**APPROXIMATION OF RELEVANT ELLIPTIC INTEGRALS IN THE SCHWARZSCHILD METRIC AND SOME ASTROPHYSICAL APPLICATIONS**

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Discovery of X-ray emission coming from the accretion disks around the black holes

Study how the matter around the black holes appears to an observer at infinity

First simulations

Luminet 1979

Image of a black hole

RAY-TRACING METHOD
INTRODUCTION

Numerical simulation of an accretion disk placed around a black hole
1.1 Geometrical structure

TARGETS

- **LIGHT BENDING** (*elliptic integral*)
- **TRAVEL TIME** (*elliptic integral*)
- **GRAVITATIONAL LENSING** (*elliptic integral*)
- **GRAVITATIONAL REDSHIFT** (exact)
- **FLUX** (derived)

\[
g = (1 + z) = \frac{E_{obs}}{E_{em}} = \frac{(V^\alpha k_\alpha)_{obs}}{(U^\alpha k_\alpha)_{em}}
\]

\[
F = \int \int R \int_\varphi I_{\nu_0} d\nu_0 d\Omega
\]
1.2 Simulations of the gravitational effects

Numerical simulations showing the gravitational effects mentioned before:

The relativistic effects become stronger and stronger
ELLiptic Integrals

Kandisky - Composition VIII (1923)
2.1 Light bending

\[
\theta = \int_R^\infty \frac{b}{r^2} \left[ 1 - \frac{b^2}{r^2} \left( 1 - \frac{2M}{r} \right) \right]^{-\frac{1}{2}} \, dr
\]

\text{Substitution ad hoc}

\[ z = 1 - \cos \alpha \]

\[ b = \frac{r \sin \alpha}{\sqrt{1 - \frac{2M}{r}}} \]  

\text{Mathematical algebra}

\[ (1 - \cos \theta)(1 - u) = 1 - \cos \alpha \]

where \( u = \frac{r_s}{r} \)

\textit{Beloborodov (2002)}

\textit{Misner, Thorne & Wheeler - Gravitation}
2.2 Time delay

\[ \Delta t = \int_R^\infty \frac{dr}{(1 - \frac{2M}{r}) \left[ 1 - \frac{b^2}{r^2} \left( 1 - \frac{2M}{r} \right) \right]^{\frac{1}{2}}} - 1 \]

\textit{Substitution ad hoc}

\[ z = 1 - \cos \alpha \]

\[ b = \frac{r \sin \alpha}{\sqrt{1 - \frac{2M}{r}}} \]  (*)

\textit{Mathematical algebra}

\[ \frac{\Delta t}{R} = y \left[ 1 + \frac{uy}{8} + \frac{uy^2}{24} - \frac{u^2y^2}{112} \right] \]

where \[ u = \frac{r_s}{r} \quad y = \frac{(1 - \cos \alpha)}{(1 - u)} \]
2.3 Solid angle

\[ d\Omega = \frac{\cos i}{D^2 \sin^2 \theta \left(1 - \frac{2M}{R}\right)} \frac{\sin^2 \alpha}{\cos \alpha} \left\{ \int_R^\infty \frac{dr}{r^2} \left[ 1 - \frac{b^2}{r^2} \left(1 - \frac{2M}{r}\right) \right]^{-\frac{3}{2}} \right\}^{-1} \]

(*) Misner, Thorne & Wheeler - Gravitation

no approximated equation found so far
CHAPTER 3

MATHEMATICAL METHOD

M. C. Escher - Relativity (1953)
3.1 Method

- Approximate the elliptic integrals with polynomial equations

\[
\int f(x) \, dx \quad \Rightarrow \quad P(x)
\]

\textit{elliptic function} \quad \textit{polynomial}

- Get rid of the square root present in the integral

\[
\sin \alpha = g(z) \quad \Rightarrow \quad z = z(\alpha)
\]

\textit{general function}

- We assume that \( \alpha \) is small, so we have:

\[
g(z) = \sin \alpha \approx \alpha \approx 0 \quad \Rightarrow \quad \text{expand in } \textit{Taylor series} \text{ the integrand } f(x)
\]
3.2 Light bending
Having even powers of $g(z)$ and researching a complete polynomial function, we choose:

$$g(z) = \sqrt{Az^2 + Bz}$$
• To find the approximation we compare the approximation with the original form in a particular computing convenience case. We choose u=0 and R=1, so we have:

\[
\psi = b \int_{R}^{\infty} \frac{d\theta}{\theta} \left[ 1 - \frac{R^2 \sin^2 \theta}{\theta^2 (1-u)} \right]^{1/2} = b \int_{1}^{\infty} \frac{d\theta}{\theta} \left[ 1 - \frac{\sin^2 \theta}{\theta^2} \right]^{1/2} = b \int_{1}^{\infty} \frac{\theta \, d\theta}{\theta \left( \theta^2 + \sin^2 \theta \right)} = \frac{b}{\sin \theta} \left| \frac{\theta^2}{\sin \theta} + 1 \right| = \frac{\theta}{2} - \text{odd} \frac{\theta \, d\theta}{\sin \theta} \quad (\text{odd} \theta) \Rightarrow \text{odd} \theta \frac{\pi}{2} = \theta
\]

• We have to get rid of the square root with an even trigonometric function, so we have:

\[
1 - \cos \psi \approx \frac{\theta^2}{2} - \frac{\theta^4}{24} \approx \left[ \frac{A Z^2 + B \hat{Z}^2}{2 (1-u)} \right] \left[ 1 + C^2 Z^2 + 2 C \dot{Z} + 2 D \dot{Z}^2 + 2 CD \dot{Z}^3 \right] - \frac{1}{24} \left[ \frac{B^2 Z^2 + 2 A B Z^3}{(1-u)^2} \right] \left[ 1 + C^2 Z^2 + 2 C \dot{Z} + 2 D \dot{Z}^2 + 2 CD \dot{Z}^3 \right] \approx \frac{B^2 Z^2}{2 (1-u)} \left[ 1 + 2 C \dot{Z} + (C^2 + 2 D) \dot{Z}^2 \right] + \frac{A Z^2}{2 (1-u)} \left[ 1 + 2 C \dot{Z} \right] - \frac{1}{24} \frac{A B \dot{Z}^3}{(1-u)^2} - \frac{1}{24} \frac{B^2 Z^2}{(1-u)^2} \left[ 1 + 2 C \dot{Z} \right] = \frac{r}{b} = \frac{\rho}{m}
\]
\[ A = -\left(\frac{B}{2}\right)^2 \]

- Testing in the particular case \((u=0 \text{ and } R=1)\) we have: \(1 - \cos \alpha = \frac{Bz}{2}\)

- Therefore choosing \(B = 2 \text{ and } A = -1\), it implies that \(z = 1-\cos\alpha\). The final approximation is:

\[
(1 - \cos \theta)(1 - u) = 1 - \cos \alpha
\]
3.3 Time delay

\[
\Delta t = \int_R^{\infty} \frac{dr}{(1 - \frac{2M}{r})} \left\{ \left[ 1 - \frac{b^2}{r^2} \left( 1 - \frac{2M}{r} \right) \right]^{-\frac{1}{2}} - 1 \right\}
\]

Substitution obtained rigorously with the mathematical method

\[ z = 1 - \cos \alpha \]

Performing the same calculations made in the case of the light bending

\[
\frac{\Delta t}{R} = y \left[ 1 + \frac{uy}{8} + \frac{uy^2}{24} - \frac{u^2y^2}{112} \right]
\]

where \( u = \frac{r_s}{r} \) \quad \( y = \frac{(1 - \cos \alpha)}{(1 - u)} \)
3.4 Solid angle

\[ d\Omega = \frac{\cos i D^2 \sin^2 \theta (1 - \frac{2M}{R}) \sin^2 \alpha}{\cos \alpha} \left\{ \int_R^\infty \frac{dr}{r^2} \left[ 1 - \frac{b^2}{r^2} \left( 1 - \frac{2M}{r} \right) \right]^{-\frac{3}{2}} \right\}^{-1} \]

Substitution obtained rigorously with the mathematical method

\[ z = 1 - \cos \alpha \]

Performing the same calculations made in the case of the light bending

\[ d\Omega = \cos \alpha \cdot R \cdot \left[ 2z + (1 - 2C') z^2 + (1 - C' + 2C'^2 - 2D) z^3 \right] \]

where

\[ C = \frac{4 - 3u}{1 - u} \quad D = \frac{39u^2 - 91u + 56}{56(1 - u)^2} \]
APPLICATIONS

Image of an accretion disk’s formation in a binary system
4.1 Iron line profile

- Flux formula
  \[ F = \int_{R} \int_{\varphi} \varphi^4 \, d\Omega \]

- Numerical code

40 millions of points

Original: more than 60 min  
Approximate: less than 1 min
4.2 Polarization

Melia, Falanga & Goldwurm 2011

Schnittman & Krolik 2010
5 CONCLUSIONS

✓ PRESENT WORKS

☒ PUBLISH THIS WORK

☒ OPTIMIZE CODES TO CALCULATE THE FLUX AND THE POLARIZATION

✗ FUTURE PROJECTS

☐ EXTENSION TO THE KERR METRIC

☐ DEVELOP FAST NUMERICAL CODES IN THE KERR METRIC
THANK YOU FOR YOUR ATTENTION!