Consistent massive graviton on general backgrounds

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Plan

Introduction to massive and bimetric gravity

Linearised field equations around a background solution

Application to the constraint analysis

Results in different cases
Motivations and history of massive gravity

Motivations

▷ Have a better understanding of massive spin-2 fields.

▷ Explain the accelerated expansion of the Universe by a modification of GR at long distance.
Motivations and history of massive gravity

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Massive gravity : a brief historical review

▷ Fierz-Pauli linear massive gravity theory (1939),
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\[ S_{h,m} = -\frac{1}{2} \bar{M}_h^2 \int d^4 x \ h_{\mu\nu} \left[ \mathcal{E}^{\mu\nu\rho\sigma} + \frac{\bar{m}^2}{2} (\eta^{\rho\mu} \eta^{\sigma\nu} - \eta^{\mu\nu} \eta^{\rho\sigma}) \right] h_{\rho\sigma} \]

\[ \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} \equiv -\frac{1}{2} \left[ \delta_\mu^\rho \delta_\nu^\sigma \Box + \eta^{\rho\sigma} \partial_\mu \partial_\nu - \delta_\mu^\rho \partial^\sigma \partial_\nu - \delta_\nu^\rho \partial^\sigma \partial_\mu - \eta_{\mu\nu} \eta^{\rho\sigma} \Box + \eta_{\mu\nu} \partial^\rho \partial^\sigma \right] h_{\rho\sigma} \]

\[ \delta \bar{E}_{\mu\nu} \equiv \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} + \frac{\bar{m}^2}{2} (h_{\mu\nu} - h \eta_{\mu\nu}) = 0 \]

▶ Field eqs. for a massive graviton that has 5 degrees of freedom.
▶ \( \partial^\nu \delta \bar{E}_{\mu\nu} \Longrightarrow 4 \) vector constraints : \( \partial^\mu h_{\mu\nu} - \partial_\nu h = 0. \)
▶ Taking another divergence : \( 2 \partial^\mu \partial^\nu \delta \bar{E}_{\mu\nu} + \bar{m}^2 \eta^{\mu\nu} \delta \bar{E}_{\mu\nu} = -\frac{3}{2} \bar{m}^4 h. \)
▶ Scalar constraint \( h = 0. \)

▶ It is the only linear massive gravity theory free of ghost.
▶ But it needs to be generalized to a non-linear theory.
Motivations and history of massive gravity

Massive gravity : a brief historical review

▸ Fierz-Pauli linear massive gravity theory (1939),

▸ van Dam, Veltman and Zakharov (vDVZ) discontinuity (1970): FP does not recover GR in the massless limit,

▸ Vainshtein mechanism (1972): have to take into account the non-linearities,

▸ Boulware Deser (BD) ghost (1972): a ghost-like 6th dof reappears in any non-linear massive gravity theory,

▸ de Rham, Gabadadze and Tolley (dRGT) theory (2011): non-linear theory free of the BD ghost.
The dRGT massive gravity theory [de Rham, Gabadadze, Tolley, 2010]

\[ S = M_g^2 \int d^4x \sqrt{|g|} \left[ R(g) - 2m^2V(S; \beta_n) \right], \]

\[ V(S; \beta_n) = \sum_{n=0}^{3} \beta_n e_n(S), \]

\[ \text{\large∇} \text{ Square-root matrix } S^\mu_\nu = \left[ \sqrt{g^{-1}f} \right]^\mu_\nu, \]

\[ \text{\large∇} \text{ } e_n(S) \text{ elementary symmetric polynomials:} \]

\[ e_0(S) = 1, \quad e_1(S) = \text{Tr}[S], \quad e_2(S) = \frac{1}{2} \left( \text{Tr}[S]^2 - \text{Tr}[S^2] \right), \]

\[ e_3(S) = \frac{1}{6} \left( \text{Tr}[S]^3 - 3\text{Tr}[S]\text{Tr}[S^2] + 2\text{Tr}[S^3] \right), \]

\[ \text{\large∇} \text{ No BD ghost.} \]
Bimetric theory (1) [Hassan, Rosen 2012]

\[
S = M_g^2 \int d^4 x \left[ \sqrt{|g|} R(g) + \alpha^2 \sqrt{|f|} R(f) - 2m^2 \sqrt{|g|} V(S; \beta_n) \right],
\]

- \( V(S; \beta_n) = \sum_{n=0}^{4} \beta_n e_n(S) \),
- Square-root matrix \( S^\mu_{\nu} = \left( \sqrt{g^{-1} f} \right)^\mu_{\nu} \),
- \( e_n(S) \) elementary symmetric polynomials:
  \[
e_0(S) = 1, \quad e_1(S) = \text{Tr}[S], \quad e_2(S) = \frac{1}{2} \left( \text{Tr}[S]^2 - \text{Tr}[S^2] \right), \quad e_3(S) = \frac{1}{6} \left( \text{Tr}[S]^3 - 3 \text{Tr}[S] \text{Tr}[S^2] + 2 \text{Tr}[S^3] \right), \quad e_4(S) = \det(S).
  \]
- No BD ghost.
- Interchange symmetry:
  \[
  \alpha^{-1} g_{\mu\nu} \leftrightarrow \alpha f_{\mu\nu}, \quad \alpha^{4-n} \beta_n \leftrightarrow \alpha^n \beta_{4-n}.
  \]
Bimetric theory (2)

\[ S = M_g^2 \int d^4 x \left[ \sqrt{|g|} R(g) + \alpha^2 \sqrt{|f|} R(f) - 2m^2 \sqrt{|g|} V(S; \beta_n) \right], \]

\[ V(S; \beta_n) = \sum_{n=0}^4 \beta_n e_n(S) \text{ and } S^{\mu}_{\nu} = [\sqrt{g^{-1}f}]^{\mu}_{\nu}. \]

Field equations

\[
\begin{cases}
E_{\mu\nu} \equiv G_{\mu\nu} + m^2 V_{\mu\nu} = 0, \\
\tilde{E}_{\mu\nu} \equiv \tilde{G}_{\mu\nu} + \frac{m^2}{\alpha^2} \tilde{V}_{\mu\nu} = 0.
\end{cases}
\]

\[
V_{\mu\nu} = g_{\mu\rho} \sum_{n=0}^3 (-1)^n \beta_n \sum_{k=0}^n (-1)^k e_k(S) [S^{n-k}]_{\rho}\nu, \\
\tilde{V}_{\mu\nu} = f_{\mu\rho} \sum_{n=0}^3 (-1)^n \beta_{4-n} \sum_{k=0}^n (-1)^k e_k(S^{-1}) [S^{k-n}]_{\rho}\nu.
\]
Linearised field equations around a background solution

Metrics expansion: $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, $f_{\mu\nu} \rightarrow f_{\mu\nu} + \delta f_{\mu\nu}$

\[
\begin{align*}
\delta E_{\mu\nu} & \equiv \delta G_{\mu\nu} + m^2 \delta V_{\mu\nu} = 0, \\
\delta \tilde{E}_{\mu\nu} & \equiv \delta \tilde{G}_{\mu\nu} + \frac{m^2}{\alpha^2} \delta \tilde{V}_{\mu\nu} = 0
\end{align*}
\]

\[
\delta G_{\mu\nu} = -\frac{1}{2} \left[ \delta^\rho_{\mu} \delta^\sigma_{\nu} \nabla^2 + g^\rho\sigma \nabla_\mu \nabla_\nu - \delta^\rho_{\mu} \nabla^\sigma \nabla_\nu - \delta^\rho_{\nu} \nabla^\sigma \nabla_\mu - g_{\mu\nu} g^\rho\sigma \nabla^2 \\
+ g_{\mu\nu} \nabla^\rho \nabla^\sigma + g_{\mu\nu} R^\rho\sigma - \delta^\rho_{\mu} \delta^\sigma_{\nu} R \right] \delta g_{\rho\sigma},
\]

\[
\delta V_{\mu\nu} = g^\rho\sigma V_{\sigma\nu} \delta g_{\mu\rho} \\
- g_{\mu\rho} \sum_{n=1}^{3} (-1)^n \beta_n \sum_{k=1}^{n} (-1)^k \left\{ \frac{1}{2} \left[ S^{n-k} \right]^\rho_{\nu} \sum_{m=1}^{k} (-1)^m e_{k-m}(S) \left[ S^{m-2} \delta S^2 \right]^\sigma_{\sigma} \\
+ e_{k-1}(S) \sum_{m=0}^{n-k} \left[ S^m \delta S S^{n-k-m} \right]^\rho_{\nu} \right\}.
\]
Linearised field equations around a background solution

Metrics expansion: \( g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \ f_{\mu\nu} \rightarrow f_{\mu\nu} + \delta f_{\mu\nu} \)

\[
\left\{
\begin{align*}
\delta E_{\mu\nu} \equiv & \quad \delta G_{\mu\nu} + m^2 \delta V_{\mu\nu} = 0 , \\
\delta \tilde{E}_{\mu\nu} \equiv & \quad \delta \tilde{G}_{\mu\nu} + \frac{m^2}{\alpha^2} \delta \tilde{V}_{\mu\nu} = 0
\end{align*}
\right.
\]

\[
\delta G_{\mu\nu} = -\frac{1}{2} \left[ \delta^\rho_\mu \delta^\sigma_\nu \nabla^2 + g^{\rho\sigma} \nabla_\mu \nabla_\nu - \delta^\rho_\mu \nabla^\sigma \nabla_\nu - \delta^\rho_\nu \nabla^\sigma \nabla_\mu - g_{\mu\nu} g^{\rho\sigma} \nabla^2 \\
+ g_{\mu\nu} \nabla^\rho \nabla^\sigma + g_{\mu\nu} R^{\rho\sigma} - \delta^\rho_\mu \delta^\sigma_\nu R \right] \delta g_{\rho\sigma},
\]

\[
\delta V_{\mu\nu} = g^{\rho\sigma} V_{\sigma\nu} \delta g_{\mu\rho}
\]

\[
- g_{\mu\rho} \sum_{n=1}^{3} (-1)^n \beta_n \sum_{k=1}^{n} (-1)^k \left\{ \frac{1}{2} [S^{m-k}]^\rho_\nu \sum_{m=1}^{k} (-1)^m e_{k-m}(S') [S^{m-2} \delta S^2]^\sigma_\sigma \\
+ e_{k-1}(S) \sum_{m=0}^{n-k} [S^m \delta SS^{n-k-m}]^\rho_\nu \right\}.
\]
Method 1: Variation of the matrix $S$

To linearised the field equations we first need to obtain the perturbed matrix $S$.

**Sylvester equation:** $AX - XB = C$

$$S_{\mu \nu} (\delta S)^{\nu}_{\sigma} + (\delta S)^{\mu}_{\nu} S^{\nu}_{\sigma} = \delta[S^2]^{\mu}_{\sigma}.$$

- **Unique explicit solution for** $\delta S$ **iff** $S$ and $-S$ do not have common eigenvalues $\iff \mathbb{X} \equiv e_3 \mathbb{1} + e_1 S^2$ is invertible.

$$\delta S = \frac{1}{2} \mathbb{X}^{-1} \sum_{k=1}^{4} \sum_{m=0}^{k-1} (-1)^m e_{4-k}(S) S^{k-m-2} \delta S^2 S^m,$$

- $\delta S$ contains more than 30 terms in massive gravity (60 in bimetric gravity)!
Method 2: Redefined fluctuation variables

Redefinition of the perturbation variable

\[ \delta g_{\mu\nu} = (\delta^\beta_{\mu} S^\lambda_\nu + \delta^\beta_{\nu} S^\lambda_\mu) \delta g'_{\beta\lambda} \]
\[ \delta f_{\mu\nu} = (\delta^\beta_{\mu} [S^{-1}]^\lambda_\nu + \delta^\beta_{\nu} [S^{-1}]^\lambda_\mu) \delta f'_{\beta\lambda} \]

We can express all other variation of variables as a function of \( \delta g'_{\beta\lambda} \) and \( \delta f'_{\beta\lambda} \).
Method 2: Redefined fluctuation variables

Redefinition of the perturbation variable

\[
\delta g_{\mu\nu} = (\delta^\beta_{\mu} S^\lambda_{\nu} + \delta^\beta_{\nu} S^\lambda_{\mu}) \delta g'_{\beta\lambda} \\
\delta f_{\mu\nu} = (\delta^\beta_{\mu} [S^{-1}]^\lambda_{\nu} + \delta^\beta_{\nu} [S^{-1}]^\lambda_{\mu}) \delta f'_{\beta\lambda}
\]

We can express all other variation of variables as a function of \(\delta g'_{\beta\lambda}\) and \(\delta f'_{\beta\lambda}\).

Invertibility of the definitions

\[
g^{-1} \delta g = S g^{-1} \delta g' + g^{-1} \delta g' S
\]

Sylvester equation: unique solution for \(g^{-1} \delta g'\) iff \(S\) and \(-S\) do not have common eigenvalues.

\[
\delta S = -g^{-1} \delta g' S^2 + S^{-1} g^{-1} \delta f' S^{-1}: \text{only two terms in } \delta S.
\]
Search for a scalar constraint

Counting the degrees of freedom

- 4 vector constraints: $\nabla^{\nu} \delta E_{\mu\nu} = 0$
- Scalar constraint: unlike in the F-P theory, it cannot be obtained from a linear combination of $g^{\mu\nu} \delta E_{\mu\nu}$ and $\nabla^{\mu} \nabla^{\nu} \delta E_{\mu\nu}$.
Search for a scalar constraint

Counting the degrees of freedom

- 4 vector constraints: $\nabla^\nu \delta E_{\mu\nu} = 0$
- Scalar constraint: unlike in the F-P theory, it cannot be obtained from a linear combination of $g^{\mu\nu}\delta E_{\mu\nu}$ and $\nabla^\mu\nabla^\nu\delta E_{\mu\nu}$.

Generalised traces and divergences of the field equations

1. We define all possible ways of tracing $\delta E_{\mu\nu}$ with $S^\mu_{\mu}$:

\[
\Phi^{(g,f)}_i \equiv [S_i]^{\mu\nu} \delta E^{(g,f)}_{\mu\nu}, \quad 0 \leq i \leq 3
\]

\[
\Psi^{(g,f)}_i \equiv [S_i]^{\mu\nu} \nabla_\nu \nabla^\lambda \delta E^{(g,f)}_{\lambda\mu}, \quad 0 \leq i \leq 3.
\]

2. Find a linear combination of these 16 scalars for which the 2nd derivative terms vanish:

\[
\sum_{i=0}^{3} \left( u_i \Phi^{(g)}_i + v_i \Psi^{(g)}_i \right) + \sum_{i=0}^{3} \left( U_i \Phi^{(f)}_i + V_i \Psi^{(f)}_i \right) \sim 0,
\]
A particular case in massive gravity: the beta 1 model

We assume $\beta_2 = \beta_3 = 0$ and $f_{\mu\nu}$ arbitrary but non-dynamical.

Field equations

$$G_{\mu\nu} + m^2 \left[ \beta_0 g_{\mu\nu} + \beta_1 g_{\mu\rho} \left( e_1(S) \delta^\rho_\nu - S^\rho_\nu \right) \right] = 0,$$

It can be solved for $S^\mu_\nu$:

$$S^\rho_\nu = \frac{1}{\beta_1 m^2} \left[ R^\rho_\nu - \frac{1}{6} \delta^\rho_\nu R - \frac{m^2 \beta_0}{3} \delta^\rho_\nu \right].$$

- It is only possible in the $\beta_1$ model.
- It can be used to eliminate any occurrences of $S$ (or $f$) in the linearised field equations.
A particular case: the beta 1 model

▷ In the $\beta_1$ model, we can express the linearised field equations as a function of $g_{\mu\nu}$ and its curvature.

▷ We now take these equations as our starting point, no more assuming that $g_{\mu\nu}$ is a background solution.
In the $\beta_1$ model, we can express the linearised field equations as a function of $g_{\mu\nu}$ and its curvature.

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The fifth scalar constraint

$$-\frac{m^2 \beta_1 e_4}{2} \Phi_0 - e_3 \Psi_0 + e_2 \Psi_1 - e_1 \Psi_2 + \Psi_3 = 0.$$ 

Massive graviton (with 5 dof) propagating on a single arbitrary background.
Beyond the beta 1 model: general massive gravity

$$\bar{\Psi} = [S^{-1}]^{\mu \nu} \nabla_\nu \nabla^\lambda \delta E_{\lambda \mu} = \frac{1}{e_4} (e_3 \Psi_0 - e_2 \Psi_1 + e_1 \Psi_2 - \Psi_3).$$

1. $\beta_3 = 0$

$$\frac{m^2}{2} \beta_1 \Phi_0 + m^2 \beta_2 \Phi_1 + \bar{\Psi} = 0.$$
Beyond the beta 1 model: general massive gravity

\[ \overline{\Psi} = [S^{-1}]^{\mu\nu} \nabla_\nu \nabla^\lambda \delta E_{\lambda\mu} = \frac{1}{e_4} (e_3 \Psi_0 - e_2 \Psi_1 + e_1 \Psi_2 - \Psi_3) . \]

1. \( \beta_3 = 0 \)

\[ \frac{m^2 \beta_1}{2} \Phi_0 + m^2 \beta_2 \Phi_1 + \overline{\Psi} = 0 . \]

2. \( \beta_3 \neq 0 \)

\[ \frac{m^2 \beta_1}{2} \Phi_0 + m^2 \beta_2 \Phi_1 - m^2 \beta_3 (\Phi_2 - e_1 \Phi_1 + \frac{1}{2} e_2 \Phi_0) + \overline{\Psi} \]

\[ \sim m^2 \beta_3 (S^{\mu\lambda} [S^2]^\nu_\beta - S^{\mu\nu} [S^2]^\beta_\lambda) \nabla_\mu \nabla_\nu \delta g'_{\beta\lambda} . \]

- It is not a covariant constraint but all the second time derivatives acting on the lapse and shifts vanish.
Going back to bimetric theory

- Problematic terms: 
  \[ [S^{-1}]^\nu_\kappa \nabla^\kappa \nabla^\mu \delta_f V_{\mu\nu} + [S]^\nu_\kappa \tilde{\nabla}^\kappa \tilde{\nabla}^\mu \delta_g \tilde{V}_{\mu\nu}, \]

- Repeating the same analysis using the interchange symmetry, we could not find a covariant constraint.

- Performing a 3+1 decomposition \(\Rightarrow\) all the 2nd-time derivative acting on the lapse or shifts in the problematic terms disappear.

\[ \frac{m^2}{2} \beta_1 \Phi_0^{(g)} + m^2 \beta_2 \Phi_1^{(g)} - m^2 \beta_3 \left( \Phi_2^{(g)} - e_1 \Phi_1^{(g)} + \frac{1}{2} e_2 \Phi_0^{(g)} \right) + \bar{\Psi}^{(g)} + \text{(interchange symmetry } g \leftrightarrow f) = 0 \]
Conclusion

- Linearised equations of bimetric theory in the general case,

- Condition for linearisation: $S$ and $-S$ do not have common eigenvalues.

- Consistent theory for a massive graviton propagating in a single arbitrary background metric ($\beta_1$ model).

- Five covariant constraints in the metric formulation of massive gravity, when $\beta_3 = 0$.

- Non-covariant scalar constraint when $\beta_3 \neq 0$ and in bimetric theories.

- Interesting applications: cosmological pertubations, Higuchi bound, partial masslessness...