Transient dynamics of vortices in relativistic regions of accretion discs around black holes

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Outline

- **Context**
  i) Turbulence in discs: its driver is linear
  ii) Non-modal growth of perturbations and its species
  iii) Fusion of HD and MHD cases

- **Motivation**
  i) Optimals beyond the sound horizon

- **Model and Results**
  i) Basic and adjoint equations
  ii) $G_{\text{max}}(k_y)$
  iii) Some analytics and $Re_T(r)$

- **Conclusions**
What makes disc accreting?

- Laminar differentially rotating flow with \( \frac{dL}{dr} > 0 \) and \( \frac{d\Omega}{dr} < 0 \) is called quasi-Keplerian (or 'anticyclonic') shear flow.

Keplerian:
\[ \Omega \sim r^{-3/2}, \quad L \sim r^{1/2}. \]

Schwarzschild:
\[ \Omega \sim r^{-3/2}, \quad L \sim r^{1/2}/(1 - r_g/r). \]

- Accretion time is \( t_\nu \sim \frac{r^2}{\nu} = \frac{(10^{10} \text{ cm})^2}{10^5 \text{ cm}^2 \text{ c}^{-1}} > 10^7 \text{ yr}. \)

⇒ disc must acquire strongly enhanced effective viscosity via turbulent motions: \( \langle \delta v_r \delta v_\phi \rangle \sim \nu_t r \frac{d\Omega}{dr}. \)
Turbulence is a path of rapid energy conversion from large-scale fluid motions (with $\lambda < \text{and/or} > h$?) into the heat by means of non-linear interactions.

- The question: How do perturbations with the largest $\lambda$’s extract energy from the shear?

- The answer: This must be a linear process anyway.
  Strictly proved for Navier-Stockes equations — the Reynolds-Orr equation for $dE/dt$ has identical forms for both linear and finite-amplitude perturbations.
  Henningson & Reddy (1994)

- The objective: Foremost, one must look for any type of small growing perturbations solving a linear problem

$$\frac{\partial q}{\partial t} = Aq \quad \Rightarrow \quad q(t) = U(t)q_0, \quad U \equiv e^{At}$$
What feeds turbulence?

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\]

Henceforth, let us consider a **small** patch of disc with size \( L \ll r \) filled with homogeneous \((\rho, s, p = \text{const})\) matter with \( \nu \neq 0 \): the only inhomogeneity will be the **shear** (shearing sheet approximation).
Growing linear modes \( Uq = e^{-i\omega t}q \)?

**HD case**
Rayleigh’s criterion,
\[ \Im[\omega] \leq 0 \quad \text{since} \quad dL/dr > 0 \]

...and \( d\Omega/dr < 0 \): **Edlund & Ji (2014)**

The absence of subcritical transition up to \( Re = 10^6 \) ... (but actual \( Re \geq 10^{10} \))

**MHD case**
MRI with \( B_z \neq 0 \),
\[ \Im[\omega] > 0 \quad \text{as} \quad 2r\Omega d\Omega/dr < -(k_zv_A)^2 \]

Vigorous supercritical transition and normalised azimuthal stress \( \alpha \sim 0.001 \div 0.1 \).

FIG. 1. (color online) Drawing of the HTX device illustrating the segmented axial boundaries: the inner-most element (yellow) is connected to the inner cylinder (black), the -most element (green) is connected to the outer cylinder (green) and the middle components (blue) are rings that can rotate differentially with respect to the cylinders.
### HD case

- Subcritical transition in the Taylor-Couette experiment: ‘cyclonic’ rotating shear flow with \( \frac{dL}{dr} > 0 \) and \( \frac{d\Omega}{dr} > 0 \).

  e.g. [Taylor (1936)](#) himself, [Burin & Czarnocki (2012)](#) recent.

- Subcritical transition in non-rotating shear flows, e.g. plane Couette & Poiseille flows.

  see book by [Schmid & Henningson (2001)](#).

### MHD case

- Subcritical dynamo in Keplerian flow: transition to turbulence from zero net magnetic flux initial state.

  see [Rincon et al. (2008)](#), [Riols et al. (2013)](#) and e.g. [Shi et al. (2015)](#) recent.

- Subcritical transition in non-rotating magnetised shear flow

  [Mamatsashvili et al. (2014)](#).

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**Overall**: ’non-linear’ character of the transition: \( Re_T \) depends on the amplitude of the initial perturbations.
Nonmodal linear growth  \( q_{n+1} = U^\dagger U \cdot q_n \)

- The question: What linear perturbations extract energy from the flow while all modes are damping?

- The answer: The existence of a non-zero shear makes dynamical operator \( U \) non-normal, i.e. \( UU^\dagger \neq U^\dagger U \), and modes become non-orthogonal making possible the transient growth of their combinations at finite time intervals.

- The objective: The most growing transient perturbation is the 1st singular vector of \( U \) which satisfies

\[
U^\dagger Uq_{opt} = Gq_{opt}
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with \( G = G(t) \) called the optimal growth and being the highest eigenvalue of \( U^\dagger U \).
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Since $U^\dagger U$ is positive-definite Hermitian operator, $q_{opt}$ can be found by power iterations (Luchini (2000)).
Iterative scheme

- Launched from an arbitrary initial state vector $q_0(t = 0)$,

$$ q_{n+1} = U^\dagger U \cdot q_n, $$

converges to an optimal initial state vector, $q_{opt}(t = 0)$, as $n \to \infty$.

- Then, $q_{opt}(t = 0)$ grows by factor $G(t)$ up to time $t$. 
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- Since \( U^\dagger = e^{A^\dagger t} \) each iteration is equivalent to the advance of \( q_n \) forward in time solving the basic equations,
  \[
  \dot{q} = A q,
  \]
  and, subsequently, backward in time solving the adjoint equations,
  \[
  \dot{q} = -A^\dagger q.
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...to be written explicitly few slides below (Zhuravlev & Razdoburdiin (2014)).
A simple example of transient growth

Schmid (2007)
Two kinds of transient growth — two scenarios of non-linear feedback

Lift up amplification of streamwise rolls

- Subcritical transition in non-rotating HD shear flows.
- Subcritical dynamo in Keplerian flow.

Swing amplification of shearing vortices

- Subcritical transition in 2D rotating and non-rotating HD and MHD shear flows.
- Subcritical transition in 'cyclonic' and 'anticyclonic' (quasi-Keplerian) HD rotating shear flows?

Waleffe (1997)

Horton et al. (2010)
Transient growth vs. exponential growth

What variant of linear growth feeds the supercritical MHD-turbulence in Keplerian disc?

Squire & Bhattacharjee (2014)

- Transient growth dominates!
- Has a distinct 3D structure (lift up?) but strongly resembles shearing vortices (swing?)
In this study we focus on the HD case only.

Since shearing vortices are mostly columnar structures (Maretzke et al. (2014)), we consider 2D perturbations only.

Regard relativistic corrections to shear motion in a simple Newtonian framework: the Paczynski-Wiita grav. potential $\Phi \sim -1/(r - r_g)$.

In order to reproduce $\Omega(r)$ & $\kappa(r)$ in the shearing sheet model, we enter the profile $q(r)$, where $q$ is constant shear rate entering equations for perturbations: $q = 1/2 + r/(r - r_g)$.

(Here is the region, where standard relativistic disc emits most of its energy)
Motivation

Mukhopadhyay et al. (2005)

- Incompressible dynamics
- Computational domain confined by walls with no-slip boundary conditions
- ⇒ Not quite adequate normalisation of the problem: $k$ is given in units of the inversed domain size, $L^{-1}$.

Contours of $\max\{G(t)\}$ for $Re=2000$. 

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Transient dynamics in relativistic discs
Shearing sheet equations for transient growth

- Define the Cartesian coordinates \( r - r_0 \rightarrow x, \ r(\varphi - \varphi_0) \rightarrow y \) and write equations for \( u_x, u_y \) & \( W \).

- Change to the dimensionless shearing coordinates

\[
x' = \Omega x/a, \ y' = \Omega(y + q\Omega x t)/a, \ t' = \Omega t,
\]

\( \Rightarrow \) the problem is reduced to the set of ODE's for a single Fourier harmonics of shearing vortex

\[
f = \hat{f}(k_x, k_y, t) \exp(i\tilde{k}_x x + ik_y y)
\]

(see e.g. Heinemann & Papaloizou (2009)).

- \( \tilde{k}_x \equiv k_x + k_y qt. \)

\( k_x, k_y \) and \( \hat{u}_x, \hat{u}_y \) are measured in units of \( H^{-1} = \Omega/a \) and \( a \), respectively. \( \hat{W} \) is measured in units of \( a^2 \).

- As a norm for shearing vortex we take its spatially averaged acoustic energy what leads to the following expression for Fourier amplitudes:

\[
||q||^2 = \frac{1}{2} \left( |\hat{u}_x|^2 + |\hat{u}_y|^2 + |\hat{W}|^2 \right)
\]
The basic and the adjoint equations

\[
\begin{align*}
\frac{d\hat{u}_x}{dt} &= 2\hat{u}_y - i \tilde{k}_x \hat{W} - Re^{-1}(\tilde{k}_x^2 + k_y^2)\hat{u}_x - f_\varsigma, \\
\frac{d\hat{u}_y}{dt} &= -(2 - q)\hat{u}_x - i k_y \hat{W} - Re^{-1}(\tilde{k}_x^2 + k_y^2)\hat{u}_y - g_\varsigma, \\
\frac{d\hat{W}}{dt} &= -i (\tilde{k}_x \hat{u}_x + k_y \hat{u}_y).
\end{align*}
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\frac{d\hat{W}}{dt} &= -i (\tilde{k}_x \hat{u}_x + k_y \hat{u}_y).
\end{align*}
\]
Shearing sheet equations for transient growth

We check that viscous terms \( \propto \text{div}\mathbf{u} \):

\[
\begin{align*}
f_\zeta &= (Re_b^{-1} + Re^{-1}/3)\tilde{k}_x[\tilde{k}_x \hat{u}_x + k_y \hat{u}_y], \\
g_\zeta &= (Re_b^{-1} + Re^{-1}/3)k_y[\tilde{k}_x \hat{u}_x + k_y \hat{u}_y]
\end{align*}
\]

suppress additional excitation of shearing sound waves (see Heinemann & Papaloizou (2009)) as the shearing vortex swings (i.e. as \( \tilde{k}_x = 0 \)) but do not affect the transient growth of vortex almost at all.
Incompressible unbounded shear

i) Hold the restriction $\tilde{k}_x \hat{u}_x + k_y u_y = 0$ (normalisation of $k_{x,y}$ by arbitrary $L^{-1}$) but remove the boundaries:

Mukhopadhyay et al. (2005)

Our calculations

\[
G_{\text{max}}(k_y) \equiv \max \{ G(t, k_y) \} \quad \text{and corresponding } t_{\text{max}}(k_y) \quad \text{for } q = 1.5 \text{ and } Re = 2000.
\]
ii) Relax $\tilde{k}_x \hat{u}_x + k_y u_y = 0$ (normalisation of $k_{x,y}$ by $H^{-1}$):

$$G_{\text{max}}(k_y) \equiv \max\{G(t, k_y)\} \text{ for } Re = 2000$$

(sound shearing waves are suppressed by large bulk viscosity)

**Light red:** Transition is ruled out by correlation of $G_{\text{max}}$ with $Re_T$ in Taylor-Couette experiments *Meseguer (2002).*

**Light green:** HD simulations in the Keplerian flow *Shen et al. (2006).*
i) It follows from the basic equations that

\[ \frac{d^2 \hat{u}_y}{dt^2} + \omega^2 \hat{u}_y = \tilde{k}_x l, \]

where \( \omega^2 \equiv \kappa^2 + \tilde{k}_x^2, \) \( \kappa^2 = 2(2 - q), \)

\[ l \equiv \tilde{k}_x \hat{u}_y - k_y \hat{u}_x + i(2 - q) \hat{W} = \text{const} \] — the Fourier harmonics of the potential vorticity (Bodo et. al 2005, Heinemann & Papaloizou (2009)).

ii) The vortex solution, when \( \frac{d^2 \hat{u}_y}{dt^2} \), gives:

\[ \hat{u}_y^{(\text{vortex})} = \frac{\tilde{k}_x}{k_y^2 + \tilde{k}_x^2 + \kappa^2} l. \]
iii) In the short wavelength limit, \( k_y \gg 1 \), it follows
\[
G_1 \approx (qt)^2 \quad \text{(see Afshordi et. al (2005)).}
\]

iv) For long wavelengths \( k_y \ll 1 \), however,
\[
G_2 \approx \frac{4}{\kappa^4} (q k_y t)^2 \quad \text{(see Zhuravlev & Razdoburden (2014)).}
\]

v) \( t_{\text{max}} \): Viscous dissipation time \( \sim \) TG time is
\[
t_{\text{max}} \approx (Re)^{1/3} (q k_y)^{-2/3}.
\]

vi) For \( k_y \gg 1 \)
\[
G_{\text{max}} \approx (q Re)^{2/3} k_y^{-4/3},
\]

vii) For \( k_y \ll 1 \)
\[
G_{\text{max}} \approx (2 - q)^{-2} (q Re k_y)^{2/3}.
\]
i) Assume that $Re_T \sim 10^7$ for Keplerian shear.

ii) Then, if $G_{\text{max}}(Re_T) = \text{const}$, we find:

$$Re_T(q) \sim 10^8 \frac{(2 - q)^3}{q}.$$
Prominent transient growth of large-scale shearing vortices is found in relativistic regions of accretion discs. (may also be important for non-stationary appearance of turbulent discs)

Whether it leads to turbulence at lower $Re_T$, must be checked in HD simulations with $L > H$.

If, probably, $\alpha \propto Re_T^{-1}$ (Lesur & Longaretti (2005)), a steep dependence of $\alpha(r)$ for non-magnetic accretion follows.

If actual Reynolds number $Re = Re_T(r)$ at some moment somewhere in a disc this can lead to outer zone free of turbulence akin dead zones in protoplanetary discs (Gammie (1996)).

The present result must be checked for magnetized accretion. If it remains valid, this could become an alternative to MRI explanation of $\alpha(r)$ revealed previously in MHD-turbulence simulations (Abramowicz et al. (1996), Penna et al. (2013)).
Thank you