Covariant Perturbations of Schwarzschild in the 1+1+2 Formalism

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The 1+3 Formalism

- Partial tetrad formalism[1] (Ehlers, Ellis, Hawking, . . .)
- Schematically:
  - Introduce a preferred timelike congruence: u^a
  - Project onto surfaces orthogonal to fluid flow

$$h_{ab} = g_{ab} + u_a u_b$$

- 1+3 variables are physically and geometrically meaningful.
- System of 1+3 equations:
  - Algebraic constraints on R_{ab} via EFE:

$$R_{ab} = \kappa \left(T_{ab} - \frac{1}{2} T g_{ab} \right) + \Lambda g_{ab}$$

  - Ricci identities applied to u^a

$$2\nabla_{[a}\nabla_{b]} u_c = R_{abcd} u^d \leftrightarrow \text{Kinematic evolution}$$

  - Twice contracted Bianchi identities:

$$\nabla_b T^{ab} = 0 \leftrightarrow \text{Conservation equations}$$

  - Bianchi identities:

$$\nabla_{[a} R_{bc]de} = 0 \leftrightarrow \text{Evolution of Weyl curvature}$$

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[1] Indices here are defined as $a, b, \dots \in \{0, 1, 2, 3\}$.

- Derivatives:

$$\dot{T}^{ab\dots c}{}_{de\dots f} = u^k \nabla_k T^{ab\dots c}{}_{de\dots f}$$

$$D_g T^{ab\dots c}{}_{de\dots f} = h^a{}_i h^b{}_j \dots h^c{}_k h_d{}^l h_e{}^m \dots h_f{}^n h_g{}^r \nabla_r T^{ij\dots k}{}_{lm\dots n}$$

- Kinematics:

$$\nabla_a u_b = \overbrace{-u_a \dot{u}_b}^{\text{Acceleration}} + \overbrace{\frac{1}{3}\Theta h_{ab}}^{\text{Expansion}} + \overbrace{\sigma_{ab}}^{\text{Shear}} + \overbrace{\omega_{ab}}^{\text{Vorticity}}$$

- Energy-Momentum:

$$T_{ab} = \overbrace{\mu u_a u_b}^{\text{Energy Density}} + \overbrace{p h_{ab}}^{\text{Isotropic Pressure}} + \overbrace{2q_{(a} u_{b)}}^{\text{Momentum Flux}} + \overbrace{\pi_{ab}}^{\text{Anisotropic Pressure}}$$

1+3 Formalism

Why 1+3 Formalism?

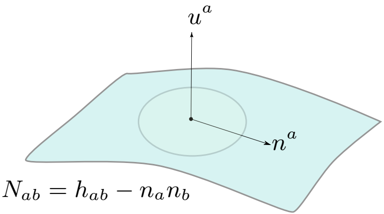
- 3-surfaces of homogeneity $\rightarrow$ System of ODEs involving scalar quantities
- Inhomogeneity $\rightarrow$ Breaks this simple structure
- Non-zero vectors and tensors $\rightarrow$ coupling terms!
- We can recover the simple structure by introducing another vector field ...

1+1+2 Formalism

Geometrical Picture

- Introduce spacelike congruence $n^a \to$ *further* split of $1+3$ equations[2]
- Projection tensor onto 2-sheets orthogonal to u^a and n^a: $N_{ab} = h_{ab} - n_a n_b$

1+1+2 Splitting of Spacetime



- All spacetime objects can be split into:
  - Scalars
  - 2-Vectors in the sheet
  - Transverse-Traceless 2-tensors in the sheet
- Supplement 1+3 equations with Ricci identity applied to n^a

$$R_{abc} \equiv 2\nabla_{[a}\nabla_{b]} n_c - R_{abcd} n^d = 0$$

[2] Clarkson and Barret, (2003), CQG, 20, 3855

1+1+2 Formalism

Splitting Spacetime *Again*

- Example: Decomposition of 3-Vectors:

$$\psi_a = \underbrace{\mathbf{\Psi} n^a}_{} + \mathbf{\Psi}^a$$

- Term parallel to n^a
- Term lying in sheet orthogonal to u^a and n^a

- New Derivatives:

$$\hat{\psi}_{a\ldots b} = n^e D_e \psi_{a\ldots b}$$

$$\delta_e \psi_{a\ldots b} = N_e{}^j N_a{}^f \ldots N_b{}^g D_j \psi_{f\ldots g}$$

- Kinematics of spacelike congruence:

$$D_a n_b = \overbrace{n_a a_b}^{\text{Acceleration}} + \overbrace{\frac{1}{2}\phi N_{ab}}^{\text{Expansion}} + \overbrace{\zeta_{ab}}^{\text{Shear}} + \overbrace{\xi \eta_{ab}}^{\text{Vorticity}}$$

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A Useful Dictionary...

Θ	Expansion of u^a
a^a	Sheet Acceleration
ϕ	Sheet Expansion
ξ	Rotation of $n^a \rightarrow$ Twisting of Sheet
ζ_{ab}	Shear of $n^a \rightarrow$ Distortion of Sheet
$\mathcal{A}$	Radial component of acceleration of u^a
$\mathcal{A}^a$	Acceleration of u^a lying in the sheet orthogonal to n^a
α^a	Acceleration of n^a
$\{\mathcal{E}, \mathcal{E}_a, \mathcal{E}_{ab}\}$	Projections of Electric Weyl Tensor E_{ab}
$\{\mathcal{H}, \mathcal{H}_a, \mathcal{H}_{ab}\}$	Projections of Magnetic Weyl Tensor H_{ab}
$\{\Sigma, \Sigma_a, \Sigma_{ab}\}$	Projections of Shear Tensor σ_{ab}
$\{\Omega, \Omega_a\}$	Projections of Vorticity Vector ω^a

Plus 1+1+2 generalisations of the 1+3 Energy-Momentum variables...

1+1+2 Formalism

Schwarzschild Spacetime: Background

- Schwarzschild covariantly characterised by

$$\mathbf{X} = \{\mathcal{E}, \phi, \mathcal{A}\}$$

$$\hat{\mathcal{E}} = -\frac{3}{2}\phi\mathcal{E} \qquad \hat{\phi} = -\frac{1}{2}\phi^2 - \mathcal{E} \qquad \mathcal{E} = -\mathcal{A}\phi$$

- Can relate 1+1+2 variables to metric functions

$$\mathcal{E} = -\frac{2m}{r^3}$$

$$\mathcal{A} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1/2}$$

$$\phi = \frac{2}{r}\left(1 - \frac{2m}{r}\right)^{1/2}$$

- Misner-Sharp mass[3]: $M_{\rm MS} = \frac{r}{2}\left[1 - \frac{r^2}{4}\phi^2\right]$

[3] Apparent horizon at $r = 2m$, $H = \phi \to 0$ at $r = 2m$

- Linear perturbations: $g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)}$

- Full system of 1+1+2 equations can be split into:

$$\begin{aligned} \text{Propagation Equations:} \quad & \hat{\mathcal{P}} = \ldots \\ \text{Evolution Equations:} \quad & \dot{\mathcal{E}} = \ldots \\ \text{Constraint Equations:} \quad & \mathcal{C} = 0 \end{aligned}$$

- Linearise all equations $\rightarrow$ Discard terms $\mathcal{O}[2]$ and higher
- Use spherical symmetry to construct gauge-invariant quantities:

$$\Psi_a = \delta_a \mathbf{X} \qquad \Rightarrow \qquad \Psi_a = \{X_a = \delta_a \mathcal{E}, Y_a = \delta_a \phi, Z_a = \delta_a \mathcal{A}\}$$

Recipe for deriving a master equation for gravitational perturbations:

- 1) Find complete set of gauge-invariant perturbations:

$$\begin{aligned} \mathbf{\Psi}_a &= \delta_a \mathbf{X} \\ \chi_a &= (\mathcal{E}_a, a_a, \dots) \\ \chi_{ab} &= (\mathcal{E}_{ab}, \zeta_{ab}, \dots) \end{aligned}$$

- 2) Re-write linearised 1+1+2 equations in terms of gauge-invariant perturbations
- 3) Harmonic analysis $\rightarrow$ Two parities can be introduced, e.g.

$$\chi_{ab} = \chi_T Q_{ab} + \bar{\chi}_T \bar{Q}_{ab}$$

- 4) System of harmonic equations linear in perturbation variables $\mathbf{\Phi}$ and $\bar{\mathbf{\Phi}}$

$$\gamma \dot{\mathbf{\Phi}} + \lambda \hat{\mathbf{\Phi}} = \Gamma \mathbf{\Phi}$$

- 5) D.o.f. governed by reduced set of frame independent master variables $\rightarrow$ Tough

- Covariant Regge-Wheeler Tensor:

$$W_{\{ab\}} = \frac{1}{2}\phi r^2 \zeta_{ab} - \frac{1}{3}\frac{r}{\mathcal{E}}\delta_{\{a}X_{b\}}$$

- Obeys a closed covariant wave equation:[4,5]

$$\ddot{W}_{\{ab\}} - \hat{\hat{W}}_{\{ab\}} - \mathcal{A}\hat{W}_{\{ab\}} + \left(\phi^2 - \mathcal{E}\right) W_{\{ab\}} - \delta^2 W_{\{ab\}} = 0$$

- Valid for both parities $\{W_T, \bar{W}_T\}$
- Introduce tortoise coordinates: $r_* = r + 2m \ln\left(\frac{r}{2m} - 1\right)$
- Wave equation reduces to $(\psi = \bar{W}_T)$

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)\psi = V\psi$$

$$V = V_{RW} = \frac{(r-2m)}{r^4}\left[\ell(\ell+1)r - 6m\right]$$

[4] Clarkson and Barret, (2003), CQG, 20, 3855
[5] Pratten, (2014), CQG 31, 3, 038001

- Can find a master variable from magnetic Weyl scalar

$$\mathcal{V}_{\{ab\}} = r^2 \delta_{\{a}\delta_{b\}}\mathcal{H}$$

- Obeys a closed, covariant wave equation:[6]

$$\ddot{\mathcal{V}}_{\{ab\}} - \hat{\hat{\mathcal{V}}}_{\{ab\}} - (\mathcal{A} + 3\phi)\,\hat{\mathcal{V}}_{\{ab\}} - \left[\delta^2 + 2K\right]\mathcal{V}_{\{ab\}} = 0$$

- $\mathcal{V}$ is purely an axial variable.
- Rescale $\mathcal{V} \to \mathcal{P} = r^3\mathcal{V}$ and introduce Tortoise coordinates
- Wave equation reduces to $(\psi = \bar{\mathcal{P}}_T)$

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)\psi = V\psi$$

$$V = V_{RW} = \frac{(r-2m)}{r^4}\left[\ell(\ell+1)r - 6m\right]$$

[6] arXiv:1503.03435

- Define a master variable from electric and magnetic Weyl 2-tensors

Master Variable

$$\mathcal{J}^{\pm}_{\{ab\}} = \mathcal{E}_{\{ab\}} \pm \epsilon_{c\{a}\mathcal{H}_{b\}}{}^{c}$$

- Obeys a closed covariant wave equation:[7]

Master Equation

$$\ddot{\mathcal{J}}^{\pm}_{\{ab\}} - \hat{\hat{\mathcal{J}}}^{\pm}_{\{ab\}} - (\mathcal{A} + 3\phi)\,\hat{\mathcal{J}}^{\pm}_{\{ab\}} \mp (4\mathcal{A} - 2\phi)\,\dot{\mathcal{J}}^{\pm}_{\{ab\}} - \left[\delta^2 + 2K - 4\mathcal{A}^2 + 4\mathcal{E}\right]\mathcal{J}^{\pm}_{\{ab\}} = 0$$

- Satisfied for both polar and axial sectors $\{\mathcal{J}^{\pm}_{T}, \bar{\mathcal{J}}^{\pm}_{T}\}$

[7] arXiv:1503.03435

- Can relate to the NP formalism

$$l_a = \frac{1}{\sqrt{2}}\left(u_a + n_a\right) \qquad k_a = \left(u_a - n_a\right) \qquad m_a = \frac{1}{\sqrt{2}}\left(v_a - i w_a\right)$$
$$g_{ab} = -l_a k_b - k_a l_b + 2 m_{(a}\bar{m}_{b)}.$$

- NP Weyl scalars:

$$\Psi_0 = \left[\mathcal{E}_{ab} + \epsilon_{r\{a}\mathcal{H}_{b\}}{}^{r}\right] m^a m^b,$$
$$\Psi_2 = \frac{1}{2}\left[\mathcal{E} - i\mathcal{H}\right],$$
$$\Psi_4 = \left[\mathcal{E}_{ab} - \epsilon_{r\{a}\mathcal{H}_{b\}}{}^{r}\right] \bar{m}^a \bar{m}^b \sim -\ddot{h}_+ + \ddot{h}_\times .$$

- Naturally identify $\mathcal{H}$ with $\Im\left[\Psi_2\right] \rightarrow$ Agreement with Price[8] that this is a RW variable

[8] Price, PRD 5 (1972) 2419

- Can relate to the 2+2 formalism

$$ds^2 = g_{AB}(x^C)\, dx^A\, dx^B + r^2(x^C)\gamma_{ab} dx^a dx^b,$$
$$h_{AB} = (\chi + \varphi)\,(n_A n_B + u_A u_B) + \varsigma\,(u_A n_B + n_A u_B)\,.$$

- Master equation for metric gauge-invariants

$$-\ddot{\chi} + \chi'' = \mathcal{S}_\chi,$$
$$-\ddot{\varphi} = \mathcal{S}_\varphi,$$
$$-\dot{\varsigma} = 2\nu\,(\chi + \varphi) + \chi'.$$

- Can show that:

$$\mathcal{E}_{ab} = -\frac{1}{2}\,(\chi + \varphi)\,Y_{ab},$$
$$\mathcal{H}_{ab} = -\frac{1}{2}\varsigma Y_{ab}.$$

Application to Scalar-Tensor Theories

Scalar-Tensor Theories

- Similarly, consider a more general scalar-tensor theory:

$$S_{st} = \int d^4x \sqrt{-g} \left[\varphi R - \frac{\omega(\varphi)}{\varphi} \nabla^a \varphi \nabla_a \varphi - V(\varphi) + \mathcal{L}_m(g_{ab}, \psi_m) \right]$$

- Resulting field equations can be re-cast in the form:

$$G_{ab} = \frac{\omega(\varphi)}{\varphi} \left[\nabla_a \varphi \nabla_b \varphi - \frac{1}{2} g_{ab} \nabla^c \varphi \nabla_c \varphi \right] + \frac{1}{\varphi} \left[\nabla_a \nabla_b \varphi - g_{ab} \Box_g \varphi \right] - \frac{V(\varphi)}{2\varphi} g_{ab},$$

$$(2\omega(\varphi) + 3) \Box_g \varphi = -\omega'(\varphi) \nabla^c \varphi \nabla_c \varphi + \varphi V'(\varphi) - 2V(\varphi).$$

- No hair theorem demands[9]:

$$\varphi \to \varphi_0,$$
$$\omega(\varphi) \to \omega(\varphi_0),$$
$$V(\varphi) \to V(\varphi_0) = 0$$

[9]Sotiriou and Faraoni, PRL, 108 (2012) 081103.

Scalar-Tensor Theories

- Effective curvature tensor:

$$T^{\varphi}_{ab} = \frac{\omega(\varphi)}{\varphi^2}\left[\nabla_a\varphi\,\nabla_b\varphi - \frac{1}{2}g_{ab}\,\nabla^c\varphi\,\nabla_c\varphi\right] + \frac{1}{\varphi}\left[\nabla_a\nabla_b\varphi - g_{ab}\,\Box_g\,\varphi\right] - \frac{V(\varphi)}{2\varphi}g_{ab}$$

- Energy-Momentum variables:

$$\mu^{\varphi} = \frac{1}{\varphi_0}\left[\hat{\hat{\varphi}} + \phi\hat{\varphi} + \delta^2\varphi + \frac{1}{2}V(\varphi)\right],$$

$$p^{\varphi} = \frac{1}{\varphi_0}\left[\ddot{\varphi} - \frac{2}{3}\hat{\hat{\varphi}} - \frac{2}{3}\phi\hat{\varphi} - \mathcal{A}\hat{\varphi} - \frac{2}{3}\delta^2\varphi - \frac{1}{2}V(\varphi)\right],$$

$$\ldots$$

- Scalar wave equation - these modes are not present in GR

$$-\ddot{\varphi} + \hat{\hat{\varphi}} + (\mathcal{A}+\phi)\,\hat{\varphi} + \delta^2\varphi = \frac{1}{2\omega(\varphi)+3}\left[\varphi_0 V'(\varphi) - 2V(\varphi)\right] \qquad (1)$$

- Recover $f(R)$ for $\varphi = f'(R)$, $\omega(\varphi) = 0$ and $V(\varphi) = Rf'(R) - f(R)$

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Scalar-Tensor Theories

- Can show that the Regge-Wheeler tensor in this class of theories is:

$$W_{ab} = \frac{1}{2}\phi r^2 \zeta_{ab} - \frac{1}{3}\frac{r^2}{\mathcal{E}}\delta_{\{a}X_{b\}} + \frac{1}{3\varphi_0} r^2 \delta_{\{a}\delta_{b\}}\varphi$$

- No-hair theorem $\rightarrow$ Restricts field configurations
- Linear order $\rightarrow$ scalar modes decouple from gravitational perturbations
- More general scalar-tensor theories $\rightarrow$ No hair? Stability? Ugly expressions!
- Second order? Expected to be suppressed in simple ST theories
- Matter content? Electrovacuum perturbations?