An Algorithmic Solution to the Problem of Compact Vector Summation with an Application to Scheduling Theory

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ABSTRACT

The problem of Compact Vector Summation (CVS) consists in finding an upper estimation for $r(x,\pi_{\min})$, the minimum of radii of spheres that contain the trajectory of partial sums for a collection of vectors $x=(x_1,...,x_n)$ of a normed space under an optimal permutation of $(x_1,...,x_n)$. Finding explicitly a permutation π that ensures an estimation found is another part of the CVS-problem.

The CVS-problem found many applications in analysis (sum range of a conditionally convergent series; Kolmogorov Conjecture on rearrangements of orthonormal systems, etc.) CVS-problem also found applications in scheduling theory (problem of reroute sequence planning in telecommunication networks; volume calendar planning, etc.) We suggest an effective algorithmic method for finding an optimal permutation in CVS and estimation of $r(x,\pi_{\min})$.

Keywords

Compact vector summation, permutation of summands, volume calendar planning

1.THE PROBLEM OF COMPACT VECTOR SUMMATION (CVS)

Given vectors $x_1,...,x_n$ from R^d and a permutation $\pi:\{1,...,n\}\to\{1,...,n\}$ define

$$r(x,\pi)=\max_{k\le n}\|x_{\pi(1)}+...+x_{\pi(k)}\|.$$

In other words, $r(x,\pi)$ is the smallest radius of spheres with the center at the origin that contain the whole trajectory of partial sums

$$s_1=x_{\pi(1)}\ ,\quad s_2=x_{\pi(1)}+x_{\pi(2)},\ ...,\quad s_n=x_{\pi(1)}+x_{\pi(2)}+...+x_{\pi(n)}\ .$$

By an optimal permutation we mean a permutation σ such that

$$r(x,\sigma)\le r(x,\pi)$$

for any permutation π.

The problem of CVS consists in finding (or estimating) $r(x,\sigma)$ for an optimal σ. It goes back to an old paper by Steinitz, 1913 solving an analysis problem (structure of the sum range of a conditionally convergent series in a multidimensional space). Regarding the development of these very interesting problems (both CVS and analysis) see the monograph [1].

Another sound analytical problem related to CVS-type inequalities is the so called Kolmogorov conjecture on existence for any orthonormal system of a rearrangement that turns it into a system of convergence. Kolmogorov conjecture stated in early 20-s is open so far. Along with the applications to purely theoretical problems, CVS found a surprising use in scheduling theory. We mention here works of the Novosibirsk school and especially papers by Sevast'janov 1988, 1993, where e.g. the Volume Calendar Planning (VCP) was treated.

In our paper we present Sevast'anov's reduction of the VCP problem to the CVS problem, as well as our new method of solving the CVS problem. The main advantage of the method is that it allows finding effectively the desired rearrangement (permutation) of summands. The correspon-ding algorithm runs in polynomial time.

2. THE VOLUME CALENDAR PLANNING (VCP) PROBLEM

The annual plan of an enterprise consists of n items. Each item is characterized by a d-dimensional vector x_k, $k=1,...,n$. The objective is to divide the annual plan into l parts as uniformly as possible.

In [2] the following model of optimality of the plan (or of a partition $P=(N_1,...,N_l)$ of the set $\{1, 2…,n\}$ of indices by l disjoint sets $N_1,...,N_l$ has been introduced. Consider a general partition

$P=(N_1,...,N_l)$, $\bigcup_1^l N_i=\{1,2\ ,...n)$. A partition (or plan) $P^O=(N^O{}_1,...,N^O{}_l)$ is called *optimal*, if it gives minimum to the following function of P

$$E(P)=\max{}_{1\le p\le l}\|\sum_{i\in N_p}x_i\ -\frac{1}{l}S\|\ ,\ (1)$$

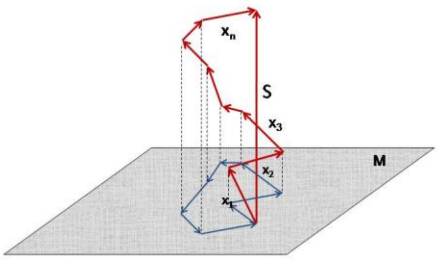
where $S=\sum_{i_p}x_i$.

Obviously, $E(P)$ can be regarded as an error, the quantity measuring the deviation from the uniformity, if a plan P is chosen. Our goal is to find an upper estimation for $E(P_O)$ and find a plan that ensures this estimation.

3. DETAILS REDUCTION TO A CVS-PROBLEM

The following steps are to be taken to reduce our VCP-problem to a CVS-problem. We use as norm $\|\cdot\|$ the usual Euclidean norm in R^d. Consider the sum $S = \sum_1^n x_i$ and denote by M the hyper-plane orthogonal to S and by x_k^* the orthogonal projection of x_k onto M, $k=1,\ldots,n$. We solve the CVS-problem for x_k^*, $k=1,\ldots,n$, i.e. the problem of finding a permutation $\pi:\{1,\ldots n\}\to\{1,\ldots n\}$ that ensures the smallness of $r(\pi, x^*)$ for the vectors x_k^*, $k=1, \ldots, n$.

An Illustration of Reduction to a CVS-problem



Denote by M_k, $k=1, \ldots, l$ the hyper-plane orthogonal to S and passing through the point $\frac{k}{l}S$; we also put formally $M_0 = M$. Consider now the following layers of R^d:

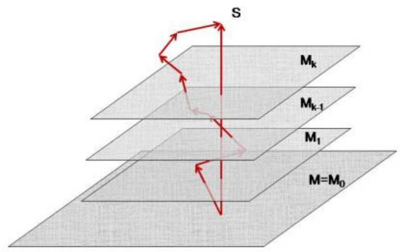
$$L_k = [M_{k-1}, M_k), \quad k=1, \ldots, l-1$$

closed from the left and open from the right, as well as the last one $L_l = [M_{l-1}, M_l]$ closed from the both sides. Finally introduce the resulting trajectory

$$S_j^\pi = \sum_1^j x_{\pi(i)}, \; j=1,\ldots, n,$$

where π is the permutation suggested by the solution to the CVS-problem. Then we form the plan $P=(N_1,\ldots,N_l)$, where N_i consists of indices of those vectors that form the fragment of the trajectory in the layer L_i, $i=1,\ldots,l$.

An Illustration of M_k, $k=1,\ldots,l-1$



To state the plan P formally let us introduce the indices $u_k = \min\{\, j: S_j^\pi \in L_k \,\},\ k=1, \ldots, l.$. Then $N_k = \{\pi(u_k), \pi(u_k+1), \ldots, \pi(u_{k+1})\},\ k=1,\ldots,l$.

Theorem 1. *The error corresponding to the constructed plan P is*

$$E(P) \le \sqrt{r^2(x^*, \pi) + \max_{1\le k\le n} \|x_k\|^2} \quad (2),$$

where $r(x^*, \pi) = \max_{1\le k\le n} \|x^*_{\pi(1)} + \ldots + x^*_{\pi(k)}\|$.

4. SOLUTION TO THE PROBLEM OF COMPACT VECTOR SUMMATION

Below we give an algorithm solving the CVS-problem which is based on the following lemma (see [3] and [4]). Given a collection of vectors $x=(x_1,\ldots,x_n)$, a permutation $\pi:\{1,\ldots,n\}\to\{1,\ldots,n\}$ and a collection of signs $\vartheta=(\vartheta_1,\ldots,\vartheta_n)$ define

$$r(x,\pi,\vartheta) = \max_{1\le k\le n} \|x_{\pi(1)}\vartheta_1 + \ldots + x_{\pi(k)}\vartheta_k\|,$$
$$r(x,\pi) = r(x,\pi,1),$$

where **1** denotes the collection consisting of +1-s only.

Lemma1 (The main lemma). *Let $(x_1,\ldots,x_n)$ be a collection of elements of a normed space X, $\sum_1^n x_i = 0$. Then for any permutation $\pi:\{1,\ldots,n\}\to\{1,\ldots,n\}$ and any collection $\vartheta=(\vartheta_1,\ldots,\vartheta_n)$ of signs $(\vartheta_i = \pm 1)$ the following inequality holds*

$$r(x,\pi) + r(x,\pi,\vartheta) \ \ge\ 2r(x,\pi^*)$$

where the permutation π^* depends only on π and ϑ, and is defined as follows: If $u_1,\ldots,u_s$ are indices for which ϑ-s are pluses, while $v_1,\ldots,v_t$ are those for which ϑ-s are minuses, then

$$\pi^* = \pi(u_1), \pi(u_2), \ldots, \pi(u_s),$$
$$\pi(v_t), \pi(v_{t-1}) \ldots, \pi(v_1)$$

(u-s and v-s are arranged in increasing order).

We also need the following lemma that states the existence of a sign algorithm.

Lemma 2. ([5], Barany, Grinberg) *If* $x_1,\dots,x_n$ *are vectors in* R^d *, then there is an algorithm defining in a polynomial time a collection of signs* $\vartheta=(\vartheta_1,\dots,\vartheta_n)$ *such that*

$$r\ (x,\vartheta)\le D, \tag{3}$$

where $D=2d\max_{1\le k\le n}\|x_k\|$.

5. ALGORITHM FOR CVS.

Lemmas 1 and 2 suggest the following algorithm for finding the desired permutation in the CVS-problem. According to Lemma 1, for any π and any $\vartheta=(\vartheta_1,\dots,\vartheta_n)$

$$r(x,\pi)+r(x,\pi,\vartheta)\ \ge\ 2\ r(x,\pi^*)\ . \tag{4}$$

On the first step we take an arbitrary π_0 and find according to Lemma 2 a ϑ^0 such that

$$r(x,\pi_0,\vartheta^0)\le D$$

and therefore

$$r(x,\pi_1)\le\frac{1}{2}\,(A+D), \tag{5}$$

where $A=r(x,\ \pi_0)$. Then we again use (4) to get

$$r(x,\pi_1)+r(x,\pi_1,\vartheta^1)\ \ge\ 2\ r(x,\pi_2)\ . \tag{6}$$

Now using (5) we obtain from (6)

$$r(x,\pi_2)\le\frac{1}{4}A+(1-\frac{1}{4})D\ ,$$

Carrying out a necessary number of iterations we come to the following assertion.

Theorem 2. *The following inequality holds true after the N-th iteration*

$$r(x,\pi_N)\le(1-\frac{1}{2^N})D+\frac{1}{2^N}A.$$

Therefore, if $N>\log_2(\frac{A}{\varepsilon})$, *then* $r(x,\pi_N)\le D+\varepsilon$.

Coming back to the VCP-problem we can give the final estimation for the error:

$$E(P)\le\max_{1\le k\le n}\|x_k\|\sqrt{4d^2+1}\,. \tag{7}$$

5. CONCLUDING REMARKS

Our main contribution is the algorithmic construction of the desired permutation in CVS based on Lemma 1. The basic idea is to reduce the rearrangement algorithm to a sign algorithm. Our Lemma 1 has been also applied by a Hungarian mathematician Makai, [6] while applying CVS to the Reroute Sequence Planning problem in telecommunication networks.

Each application of Lemma 1 to CVS is based on a sign algorithm. In this work we have used the Barany-Grinberg algorithm. Makai used the sign algorithm in l_∞^d constructed by Spencer [7].

We are going to use the method of statistical testing. Instead of using a sign algorithm we just generate sufficient amount of collections of signs $\vartheta=(\vartheta_1,\dots,\vartheta_n)$ and choose $\vartheta^{\min}$ minimizing $r(x,\vartheta)$. Then by means of our algorithm we construct a sequence (π_k) for which $r(x,\pi_k)$ decreases. Although the first results are very promising, we need a solid comparative analysis based on tail probability estimation of vector Rademacher sums.

Using a recent result by Banaszczyk [8] for the signed prob lem we can improve (7) to $E(P)\le\max_{1\le k\le n}\|x_k\|\sqrt{Cd+1}$, where C is an absolute constant. However we have to check on the algorithmic complexity of Banaszczyk's solution.

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