$  is well defined wenn  $u$  belongs to the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  of fast decreasing in  $\mathbb{R}^n$  functions.  $\mathfrak{R}$  is one-to-one operator acting from  $\mathcal{S}(\mathbb{R}^n)$  in the Schwartz space  $\mathcal{S}(Z)$ , where  $Z$  is the cylinder  $Z = \mathbb{S}^{n-1} \times \mathbb{R}$ .

The Radon transform, which is defined by (1) only for the belonging to the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  functions, admits a continuous extension in some weighted  $L_2$ -spaces. Let  $W_\nu(x) = (1 - |x|^2)^{\nu-n/2}$  be weight function, defined on the unit ball  $\Omega^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$  and let  $w_\nu(s) = (1 - s^2)^{\nu-1/2}$ ,  $s \in [-1, 1]$  be weight function, defined on the cylinder  $Z$ . It is shown that  $\mathfrak{R}$  is continuous operator, acting from the space  $H_\nu = L_2(\Omega^n, W_\nu^{-1})$  in the space  $K_\nu = L_2(Z, w_\nu^{-1})$ , which are endowed with the usually norms and  $\nu > n/2 - 1$ , the acting in this spaces operator  $\mathfrak{R}$  admits a singular decomposition, which is obtained by A.Louis.

The Radon transform is studied in many papers. Constructions of the inverse Radon transform are obtained by J.Radon, A. Louis, F. Netterer and others. Beginning from the 60-th of the past century, there exists the problem of construction of algorithms for the approximate solution of the equation

$$\mathfrak{R}u = f, \quad (2)$$

where the Radon transform  $\mathfrak{R}$ , generally speaking, acts from the Fréchet space in a such type space. Some constructions of approximate Radon inverse transform are given by R. Dietz, P. Maas and others. To this problem is devoted our reports too. If  $\mathfrak{R}$  acts from  $H_\nu$  in  $K_\nu$ , the equation (2) is ill-posed and with a view to its transformation in a well posed one, in some sense, we follow the approach of Thichonow. We have founded a set of functions  $f$  for which the equation (2) is getting well-posed. For the further consideration of our problem we reduce some definitions.

Let  $E$  be a linear space over the scalar field of real or complex numbers. Let  $F$  be an absolute convex set in  $E$ . Let us consider a linear operator  $S : E \rightarrow G$ , called a solution operator, where  $G$  is a linear metric space over the scalar field of real or complex numbers with a metric  $d$ . Elements  $f$  from  $F$  are called problem elements for solution operator and  $S(f)$  are called solution elements. For  $f$  we wish to compute  $S(f)$ . Let  $U(f)$  be the computed approximation. The distance  $d(S(f), U(f))$  between  $S(f)$  and  $U(f)$  is called the absolute error.

For the construction of a computed approximation  $U(f)$  we gather enough an information about a problem element  $f$ . Let  $y = I(f)$  be a nonadaptive computed information of cardinality  $m$ , i.e.

$$I(f) = [L_1(f), \dots, L_m(f)] , \quad (3)$$

where  $L_1, \dots, L_m$  are linear functionals on the space  $E$ . Knowing  $y = I(f)$ , the approximation  $U(f)$  is computed by combining the information to produce

an element of  $G$ , which approximates  $S(f)$ . That is  $U(f) = \varphi(I(f))$ , where  $\varphi : I(E) \rightarrow G$  is a mapping, which is called an algorithm.

The worst case error of  $U$  is defined by

$$e(\varphi, I) = \sup\{d(S(f), U(f)), f \in F\}.$$

Naturally, we are interested in algorithm with minimal error. An algorithm  $\varphi^*$  is called an optimal error algorithm if it realized inf in last equality, i.e.  $e(\varphi^*, I) = \inf\{e(\varphi, I) : \varphi \in \Phi\}$ , where  $\Phi$  is the set of all algorithms.

In the classical theory of algorithms, besides to optimal algorithms, there are considered the so-called spline and central algorithms. A central algorithm possesses even more strong property than optimal one. Every central algorithm is optimal, but obviously not every optimal error algorithm algorithm is central. Central algorithms ensure the best possible approximation to  $S(f)$  for every  $f$  from the domain of definition of  $S$ . Especially important are algorithms which are linear, spline and central simultaneously. It happens that, under the consideration of some problem, a spline does not exists. Then does not exists the corresponding spline algorithm too. For the investigation of these cases we have introduced notions of generalized spline and generalized central algorithms. The substance of these generalization is in following. These new notions are defined with the help not one set  $F \subset E$ , but with a decreasing sequence of absolutely convex absorbed sets  $\{V_n\}$ ,  $n \in N$ , belonging to  $E$ . We have not have a possibility for the formulation of these notions because of the time deficiency.

In what follows, an operator  $S$  is said to be the solution operator of an operator equation  $Au = f$ , if  $u = Sf$ . We call the central (resp. linear, spline, optimal) algorithm, approximating the solution operator  $S$ , as central (resp. linear, spline, optimal) algorithm for the equation  $Au = f$ .

Let  $H$  and  $M$  are Hilbert spaces and  $\{\varphi_k\}$ ,  $\{\psi_k\}$  are orthogonal systems in  $H$  and  $M$  respectively. For the sake of simplicity we apply the same notations  $(\cdot, \cdot)$  for the inner products in  $H$  and  $M$ . Let further  $A$  be an operator acting from  $H$  in  $M$  and having a singular decomposition

$$Au = \sum_{k=1}^m \sigma_k(u, \varphi_k)\psi_k, \quad u \in H, \quad \sigma_k > 0 \quad (4).$$

If the operator  $A$  has such decomposition, we say also that the triple  $\{\psi_k, \varphi_k, \sigma_k\}$ ,  $k = 1, 2, \dots$  represents a singular system for  $A$ . The number  $\sigma_k$  are called

singular numbers of the operator  $A$ . In general, such operators are not compact and selfadjoint and  $\text{Im}A \neq M$ . The problem of solution of equation

$$Au = f, \quad (5)$$

in general, is ill posed. For the realization of Thichonov's idea we find in  $H$  a set on which the problem is well posed. Namely, we consider the set  $D(A^{-\infty})$  in  $H$  consisting from such  $f \in H$  on which may be realized the operator  $A^{-n} \equiv A^{-1}(A^{n-1}f)$  for all  $n \in \mathbb{N}$ . On  $D(A^{-\infty})$  we consider the following sequence of Hilbertian seminorms

$$\|x\|_n = (\|x\|^2 + \|A^{-1}x\|^2 + \dots + \|A^{-n+1}x\|^2)^{1/2}, \quad n \in \mathbb{N}.$$

We seek the generalized solution of (5) in the sense of Mourie-Penrose. This means that if  $f \in \text{Im}A + (\text{Im}A)^\perp$ , as a generalized solution is considered an element  $A^+f$  which minimized the norm  $\|Au - f\|$ . If there exist a set of such elements, we take between them an element with the minimal norm. This generalized solution satisfies the equation

$$A^*Au = A^*f \quad (6).$$

It may be proved that if  $A$  possesses a decomposition (4), then the unique solution  $u^+$  of (6), in the sense of Mourie-Penrose, is given by the formula

$$u^+ = \sum_{k=1}^{\infty} (\sigma_k(\psi_k, \psi_k)(\varphi_k, \varphi_k))^{-1} (f, \psi_k) \varphi_k.$$

The operator  $A^*A$  is symmetric and positive. Let us assume yet that  $A$  is one-to-one on whole  $H$  operator. The operator  $A^*A$  is selfadjoint, having dense image and positive eigenvalues  $\sigma_k^2(\psi_k, \psi_k)$ , which correspond to the functions  $\varphi_k$ .

We have solved the equation (6) by Rietz extended method and proved that the obtained algorithm

$$u_m = \sum_{k=1}^m (\sigma_k(\psi_k, \psi_k)(\varphi_k, \varphi_k))^{-1} (f, \psi_k) \varphi_k. \quad (7)$$

is a linear, spline and central for the information  $I(f) = [(f, \psi_1), \dots, (f, \psi_m)]$ . Moreover, if the condition  $\lim_{k \rightarrow \infty} \sigma_k(\varphi_k, \varphi_k)(\psi_k, \psi_k) = 0$  holds, then the approximate solution  $u_m$  converges to the solution of the equation (6) so that  $(A^*A)^{-n}u_m$  converges to  $(A^*A)^{-n}u$ ,  $m \rightarrow \infty$ , for every  $n \in \mathbb{N}$  in  $H$ .

Now we may apply the obtained results to our main problem of the construction of approximate inverse for the Radon transform, i.e. of the

approximate solution of the equation (2). For this we apply a singular decomposition with respect to products of Gegenbauer polynomials and spherical harmonics for the Radon transform, which acts from the above mentioned weight space  $H_\nu = L_2(\Omega^n, W_\nu^{-1})$  in the weight space  $K_\nu = L_2(Z, w_\nu^{-1})$  and which is obtained by A. Louis. The two-dimensional case was considered earlier by A. Cormack. We use the form of this decomposition, which is given in the work of R.Dietz.

We introduce some notations:

$P_m^{(\alpha, \beta)}$  is the Jacobi polynomial of degree  $m$  and indices  $\alpha, \beta$ ;  $C_m^\nu$  is the Gegenbauer polynomial of degree  $m$  and index  $\nu$ ;  $\Gamma$  is the second kind Euler integral.

$\{Y_{lk}, k = 1, \dots, N(n, l)\}$  is orthonormal basis of spherical functions, defined on  $\mathbb{S}^{n-1}$ , where  $l = 0, 1, \dots$ , and  $N(n, l) = \frac{(2l+n-2)(n+l-3)!}{l!(n-2)!}$ ,  $n \geq 2$ ;

$$v_{mlk}^\nu(x) = W_\nu(x)|x|^l P_{(m-l)/2}^{(\nu-n/2, l+n/2-1)}(2|x|^2-1)Y_{lk}(x/|x|); \quad (8)$$

$$u_{mlk}^\nu(\omega, s) = d_{ml}w_\nu(s)C_m^\nu(s)Y_{lk}(\omega), \text{ where}$$

$$d_{ml} = \pi^{n/2-1}2^{2\nu-1} \frac{\Gamma((m-l)/2 + \nu - n/2 + 1)\Gamma(m+1)\Gamma(\nu)}{\Gamma((m-l)/2 + 1)\Gamma(m+2\nu)}; \quad (9)$$

$$\sigma_{mlk}^2 = \frac{2^{2\nu}\Gamma((m+l)/2 + \nu)\Gamma((m-l)/2 + \nu - n/2 + 1)\Gamma(m+1)}{\pi^{1-n}\Gamma((m+l+n)/2)\Gamma((m-l)/2 + 1)\Gamma(m+2\nu)} = \sigma_{ml}^2; \quad (10)$$

We note that in the notations (8)-(10),  $P_0^{(\alpha, \beta)} \equiv 1$ ,  $C_0^\lambda \equiv 1$  and  $Y_{0k} \equiv 1$ .

Now we formulate our main result:

**Theorem.** *Let  $\{v_{rlk}^\nu, u_{rlk}^\nu, \sigma_{rl}\}$ ,  $l \leq r$ ,  $1 \leq k \leq N(n, l)$  is a singular system for the Radon transform  $\mathfrak{R}$ , which acts from  $H_\nu = L_2(\Omega^n, W_\nu^{-1})$ ,  $\nu > n/2 - 1$ , in the space  $K_\nu = L_2(Z, w_\nu^{-1})$ . Then the algorithm*

$$\varphi^{(m)}(I(f))(x) = \sum_{r=0}^m \sum_{l \leq r} \sigma_{rl} \sum_{k=1}^{N(n, l)} (f, u_{rlk}^\nu)_{L_2(Z, w_\nu^{-1})} v_{rlk}^\nu(x), \quad x \in H_\nu, \quad (11)$$

where  $\Sigma'$  means that the summability takes place only for even  $m+l$ , is the linear generalized spline and the generalized central for the solution operator  $S = (\mathfrak{R}^* \mathfrak{R})^{-1}$  and nonadaptive information  $I(f) = [(f, u_{001}^\nu), \dots, (f, u_{mmN(n, m)}^\nu)]$ . Moreover, these approximative solutions converges to the solution of equation (2) (in the sense of Moorie-Penrose) so that  $(\mathfrak{R}^* \mathfrak{R})^{-n} \varphi^{(m)}$  converges to  $(\mathfrak{R}^* \mathfrak{R})^{-n} u$ ,  $m \rightarrow \infty$ , for every  $n \in \mathbb{N}$  in  $K_\nu$ , where  $u$  is solution of Radon equation (2).

Finally, we make one comment relative the formula (11). As it was mentioned above, with the help of tomograph we may calculate integrals of  $f$  along hyperplanes. Our formula contains not these integrals, but inner products of  $f$  and functions  $u_{rlk}^\nu$ . It may be proved that with the help of integrals over hyperplanes we may calculate these inner product. This follows from the following formula

$$(f, u_{rlk}^\nu)_{L_2(Z, w_\nu^{-1})} = \int_{-1}^1 w_\nu(s) ds \int_{S^{n-1}} u_{rlk}^\nu(s, w) dw \int_{\omega^\perp \cap \Omega^n} f(s\omega + y) dy.$$

It enters in our plane to make the programm for the algorithm (11) that is not so easy.