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High-precision cosmology and inhomogeneities: exact results in the geodesic light-cone gauge

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based on:

F, Gasperini, Marozzi, Veneziano, JCAP 1311 (2013) 019
F, Nugier, JCAP 1502 (2015) 02, 002
F, Gasperini, Marozzi, Veneziano, JCAP 1508 (2015) 08, 020

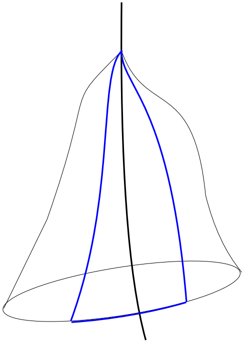
Contents

- What does GLC consist of?
- Finding a solution for the geodesic deviation equation
- Weak lensing and deflection angles
- Connecting both formalisms
- An example about exact non linear regime

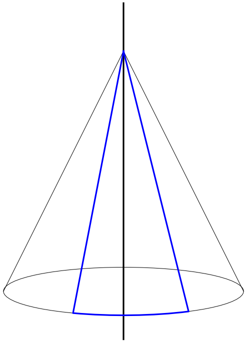
The GLC gauge (1)

The Geodesic Light-Cone (GLC) coordinates consist of a timelike coordinate τ (which can always be identified with the proper time of the synchronous gauge), of a null coordinate w and of two angular coordinates $\tilde{\theta}^a$ ($a = 1, 2$):

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)$$



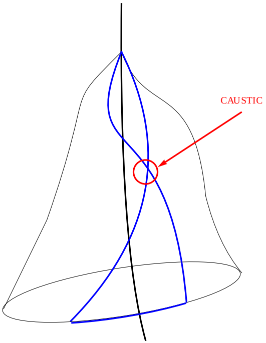
Past light-cone in FRW coordinates



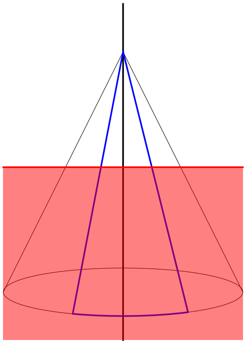
Past light-cone in GLC coordinates

The GLC gauge (2)

This identification holds if no caustics appear in light-propagation. Otherwise, an intrinsic limit breaks our description



Caustic on the inhomogeneous light-cone



Past light-cone in GLC coordinates

The GLC gauge (3)

Fundamental properties:

- $w =$ constant defines the past light-cone of the observer (ourselves)
- $u_\mu = -\partial_\mu \tau$ describes a geodesic flow (related to SG)
- $k^\mu = \omega \Upsilon^{-1} \delta^\mu_\tau$ is the quadri-momentum of the photon (constant w and θ^a)
- These properties are really useful: they allow us to express easily redshift only as function of τ:

$$1 + z = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{\Upsilon\left(\tau_o, w, \tilde{\theta}^a\right)}{\Upsilon\left(\tau_s, w, \tilde{\theta}^a\right)}$$

- Really similar to FRW metric, but exact and non perturbative!!!
- using the GLC coordinates, it is possible averaging physical observables on the light-cone in a gauge invariant way[1] which is also free from UV and IR divergences[2]

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[1] Gasperini, Marozzi, Nugier, Veneziano, JCAP 1107 (2011) 008

[2] Ben-Dayan, Gasperini, Marozzi, Nugier, Veneziano, Phys.Rev.Lett. 110 (2013) 021301

Definition of the Jacobi map

- Starting from the geodesic deviation equation:

$$\nabla^2_\lambda \xi^\mu = R_{\alpha\beta\nu}{}^\mu k^\alpha k^\nu \xi^\beta \tag{1}$$

where λ is the affine parameter and $\nabla_\lambda \equiv k^\alpha \nabla_\alpha$

- We can define the so called Sachs basis $\{s^\mu_A\}_{A=1,2}$ such that

2-D flat subspace	$g_{\mu\nu} s^\mu_A s^\nu_B = \delta_{AB}$	
Orthogonality conditions	$s^\mu_A u_\mu = 0 \qquad s^\mu_A k_\mu = 0$	(2)
Parallel transport	$\Pi^\nu_\mu \nabla_\lambda s^\mu_A = 0$	

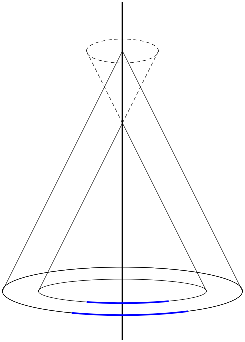
- In such a way, let us project the displacement on the Sachs basis $\xi^A \equiv \xi^\mu s^A_\mu$ the fundamental equation to solve is:

$$\frac{d^2}{d\lambda^2} J^A_B(\lambda, \lambda_o) = R^A_C J^C_B \tag{3}$$

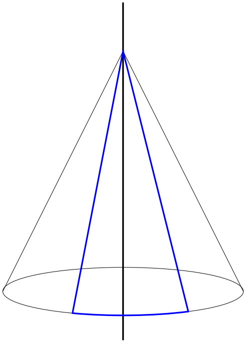
with $\xi^A = J^A_B(\lambda, \lambda_o) \left(\frac{k^\mu \partial_\mu \xi^B}{k^\mu u_\mu} \right)_o$ and initial conditions:
$J^A_B(\lambda_o, \lambda_o) = 0\,,\ \frac{d}{d\lambda} J^A_B(\lambda_o \lambda_o) = \delta^A_B (u^\mu k_\mu)_o$

The Jacobi map in the GLC gauge (1)

In GLC gauge, we have that $\xi^a =$ constant and $\xi^w = 0$ is solution of the geodesic deviation equation



$\xi^a =$ constant and $\xi^w \neq 0$



$\xi^a =$ constant and $\xi^w = 0$

The Jacobi map in the GLC gauge (2)

- Therefore, using the properties of the GLC gauge, we can easily construct the Jacobi map with the following solution

$$J^A_B(\lambda, \lambda_o) = s^A_a(\lambda) \left\{ \left[\left(u_\tau^{-1} \partial_\tau s \right)^{-1} \right]^a_B \right\}_{\lambda=\lambda_o}$$

- So the angular distance is immediately given by:

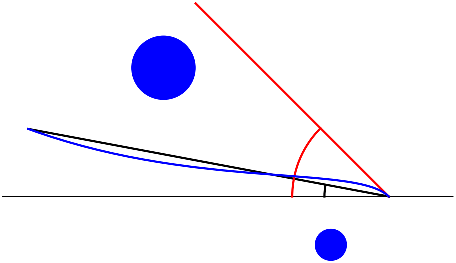
$$d_A^2 = \det \left(J^A_B(\lambda_s, \lambda_o) \right) = \frac{\sqrt{\gamma(\lambda_s)}}{\frac{1}{4} \left[\det \left(u_\tau^{-1} \partial_\tau \gamma^{ab} \right) \gamma^{3/2} \right]_{\lambda=\lambda_o}} \quad , \quad \gamma \equiv \det \gamma_{ab}$$

- In such a way, thanks to the Etherington relation, even the luminosity distance can be easily written as:

$$d_L^2 = 4 \frac{\Upsilon_s^{-4}}{\Upsilon_o^{-4}} \frac{\sqrt{\gamma_s}}{\left[\det \left(u_\tau^{-1} \partial_\tau \gamma^{ab} \right) \gamma^{3/2} \right]_o}$$

Weak lensing and deflection angles

- We know that one of the most important relativistic effects which happens in light propagation is deviating light-like signals trajectories because of the content of matter along the traveling
- Finding the relation $\theta_s^a\left(\theta_o^b\right)$ is the key point of lensing theory



- It is useful defining the well known amplification matrix $\mathcal{A}_b^a = \frac{\partial\theta_s^a}{\partial\theta_o^b}$, because of its connection with other physical observables like angular distance

Weak lensing and GLC gauge

What is the advantage in using GLC gauge for studying deflection angles?

- First of all, $\tilde{\theta}^a = \theta^a_o$ by construction
- In this way, if $x^\mu = \left(\tau, w, \tilde{\theta}^a\right)$ and $y^\mu = (\eta, r, \theta^a)$, the crucial relation between angles can be found by a simple coordinates transformation:

$$g^{\mu\nu} = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} g^{\alpha\beta}_{GLC}$$

- Solving this set of equations will give us the relation:

$$\theta^a = \theta^a\left(\tau, w, \tilde{\theta}^b\right) = \theta^a\left(z, w, \tilde{\theta}^b\right)$$

Connecting Poisson gauge to GLC gauge (1)

- Let us evaluate the deflected angles in the well known Poisson gauge
- By defining $\eta^{+} = \eta + r$, we adopt the following form for its third order expression:

$$ds^2 = g^{PG}_{\mu\nu} dy^{\mu} dy^{\nu} = a^2(\eta)\left[-2d\eta^2(\Phi+\Psi) + (1-2\Psi)\left(d\eta^{+2} - 2d\eta d\eta^{+}\right)\right.$$
$$\left. + (1-2\Psi)\left(\eta^{+}-\eta\right)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

where $\Psi = \psi + \frac{1}{2}\psi^{(2)} + \frac{1}{6}\psi^{(3)}$ and $\Phi = \phi + \frac{1}{2}\phi^{(2)} + \frac{1}{6}\phi^{(3)}$

- This way to proceed allows us to get the full deflection angles (even the non leading lensing terms) up to each desired order in perturbation theory

Amplification matrix and deflection angles (1)

- From the amplification matrix, we can try to build a general iterative approach for having the deflection angles up to each desired order

$$\mathcal{A}^a_b = \frac{\partial \theta^a}{\partial \theta^b_o} \equiv \delta^a_b - \Psi^a_b$$

- From the previous definition, we get that

$$(\Psi^a_b)^{(n)} = -\frac{\partial \theta^{a(n)}}{\partial \tilde{\theta}^b}$$

- Hence, according to the solutions that we found for angles, we obtain[3]

$$(\Psi^a_b)^{(1)} = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \frac{\eta' - \eta_s}{\eta_o - \eta'} \hat{\gamma}_0^{ac} \partial_c \partial_b \psi(\eta', \eta_o - \eta', \theta^a_o)$$

$$(\Psi^a_b)^{(2)} = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \frac{\eta' - \eta_s}{\eta_o - \eta'} \hat{\gamma}_0^{ac} \left[\partial_c \partial_b \partial_d \psi(\eta') \theta^{d(1)} - \partial_c \partial_d \psi(\eta') \Psi^{d(1)}_b \right]$$

$$(\Psi^a_b)^{(3)} = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \frac{\eta' - \eta_s}{\eta_o - \eta'} \hat{\gamma}_0^{ac} \Big[\partial_c \partial_b \partial_d \psi(\eta') \theta^{d(2)} + \frac{1}{2} \partial_c \partial_b \partial_d \partial_e \psi(\eta') \theta^{d(1)} \theta^{e(1)}$$
$$- \partial_c \partial_d \partial_e \psi(\eta') \theta^{e(1)} \Psi^{d(1)}_b - \partial_c \partial_d \psi(\eta') \Psi^{d(2)}_b \Big]$$

[3] F, Gasperini, Marozzi, Veneziano, JCAP 1508 (2015) 08, 020

Amplification matrix and deflection angles (2)

- Those expressions seem to behave perfectly for solving the so called lens equation

$$\Psi^a_b = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \frac{\eta' - \eta_s}{\eta_o - \eta'} \hat{\gamma}_0^{ac} \partial_c \partial_d \psi(\eta', \eta_o - \eta', \theta^a) \left[\delta^d_b - \Psi^d_b\right]$$

- In order to have the full agreement between our results and the lens equation, we have to expand the angles appearing in ψ

Finding a more satisfactory starting point (1)

- According to the standard definition, the indices in the amplification matrix seems to depend by the choice of the coordinates. Can we find a theoretical well posed starting point for its definition?
- In lensing theory, it is well known that the amplification matrix can be decomposed as

$$\mathcal{A}^A_B = \begin{pmatrix} 1-\kappa-\hat{\gamma}_1 & \hat{\omega}-\hat{\gamma}_2 \\ -\hat{\omega}-\hat{\gamma}_2 & 1-\kappa+\hat{\gamma}_1 \end{pmatrix}$$

  where κ is the convergence, $\hat{\omega}$ the vorticity and $|\hat{\gamma}|^2 = \hat{\gamma}_1^2 + \hat{\gamma}_2^2$ the total shear
- This decomposition leads to the relation

$$\mu^{-1} = (1-\kappa)^2 + \hat{\omega}^2 - |\hat{\gamma}|^2$$

  where $\mu = \left(\det \mathcal{A}^A_B\right)^{-1} = \left(\frac{\bar{d}_A}{d_A}\right)^2$ is the magnification

Finding a more satisfactory starting point (2)

- Let us notice that both the determinants of the Jacobi map and the amplification matrix are related to the angular distance. They just differ for the dimension
- Therefore, in order to provide a better defined starting point for the amplification matrix, let us identify[4]

$$\mathcal{A}^A_B \equiv \frac{J^A_B}{\bar{d}_A}$$

- This equality allows us to define an amplification matrix independently from the coordinates system. Moreover, in the GLC gauge, we have that

$$\mathcal{A} \sim s$$

[4] F, Nugier, JCAP 1502 (2015) 02, 002

Weak lensing in GLC coordinates

- Thanks to our identification, we can immediately furnish exact expressions for the amplification matrix entries.
- In particular, we choice the following combinations:

$$(1-\kappa)^2+\hat{\omega}^2 = \left(\frac{u_{\tau_o}}{\bar{d}_A}\right)^2\left\{\left[\frac{\gamma\,\dot{\gamma}_{ab}\gamma^{bc}\dot{\gamma}_{cd}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)^2}\right]_o\gamma\,\gamma^{ad}+2\frac{\sqrt{\gamma\,\gamma_o}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)_o}\right\}$$

$$|\hat{\gamma}|^2 = \left(\frac{u_{\tau_o}}{\bar{d}_A}\right)^2\left\{\left[\frac{\gamma\,\dot{\gamma}_{ab}\gamma^{bc}\dot{\gamma}_{cd}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)^2}\right]_o\gamma\,\gamma^{ad}-2\frac{\sqrt{\gamma\,\gamma_o}}{\left(\det^{ab}\dot{\gamma}_{ab}\right)_o}\right\}$$

  because they don't depend on the residual $U(1)$ rotation that we can perform on the Sachs basis.
- By construction of the GLC gauge, these results hold as long as no caustics appear.

Weak lensing in an exact non-linear regime (1)

- Having these results would allow us to study any kind of inhomogeneities we can think at, at least in principle.
- Unfortunately, solving Einstein's equations in the GLC gauge is not a trivial task, even if fascinating.
- In order to apply our results to a physical case, we can perform a coordinates transformation from GLC gauge to a solvable metric.
- To this end, we can consider an Lemaître-Tolman-Bondi metric

$$ds^2 = -dt^2 + X^2(t,r) + A^2(t,r)\left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

with an off-center observer.

- The angular distance can be exactly evaluated in this case, thanks to a coordinates transformation:

$$d_A^2 = \frac{A^2 X\left(r^2 + d^2 - 2\, rd\cos\theta\right)}{\sqrt{A^2\, d^2 \sin^2\theta + r^2 X^2\left(r - d\cos\theta\right)^2}}\frac{A_0(d)}{d\, X_0(d)}\frac{\sin\theta}{\sin\tilde{\theta}}$$

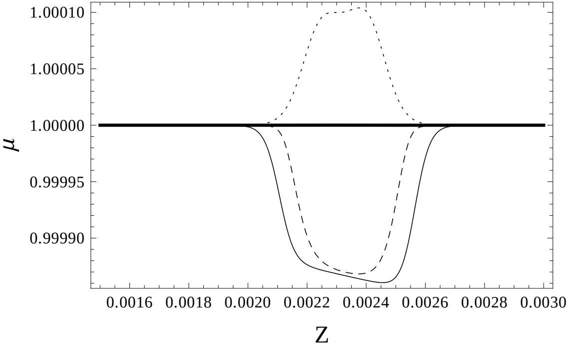
Weak lensing in an exact non-linear regime (2)

- The lensing quantities in this case are then given by

$$
\begin{aligned}
(1-\kappa)^2 = & \frac{A^2(t,r)}{4\sin\theta\, d^2 r\, a^2(t) X_0^2(d)\sqrt{d^2-2dr\cos\theta+r^2}} \Bigg[d^2\sin^2\theta X_0^2(d) \\
& + \frac{A_0^2(d) X^2(t,r)\left(d^2-2dr\cos\theta+r^2\right)^2}{d^2\sin^2\theta A^2(t,r)+r^2X^2(t,r)(r-d\cos\theta)^2} \\
& + \frac{2d\sin\theta A_0(d)X_0(d)X(t,r)\left(d^2-2dr\cos\theta+r^2\right)}{\sqrt{d^2\sin^2\theta A^2(t,r)+r^2X^2(t,r)(r-d\cos\theta)^2}} \Bigg] , \\
|\hat{\gamma}|^2 = & \frac{A^2(t,r)}{4\sin\theta\, d^2 r\, a^2(t) X_0^2(d)\sqrt{d^2-2dr\cos\theta+r^2}} \Bigg[d^2\sin^2\theta X_0^2(d) \\
& + \frac{A_0^2(d) X^2(t,r)\left(d^2-2dr\cos\theta+r^2\right)^2}{d^2\sin^2\theta A^2(t,r)+r^2X^2(t,r)(r-d\cos\theta)^2} \\
& - \frac{2d\sin\theta A_0(d)X_0(d)X(t,r)\left(d^2-2dr\cos\theta+r^2\right)}{\sqrt{d^2\sin^2\theta A^2(t,r)+r^2X^2(t,r)(r-d\cos\theta)^2}} \Bigg]
\end{aligned}
$$

Weak lensing in an exact non-linear regime (5)

- Some interesting illustrative models can be analytically solved with this metric. For instance, we can consider an inhomogeneous flat ΛCDM model with an underdensity region...

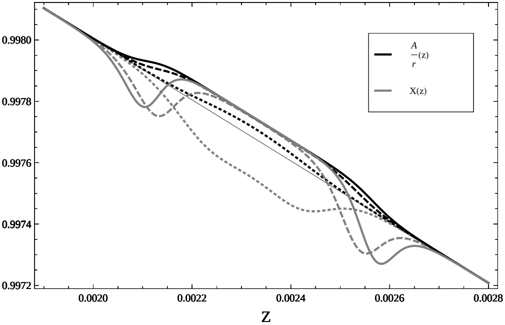


Weak lensing in an exact non-linear regime (6)

Having in mind that our exact solution is given, in general, by:

$$A(t,r) = r\left[\frac{1-\Omega_{\Lambda 0}(r)}{\Omega_{\Lambda 0}(r)}\right]^{1/3}\left(\sinh\left[\operatorname{arcsinh}\sqrt{\frac{\Omega_{\Lambda 0}(r)}{1-\Omega_{\Lambda 0}(r)}}+\frac{3}{2}\sqrt{\Omega_{\Lambda 0}(r)}\,H_0(r)\,t\right]\right)^{2/3}$$

with $X(t,r) = \partial_r A(t,r)$, we can understand our result by looking at the following plots:



Conclusions and perspectives (1)

- With the aim of the GLC, we have provided exact model independent expressions for some relevant physical observables.
- This is remarkable from both the theoretical (exact solution for the Sachs equation, iterative approach for deflection angles) and the phenomenological (model independence of the results) point view
- Formulating lensing theory in term of the Jacobi map helps us in taking into account all the effects, even the non linear ones. Within the GLC gauge, this relation becomes fundamental, because of the knowledge of the Jacobi map.

Conclusions and perspectives (2)

- This new approach allows to get directly the expression of the desired physical quantities in terms of the observer angles and redshift, i.e. other physical observables
- The same evaluation has been applied for deriving the angular/luminosity distance up to second order in perturbation theory. Result agrees with other evaluation given in literature, within the framework of the GLC gauge
  (Ben-Dayan, Marozzi, Nugier, Veneziano, JCAP 1211 (2012) 045)
  (Ben-Dayan, Gasperini, Marozzi, Nugier, Veneziano, JCAP 1306 (2013) 002)
  (F, Gasperini, Marozzi, Veneziano, JCAP 1311 (2013) 019)
- This evaluation directly allows to get the (full) deflection angles up to the desired order

Conclusions and perspectives (3)

- The exact knowledge of the deflection angles up to each desired order certainly helps us in evaluating lensing corrections to some relevant spectra (for instance the CMB's ones) beyond the first order approximation [WORK IN PROGRESS]
- This formalism seems to be promising even in the description of Ultra-Relativistic particles [WORK IN PROGRESS]
- An interesting goal to reach is solving Einstein's equations with this gauge, at least perturbatively [WORK IN PROGRESS]
- Recovering the standard definition of the Amplification matrix and understanding if different choices can be done [FUTURE TASK]
- Extending this formalism in order to include caustics is one of the most interesting goal to reach [FUTURE TASK]