Solving the Einstein-Maxwell Equations for the Propagation of Electromagnetic Radiation during Early Universe Kasner-like Epochs

Brett Bochner, Hofstra University, Hempstead, USA

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Abstract

The pre-homogenized very early universe generically experiences Mixmaster-like behavior as it approaches the Big Bang, featuring a sequence of anisotropically and nonadiabatically expanding Kasner epochs, for which the geometrical optics approximation substantially breaks down. To gain a better understanding of the transport of relativistic energy in such environments, we solve for the propagation of (test particle) electromagnetic fields through background spacetimes with various sets of Kasner expansion indices. In solving the Einstein-Maxwell equations, we obtain fourth-order differential equations independently for each of the electric and magnetic fields, which can be solved to yield interesting information about how they are parametrically driven by the asymmetrically expanding early universe.

1 Introduction

An often unspoken cosmological truism, is that before the universe was homogeneous and isotropic, it was *inhomogeneous* and *anisotropic*. The focus of many cosmological studies is simply to get rid of this pre-Homogenized epoch as soon as possible, by erasing it using some form of inflation, thus eliminating the relevance of the very earliest cosmic epochs from practically all later developments.

But whether the "initial" distribution of matter and energy is simply a source of philosophical (e.g., aesthetic or fine-tuning) consternation, of if in fact the detailed conditions of the very earliest epochs really can be communicated through to more observable epochs, it remains in either case an interesting problem to study the propagation of relativistic mass-energy (e.g., light) during this most highly nonuniform epoch, so soon after the Big Bang.

A useful and well known model of the anisotropic early universe is the Mixmaster model [1], consisting primarily of a series of anisotropically expanding (though homogeneous) "Kasner" epochs; during a specific Kasner epoch, each spatial axis expands as a polynomial with a constant but potentially distinct power of cosmic time t.

Previous Kasner energy propagation studies (e.g., [2, 3, 4, 5, 6]) have often been restricted to highly simplified situations – such as obtaining just the first few terms of approximate series solutions to the wave equations, or considering wave propagation along only a single axis, or limiting the study to only a single set of Kasner expansion rates (often the axisymmetric case with Kasner coefficients $[2/3, 2/3, -1/3]$). For the research introduced in this paper, we obtain exact solutions of propagating electromagnetic (test) fields in Kasner metrics for a wider range of cases, by obtaining full solutions of the Einstein-Maxwell equations for as varied a selection of Kasner indices as possible.

2 Formalism

The Kasner metric is specified as [1]:

$$ds^2 = -\,dt^2 \;+\; t^{2p_1}dx^2 \;+\; t^{2p_2}dy^2 \;+\; t^{2p_3}dz^2 \;, \tag{1}$$

where, say, $p_1 \geq p_2 \geq p_3$. Furthermore, to ensure the vacuum background conditions of the standard Kasner model, one usually requires the further conditions, $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$. But, there is no particular reason why we must enforce these exact Kasner conditions; non-vacuum metrics (e.g., FRW) are perfectly acceptable (whether or not mass-energy actually dominates the cosmological evolution), as long as the physically reasonable energy conditions [7] are satisfied. And here, as we are specifically interested in studying *energy propagation* in Kasner-like cosmologies, the assumption of pure-vacuum conditions is not particularly useful. We will therefore consider more general selections of (real) indices – and rename them $p_x \geq p_y \geq p_z$ – to indicate the distinction for these "Kasner-like" metrics, which do not necessarily possess vacuum backgrounds.

The Einstein-Maxwell equations in curved spacetime (for charge-free and current-free volumes) are:

$$\partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} + \partial_\alpha F_{\beta\gamma} = 0 \ , \tag{2}$$

and:

$$\partial_\alpha \{ \sqrt{-|g_{\mu\nu}|} F^{\alpha\beta} \} = 0 \ . \tag{3}$$

Furthermore, the observable electromagnetic fields can be related to the electromagnetic tensor $F_{\alpha\beta}$ as follows [8]:

$$E_\alpha^{\mathrm{Obs}} = F_{\alpha\beta} U^\beta \ , \tag{4}$$

$$B_\alpha^{\mathrm{Obs}} = -\frac{1}{2} \epsilon_{\alpha\beta}^{\gamma\delta} F_{\gamma\delta} U^\beta \ , \tag{5}$$

where we will ultimately define the electromagnetic tensor components in terms of the observed fields by considering a *rest* observer in the Kasner-like expansion, with 4-velocity $U^\beta = [1, 0, 0, 0]$.

Lastly, we re-scale each of the fields (by appropriate powers of t), to eventually simplify the Maxwell-like equations that will be derived below: e.g., $E_z \equiv (t^{p_x + p_y - p_z}) E_z^{\mathrm{Obs}}$, plus cyclic permutations over $\{x, y, z\}$ (with identical re-scaling definitions for the B-fields).

Plugging this electromagnetic tensor and the general Kasner-like metric above into the Einstein-Maxwell equations, we obtain the divergence equations:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0 \ , \tag{6}$$

which for these re-scaled fields look exactly like the usual flat-spacetime versions; along with the following curl-like equations (which are *not* identical to the usual flat-spacetime ones):

$$\partial_t E_x = (t^{-p_x - p_y + p_z}) \partial_y B_z - (t^{-p_x + p_y - p_z}) \partial_z B_y \ , \tag{7}$$

plus cyclic permutations (and substitutions $\{E_i \to B_i, B_i \to -E_i\}$) to obtain the rest of the curl-like Maxwell equations for all six electromagnetic field components.

Now, in the typical (flat spacetime) case, it is a simple matter to use basic vector identities and the divergence equations above to obtain a set of completely decoupled (second-order differential) wave equations for each of the 6 E- and B-fields, independently. But in this case, where the curl-like equations in Eq. 7 contain additional powers of t as coefficients, it is *not* possible (for general polarizations and propagation directions) to obtain fully-decoupled second-order wave equations.

For example, defining the operator:

$$\{T\nabla^2\} \equiv [t^{-2p_x} \partial_x^2 + t^{-2p_y} \partial_y^2 + t^{-2p_z} \partial_z^2] \ , \tag{8}$$

we can obtain one possible second-order equation as:

$$\partial_t^2 E_x + \frac{p_x}{t} \partial_t E_x - \{T\nabla^2\} E_x = \frac{(p_z - p_y)}{t} [(t^{-p_x - p_y + p_z}) \partial_y B_z + (t^{-p_x + p_y - p_z}) \partial_z B_y] \ . \tag{9}$$

This resembles a nonhomogeneous differential equation in which the (orthogonal-axis) magnetic fields are the source of driving terms for this electric field. Unlike the flat-spacetime case, such driving terms on the right-hand side *cannot* generally be eliminated. They can, however, be modified (using the curl-like equations) into a variety of forms, with some of them being more useful for obtaining specific solutions (e.g., for particular sets of Kasner indices, or E- and $B-$field polarizations and/or wave propagation directions). Thus, Equation 9 for E_x can be expressed in two distinctly useful alternative forms:

$$\partial_t^2 E_x + \frac{p_x + (p_z - p_y)}{t}\partial_t E_x - \{T\nabla^2\}E_x = 2\frac{(p_z - p_y)}{t}[(t^{-p_x - p_y + p_z})\partial_y B_z] \ , \tag{10}$$

or:

$$\partial_t^2 E_x + \frac{p_x - (p_z - p_y)}{t}\partial_t E_x - \{T\nabla^2\}E_x = 2\frac{(p_z - p_y)}{t}[(t^{-p_x + p_y - p_z})\partial_z B_y] \tag{11}$$

Sets of formulas similar to Equations 9-11 can be obtained all of the other E- and $B-$field components as well, through cyclic permutations over $\{x, y, z\}$, in addition to employing the substitutions $\{E_i \rightarrow B_i, B_i \rightarrow -E_i\}$.

Although these second-order differential equations cannot be made homogeneous by zeroing out their right hand sides – i.e., the E- and $B-$fields cannot yet be completely decoupled – we can simplify the coupling between these fields by choosing convenient forms of the second-order equations for each of them. Thus we can break up the six E- and B-fields into three groups of coupled pairs – such as, $\{E_x \Leftrightarrow B_y, E_y \Leftrightarrow B_z, E_z \Leftrightarrow B_x\}$. (For example, Equation 10 is an example of $E_x \Leftrightarrow B_z$ coupling, but Equation 11 is an example of $E_x \Leftrightarrow B_y$ coupling.) Then, for each coupled pair of fields (e.g., $E_x \Leftrightarrow B_y$), we can uncouple them by taking *two more* time derivatives, plugging the curl-like equations for E_x and B_y into one another repeatedly, ultimately resulting in a fourth-order, *homogeneous* differential equation for E_x (or for B_y, depending upon whether we *started* with the equation: $\partial_t^2 E_x + \ldots$, or with the equation: $\partial_t^2 B_y + \ldots$). The resulting fourth-order equation for E_x will now be completely decoupled from B_y (and obviously from all of the other fields). This process can be done for all six of the E- and B-field components, rendering them entirely independent of one another for the purpose of solving each of their fourth-order differential equations.

As part of this process, we assume separation of spatial variables, with spatial functions of the form, $\sin/\cos(k_x x + k_y y + k_z z)$, so that we can replace all spatial derivatives (in their action upon the electric or magnetic fields, $F_i \equiv E_i$ or B_i), via the prescription:

$$-\{T\nabla^2\}F_i = [\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}]F_i \ . \tag{12}$$

The resulting fourth-order differential equations are cyclic in $\{x, y, z\}$, and are exactly the same (interchanging $E_i \leftrightarrow B_i$) for $F_i \equiv E_i$ or B_i. Thus all six of these equations can be specified by writing a single one of them out; for example:

$$\begin{aligned}
0 = {} & \partial_t^4 F_x + \frac{2(1+2p_x)}{t}\partial_t^3 F_x \\
& + \{2[\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}] + \frac{1}{t^2}[2p_x + 5p_x^2 - (p_y - p_z)^2]\}\partial_t^2 F_x \\
& + \{\frac{2}{t}[\frac{k_x^2}{t^{2p_x}} + (1 + 2p_x - 2p_y)\frac{k_y^2}{t^{2p_y}} + (1 + 2p_x - 2p_z)\frac{k_z^2}{t^{2p_z}}] + \frac{1}{t^3}[(2p_x - 1)(p_x^2 - [p_y - p_z]^2)]\}\partial_t F_x \\
& + \{[\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}]^2 \\
& + \frac{2}{t^2}[(p_x - p_y)(1 + p_x + p_z - 3p_y)\frac{k_y^2}{t^{2p_y}} + (p_x - p_z)(1 + p_x + p_y - 3p_z)\frac{k_z^2}{t^{2p_z}}]\}F_x \ .
\end{aligned} \tag{13}$$

The necessity of going to fourth-order differential equations to disentangle the fields – for cases *not* involving special Kasner indices, propagation or polarization directions (which may sometimes eliminate the driving terms shown above and collapse the fourth-order equations to homogeneous second-order equations) – implies that there are (in general) *four* different functional solutions to each of the equations for the E- and B-fields.

3 Sample Complete Solution for the Simplest Kasner Metric

Fortunately, the solutions to the full fourth-order electromagnetic field equations can often be guessed from the solutions to the second-order equations, especially when the second-order equations can (in specific cases) be made homogeneous and solvable. One interesting example in which the fields can be completely solved for – for fully general propagation directions and polarizations – is that of $\vec{p} \equiv \{p_x, p_y, p_z\} = [1, 0, 0]$. (Though this particular Kasner metric is actually conformally flat, the solutions for this case are instructive.)

In this $\vec{p} = [1, 0, 0]$ Kasner case, the second-order equations for F_x ($\equiv E_x$ or B_x) actually *are* homogeneous (recall Equations 9-11), uniquely simplifying to:

$$\partial_t^2 f_x + \frac{1}{t}\partial_t f_x + [\frac{k_x^2}{t^2} + (k_y^2 + k_z^2)]f_x = 0 \ . \tag{14}$$

This differential equation is immediately recognizable [9], and solvable in terms of Bessel functions of imaginary order (from which real combinations can straightforwardly be constructed [10]):

$$F_x \propto J_{\pm ik_x}[\sqrt{k_y^2 + k_z^2}\ t] \ . \tag{15}$$

For F_y and F_z, however, the second-order differential equations (derived via cyclic permutations of any of Equations 9-11, and substitution with $\vec{p} = [1, 0, 0]$) are *not* homogeneous, so we need four independent functions for the full solution. A clue to find them is obtained from the well known Bessel recursion relation:

$$J_{\pm ik_x}[\sqrt{k_y^2 + k_z^2}\ t] \propto t\{J_{(\pm ik_x - 1)}[\sqrt{k_y^2 + k_z^2}\ t] + J_{(\pm ik_x + 1)}[\sqrt{k_y^2 + k_z^2}\ t]\} \ . \tag{16}$$

This suggests that the four functions needed for the solutions are:

$$t\{J_{(\pm ik_x \pm 1)}[\sqrt{k_y^2 + k_z^2}\ t]\} \ , \tag{17}$$

from which we can construct the necessary linear combinations. In that regard, we recall one other Bessel recursion relation:

$$t\partial_t\{J_{\pm ik_x}[\sqrt{k_y^2 + k_z^2}\ t]\} \propto t\{J_{(\pm ik_x - 1)}[\sqrt{k_y^2 + k_z^2}\ t] - J_{(\pm ik_x + 1)}[\sqrt{k_y^2 + k_z^2}\ t]\} \ . \tag{18}$$

With these Bessel function combinations, we can construct the full electromagnetic solutions for the $\vec{p} = [1, 0, 0]$ Kasner metric – for *general* propagation direction, assuming nonzero k_x, k_y, and k_z – by splitting the problem into two different polarization cases: (i) $B_x = 0$, with *nonzero* E_x driving B_y and B_z; (ii) $E_x = 0$, with *nonzero* B_x driving E_y and E_z. Fully consistent solutions (with appropriate, *finite* coefficients, though not shown here) can then be found for both cases, noting first that all of the solutions for this Kasner metric turn out to have the same Bessel function argument, $[\sqrt{k_y^2 + k_z^2}\ t]$.

In Case (i), the second-order differential equations for all of the E-fields can be brought into homogeneous form (since $B_x = 0$, and $p_y = p_z = 0$ for this Kasner metric), and thus they have *only* the two main solutions, $\{E_x, E_y, E_z\} \propto J_{\pm ik_x}$. The magnetic fields (which are driven by $E_x \neq 0$) can be shown to have the form: $B_x = 0$, $\{B_y, B_z\} \propto t\partial_t[J_{\pm ik_x}]$. In Case (ii), alternatively, the roles for the electric and magnetic fields are obviously switched, with the functional forms of their solutions being the same as in Case (i), but with $E_i \leftrightarrow B_i$.

4 Selected Solutions for More General Kasner-like Metrics

Having completely solved the Einstein-Maxwell equations in this very simple Kasner case, it would be desirable to solve for the electromagnetic fields for all possible Kasner-like metrics, given *general* Kasner coefficients $\{p_x, p_y, p_z\}$. But this happens to be a highly nontrivial problem, since even the second-order equations are usually not solvable in terms of closed-form functions (Bessel or otherwise).

To clarify the problem, we note that the second-order equations for the electromagnetic fields always reduce to something of the form:

$$\partial_t^2 F_i + \frac{N[p_x, p_y, p_z]}{t}\partial_t F_i + [\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}]F_i = \text{“R.H.S.''} \ , \tag{19}$$

with $N[p_x, p_y, p_z]$ representing some numerical expression depending upon the (constant) Kasner p-indices, and the particular form of the "Right-Hand Side" ("R.H.S.") preferred. Even if the R.H.S. can be somehow set (or forced) to zero, this equation is *not* generally solvable (to this author's knowledge), unless the $[\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}]$ coefficient is considerably simplified. This can be done in a variety of ways – such as by setting some of the p-indices equal to each other (i.e., a "cylindrical" metric), and/or equal to unity; or by dropping one or two of the k_i values (planar or on-axis propagation, respectively); or by using early-time ($\frac{k_x^2}{t^{2p_x}} >> [\frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}]$, for $p_x \geq p_y \geq p_z$) or late-time ($[\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}}] << \frac{k_z^2}{t^{2p_z}}$) approximations. Unfortunately, any of these approximations (essentially equivalent to one another) are *highly* restrictive of the physical possibilities of interest.

Nevertheless, there are selected cases for which we are able to obtain exact closed-form solutions to the full Einstein-Maxwell equations. As noted above, the early- and late-time approximations are equivalent to propagation along a single axis (which is the actual asymptotic physical behavior of the propagating fields [3]); and though not shown here, we have obtained explicit solutions for every possible case involving a single propagation axis (e.g., $k_x \neq 0$, $\{k_y, k_z\} = 0$).

It is also possible to obtain exact solutions for two-dimensional propagation – i.e., $\{k_x, k_y\} \neq 0$, $k_z = 0$ – where at least one of $\{p_x, p_y\}$ (or both) are equal to unity. We divide the solutions up into three different index cases:

Case (I): $p_x = 1$, $p_y < 1$. Here, the electromagnetic field solutions are in terms of Bessel functions, $F_i \propto [(t^{-\Delta p})J_{\pm\Omega}(\alpha)]$, with the Bessel order and argument given by:

$$\Delta p = \pm\frac{1}{2}(p_y - p_z) \ , \ \Omega = \frac{\sqrt{(\Delta p)^2 - k_x^2}}{1 - p_y} \ , \ \alpha = \frac{k_y}{1 - p_y}t^{(1-p_y)} \ . \tag{20}$$

Note the interesting result that the order is only complex – and thus, the Bessel function is only oscillatory as $t \to 0$ – for $k_x > |\Delta p|$. In a sense, the anisotropically-expanding Kasner spacetime acts like a "waveguide", only permitting the propagation of waves with a sufficiently small projected wavelength in the x-direction (i.e., sufficiently large k_x).

Case (II): $p_x > 1$, $p_y = 1$. Here, for the field solutions $F_i \propto [(t^{-\Delta p})J_{\pm\Omega}(\alpha)]$, the Bessel order and argument are given by:

$$\Delta p = \frac{1}{2}[p_x - 1 \pm (1 - p_z)] \ , \ \Omega = \frac{\sqrt{(\Delta p)^2 - k_y^2}}{1 - p_x} \ , \ \alpha = \frac{k_x}{1 - p_x}t^{(1-p_x)} \ . \tag{21}$$

In this case (analogously with the previous one, but now with p_y equal to unity rather than p_x), the Bessel order is only complex for $k_y > |\Delta p|$. But another significant difference here, since $(1 - p_x) < 0$, is that the Bessel argument actually goes to $\alpha \to -\infty$ as $t \to 0$. This means that the electromagnetic waves will experience an *infinite* number of oscillations (in the x- and y-directions, since $\{k_x, k_y\} \neq 0$), in the finite period of time going back to the Big Bang; but that result actually makes sense in this case, since with $\{p_x, p_y\}$ both ≥ 1, there is no "Horizon Problem" in the x- or y-directions anyway, since they have formally infinite particle horizons even in the geometrical optics (light ray) approximation.

Case (III): Lastly, we consider the double special case of $p_x = p_y = 1$, which also has infinite particle horizons in the x- and y-directions. (Note that we could also solve for this case with $k_z \neq 0$, but that would not be essentially different from Case (I).) For this case, the solutions are not given in terms of Bessel functions; instead, in terms of $k \equiv \sqrt{k_x^2 + k_y^2}$ and $\Delta p = [(1-p_z)/2]$, the electromagnetic fields are of the form:

$$F_i \propto \{(t^{\mp\Delta p}) \cos / \sin [(\sqrt{k^2 - (\Delta p)^2})(\ln t)]\} \ . \tag{22}$$

These solutions, similar to the above cases, are only oscillatory for $k > |\Delta p|$; but when that condition is satisfied, there are an infinite number of oscillations in both the limits $t \to 0$, and $t \to \infty$.

5 Conclusions

We have studied the propagation of (test-field) electromagnetic waves in anisotropic, Kasner-like metrics by obtaining exact solutions for the Einstein-Maxwell equations for a variety of propagation directions, polarizations, and Kasner expansion indices. These solutions allow us to go beyond the geometrical optics (light ray) approximation, revealing interesting behaviors of the fields, such as (in several cases) Bessel function solutions, and "waveguide"-like behavior for which the unequal expansion rates in different directions places a lower limit on certain wavenumbers of the light, preventing it from propagating freely if the wavelength (projected along certain axes) is too large. Furthermore, we find that the oscillatory behavior of the waves going back towards the Big Bang ($t \to 0$) generally agrees with expectations stemming from the finite or infinite nature of the formal particle horizons for the various axis directions, given the power of cosmic time t governing the expansion rates of each of these axes.

In contrast to previous studies [11], we demonstrate that there are, in fact, nonhomogeneous driving terms coupling the various electric and magnetic fields together in the second-order differential wave equations, forcing us to obtain fourth-order differential equations for the fields in order to obtain independent equations (and solutions) for each of them. This implies that there are actually *four* solutions for each of the fields, rather than two solutions each (except for certain propagation directions and polarizations for which the coupling terms in the second-order equations get zeroed out).

Fortunately, the solutions to the full fourth-order equations can often be "guessed" from the solutions to the second-order equations, obtainable when the nonhomogeneous driving terms can somehow be discarded; but the biggest obstacle in the way of solving for all possible cases, is due to the lack of closed-form (Bessel-type function) solutions for the most general sets of Kasner indices $\{p_x, p_y, p_z\}$ and propagation wavenumbers $\{k_x, k_y, k_z\} \neq 0$. Obtaining fully general and complete solutions to *all* Kasner-like metrics would require us to be able the solve even those second-order differential equations which do not correspond to varieties of Bessel-type equations with currently known exact solutions.

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