Backrection of voids in a Friedman background with constant w equation of state

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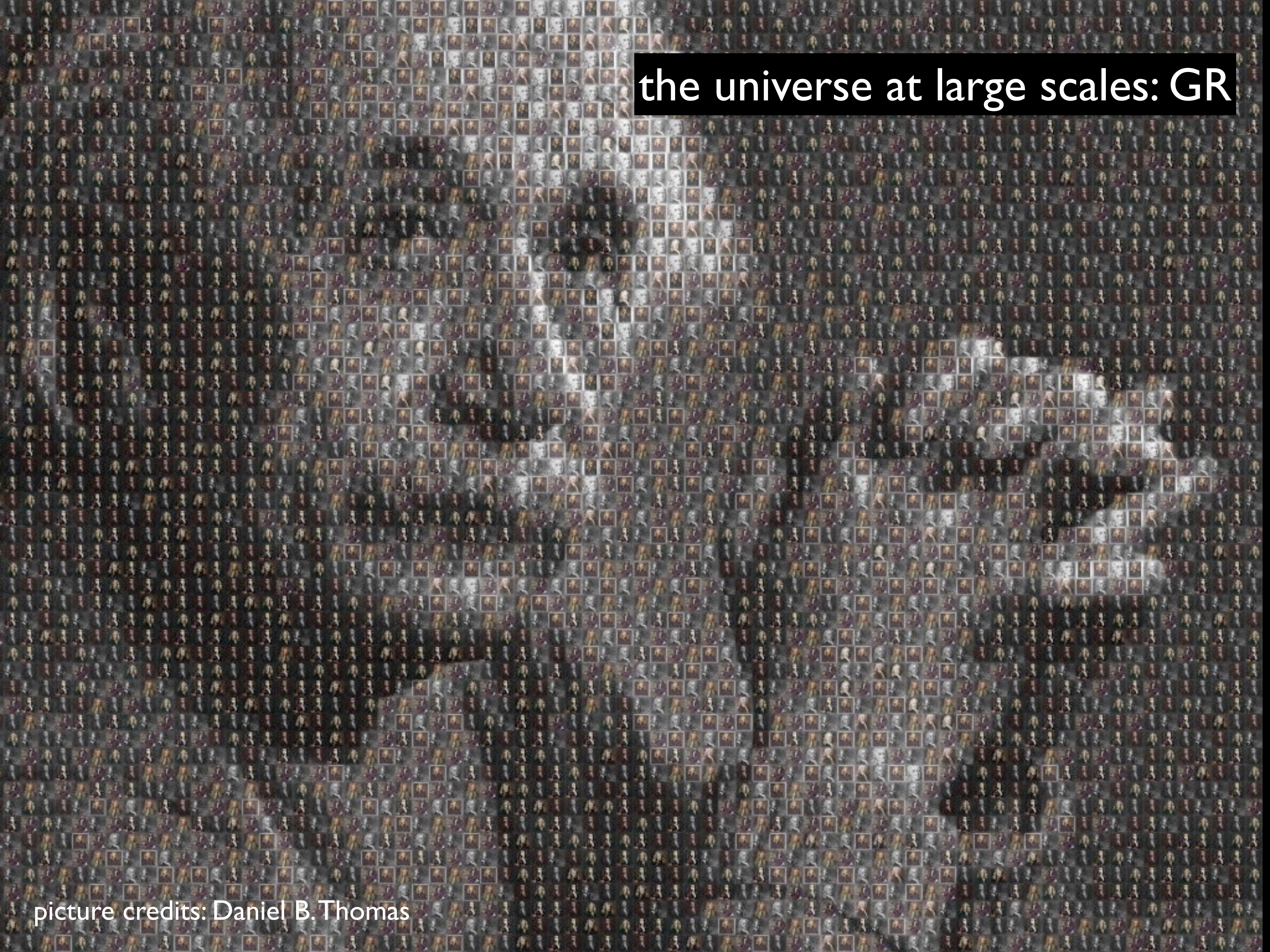
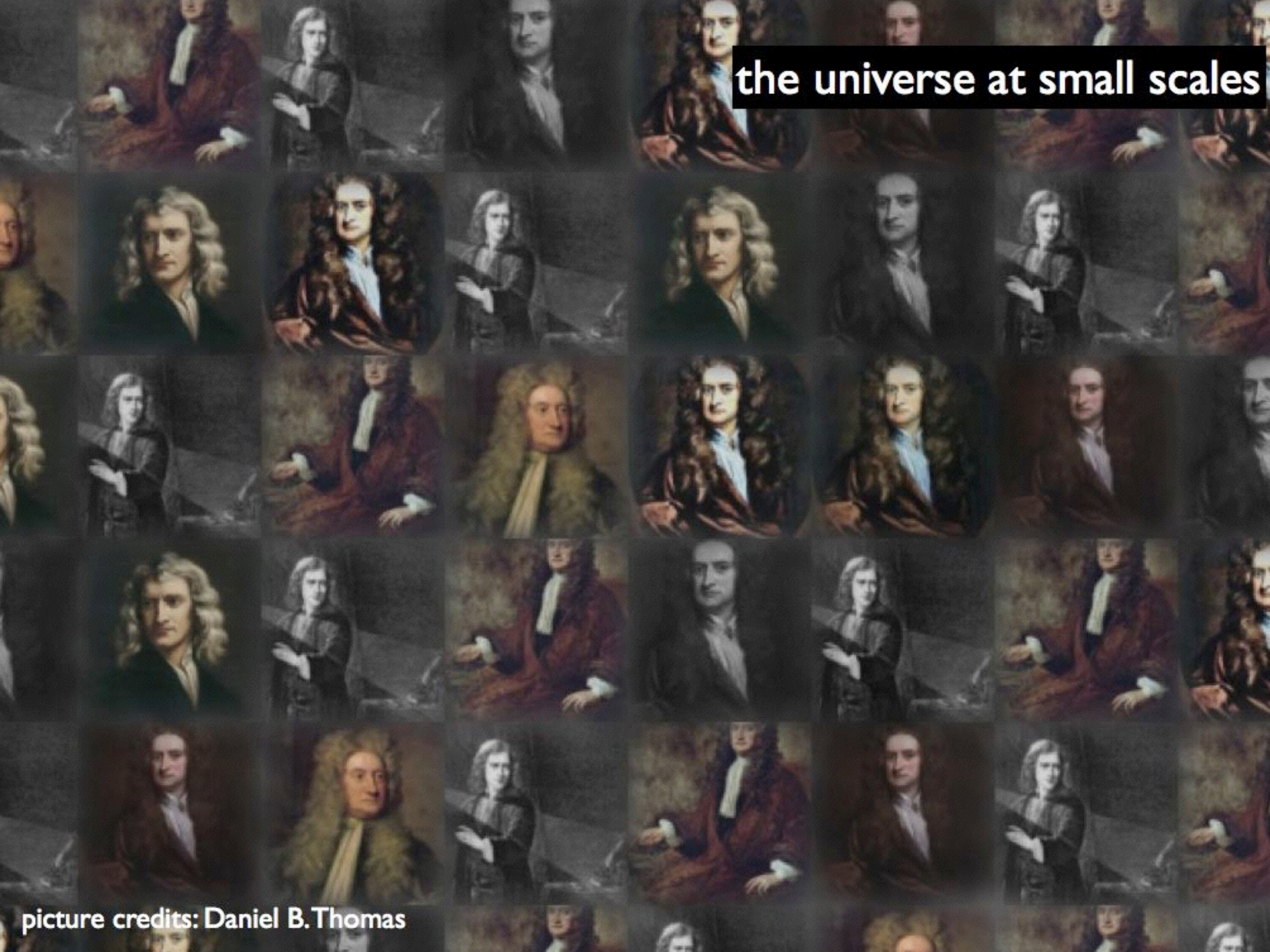
the universe at large scales: GR



picture credits: Daniel B. Thomas

the universe at small scales



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Newtonian Cosmology

1. Newtonian self-gravitating fluid: described by the continuity, Euler and Poisson equations

2. rescale physical coordinates to comoving coordinates

$$\dot{\vec{r}} = H\vec{r} + a\vec{v}$$

dust: p=0

$$\frac{d\delta}{dt} + \frac{\vec{\nabla}\cdot\vec{v}}{a}(1+\delta)$$

$$\frac{d\vec{v}}{dt} + \frac{\dot{a}}{a}\vec{v} = -\vec{\nabla}\phi$$

$$\nabla^2\phi = 4\pi G\rho_b\delta$$

note: convective time derivative

Linear perturbations

- for dust, linearise, combine continuity and Euler, substitute from Poisson, to get

$$\delta'' + \frac{3}{2a}\delta' - \frac{3}{2a^2}\delta = 0\,,$$

- In GR, for a w=constant fluid, use energy and momentum conservation equations, and the Energy constraint, to get (Δ gauge-invariant)

$$\Delta'' + \frac{3}{2S}(1-3w)\Delta' + \frac{3}{2S^2}(3w^2-2w-1)\Delta - \frac{wD^2\Delta}{H_0^2\Omega_0}S^{1+3w} = 0$$

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Solution in EdS and top-hat

$$a(t) = a_i \left(\frac{t}{t_i}\right)^{2/3},$$

$$\delta(t) = \delta_+ a(t) + \delta_- a(t)^{-3/2}$$

- top-hat turnaround and collapse time: characterized by the value of δ at these events:

$$\delta_T = 1.06 \qquad \delta_c = 1.696$$

the Averaging, BR & Fitting program

- Strictly speaking, Einstein Field Equations (EFE) describe the fundamental interaction, gravity.
- Only the truly inhomogeneous universe obeys EFE, precisely in the same way that in the Newtonian N-body problem each particle interact will all others
- Thus, in principle we should simulate inhomogeneous models and extract an average expansion a-posteriori
- Instead, we first ***assume*** the existence of a ***fitting*** homogeneous isotropic metric, ***then*** solve EFE for this.
- We should instead average EFE, obtaining an effective homogeneous limit that satisfies EFE with effective back-reaction terms.

Bucherts approach to the averaging problem(*)

- consider an irrotational dust spacetime [(-,+,+,+) and c=1] and adopt synchronous comoving coordinates, so that the line element reads

$$ds^2 = -dt^2 + h_{ab}(\vec{x}, t) dx^a dx^b,$$

where h_{ab} is the spatial metric of the constant t hypersurfaces, with determinant h.

then we define the average of a scalar Ψ on a compact coordinate domain $\mathcal{D}$ and the proper volume $V_{\mathcal{D}}$ as

$$\langle \Psi \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} d^3x \sqrt{h} \Psi . \qquad V_{\mathcal{D}} := \int_{\mathcal{D}} d^3x \sqrt{h}$$

(*) see e.g.: Buchert (2008), GRG 40(2), pp.467-527
Buchert (2011) CQG 28(1), p.4007.

Bucher't's averaging

- From V, we can then define the average scale factor

$$V_{\mathcal{D}} := \int_{\mathcal{D}} d^3x\sqrt{h}$$

$$a_{\mathcal{D}} \equiv (V_{\mathcal{D}}/V_{\mathcal{D}ini})^{1/3}$$

- then, the key to getting BR through averaging is the non-commutativity of the time derivative and the spatial averaging

$$\partial_t\langle\Psi\rangle_{\mathcal{D}} - \langle\partial_t\Psi\rangle_{\mathcal{D}} = \langle\Theta\Psi\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}\langle\Psi\rangle_{\mathcal{D}}$$

- then, averaging the continuity equation, Hamiltonian constraints and the Raychaudhuri equation gives effective Friedmann equations

$$\dot{\langle\rho\rangle}_{\mathcal{D}} = -3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle\rho\rangle_{\mathcal{D}}$$

$$\begin{aligned}\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 &= \frac{8\pi G}{3}\langle\rho\rangle_{\mathcal{D}} - \frac{1}{6}(\mathcal{Q}_{\mathcal{D}} + \langle\mathcal{R}\rangle_{\mathcal{D}})\\ \left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) &= -\frac{4\pi G}{3}\langle\rho\rangle_{\mathcal{D}} + \frac{1}{3}\mathcal{Q}_{\mathcal{D}},\end{aligned}$$

Bucher't's averaging

- in the effective Friedmann equations

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = \frac{8\pi G}{3}\langle\rho\rangle_{\mathcal{D}} - \frac{1}{6}(\mathcal{Q}_{\mathcal{D}} + \langle\mathcal{R}\rangle_{\mathcal{D}})$$

$$\left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) = -\frac{4\pi G}{3}\langle\rho\rangle_{\mathcal{D}} + \frac{1}{3}\mathcal{Q}_{\mathcal{D}},$$

- the term $\langle\mathcal{R}\rangle_{\mathcal{D}}$ represents the average of the spatial Ricci scalar, while

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\langle\Theta^2\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}}.$$

- is the back-reaction term, which can be positive. If this term satisfies $\mathcal{Q}_{\mathcal{D}} > 4\pi G\langle\rho\rangle_{\mathcal{D}}$ then clearly it can act as Dark Energy

Bucherts averaging

$$\mathcal{Q}_\mathcal{D} \equiv \frac{2}{3}\left(\langle\Theta^2\rangle_\mathcal{D} - \langle\Theta\rangle_\mathcal{D}^2\right) - 2\langle\sigma^2\rangle_\mathcal{D}.$$

- So, we can get an accelerated expansion of the averaged volume if $\mathcal{Q}_\mathcal{D} > 4\pi G\langle\rho\rangle_\mathcal{D}$, i.e. if the non-local variance of the local expansion dominates.
- Even if the local expansion rate is slowing down, this non-local effects may cause acceleration.
- This non local effect is in essence the main argument of those supporting the idea that back-reaction can be important against the argument - used by detractors - that local perturbations are always very small.
- Big bonus: there is no coincidence problem. Not only because there isn't a real additional DE, but really because the effective BR DE, the variance of Θ, grows naturally as structure grows.

Szekeres-Szafron models

Meures & Bruni arXiv:1103.0501

We reconsider Szekeres (1975) models including Λ, first considered by Barrow & Stein-Schabes (1984) - Cosmic No-Hair

Rewriting and classifying

Goode and Wainwright 1982 - splitting in background and deviations possible

$$ds^2 = -dt^2 + S^2\left[e^{2\alpha(\mathbf{x})}\left(dx^2 + dy^2\right) + Z(\mathbf{x}, t)^2 dz^2\right]$$

Class 1

- $S = S(z, t)$
- e.g. LTB
- can get spherical symmetry

Class 2

- $S = S(t)$
- e.g. FRW
- can get axial symmetry

we will focus on this subclass

$\alpha=0$

We find

$$e^{\alpha} = \frac{1}{1 + \frac{1}{4}k(x^2 + y^2)}.$$

Restricting ourselves to $k = 0$ implies

$$ds^2 = -dt^2 + S(t)^2 \left[dx^2 + dy^2 + Z(\mathbf{x}, t)^2 dz^2 \right].$$

Note:

- For $Z = 1$ this is flat FRW
- How much choice in $Z(\mathbf{x}, t)$?

EFE dictate $Z(\mathbf{x},t) = A(\mathbf{x}) + F(t,z)$

BR in Szekeres-Szafron

- Szekeres models have been generalised by Szafron to include a homogeneous pressure $p=p(t)$, a free function
- we are free to choose $p(t)$ as $p=w\rho_0 S^{-3(1+w)}$, i.e. as the pressure of a FLRW model with $p=w\rho_b$ equation of state, with w=const, where $S=S(t)$ is the scale factor
- it then turns out that again we have that $S(t)$ can be interpreted as the scale factor of the FLRW background, and $Z=A+F$ the exact perturbation
- we are interested in the late time behaviour; therefore we neglect a possible extra $p=0$ component and we restrict to the case of $-1/3<w\leq 0$, so that the w-fluid component dominates at late times but is not a DE

BR in Szekeres-Szafron

- gauge-invariant density perturbations in a w=const FLRW universe obey the equation

$$\Delta'' + \frac{3}{2S}(1-3w)\Delta' + \frac{3}{2S^2}(3w^2-2w-1)\Delta - \frac{wD^2\Delta}{H_0^2\Omega_0}S^{1+3w} = 0$$

- it turns out that with our choice of p(t), the F function in the Szekeres metric satisfies

$$F'' + \frac{3}{2S}(1-3w)F' + \frac{3}{2S^2}(3w^2-2w-1)F = 0\ ,$$

- we may therefore interpret F as a large-scale perturbation on top of the FLRW background

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BackReaction in Szekeres-Szafron models

- Indeed $$F = \beta_+ S^{1+3w} + \beta_- S^{-\frac{3}{2}(1-w)} \,.$$
- and $$\delta = -\frac{F}{A+F}(w+1)(3w+1)$$
- so that, as in the w=0 case, for the growing mode $\delta \propto -F$ at early times
- focusing on the case of voids, for F>0, at late times $\delta \propto$ constant

BackReaction in Szekeres-Szafron models

- In this model, we now focus on voids and we consider a comoving coordinate volume $\mathcal{D}$

- we can now perform a Buchert averaging and, crucially, compute the back-reaction term in Buchert's equations due to the growing mode

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\langle\Theta^2\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}} \,.$$

- We therefore average the expansion and shear, which are

$$\Theta = 3\frac{\dot{S}}{S} + \frac{\dot{Z}}{Z}\,, \qquad \sigma^2 = \frac{1}{3}\left(\frac{\dot{Z}}{Z}\right)^2$$

Strong BR in Szekeres-Szafron models

- we g

$$\langle\Theta\rangle_{\mathcal{D}} = 3\frac{\dot S}{S} + \left\langle \frac{\dot Z}{Z} \right\rangle_{\mathcal{D}} = \frac{\dot V_{\mathcal{D}}}{V_{\mathcal{D}}}$$

- so that the back-reaction term i

$$\mathcal{Q}_{\mathcal{D}} = -\frac{2}{3}\left\langle \frac{\dot Z}{Z} \right\rangle_{\mathcal{D}}^{2} .$$

- and the average curvatu

$$\langle R\rangle_{\mathcal{D}} = -\frac{2}{3}(5+3w)(1+3w)\rho_0 \left\langle \frac{\beta_+}{ZS^2} \right\rangle_{\mathcal{D}}$$

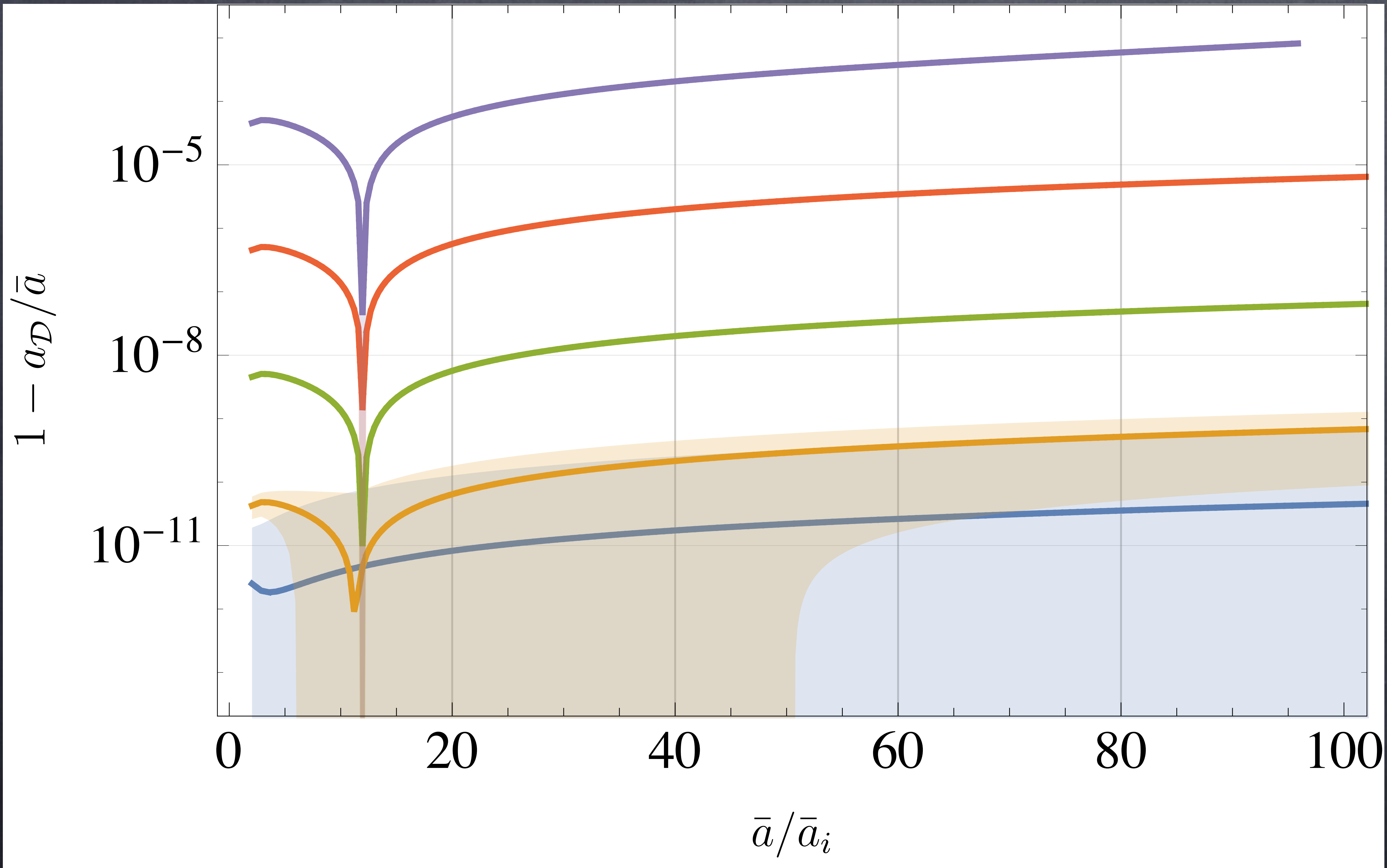
- to find the averaged scale factor, we have to insert these into Buche

$$\left(\frac{\dot a_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = \frac{\langle\rho\rangle_{\mathcal{D}}}{3} - \frac{1}{6}(\mathcal{Q}_{\mathcal{D}} + \langle\mathcal{R}\rangle_{\mathcal{D}}) ,$$

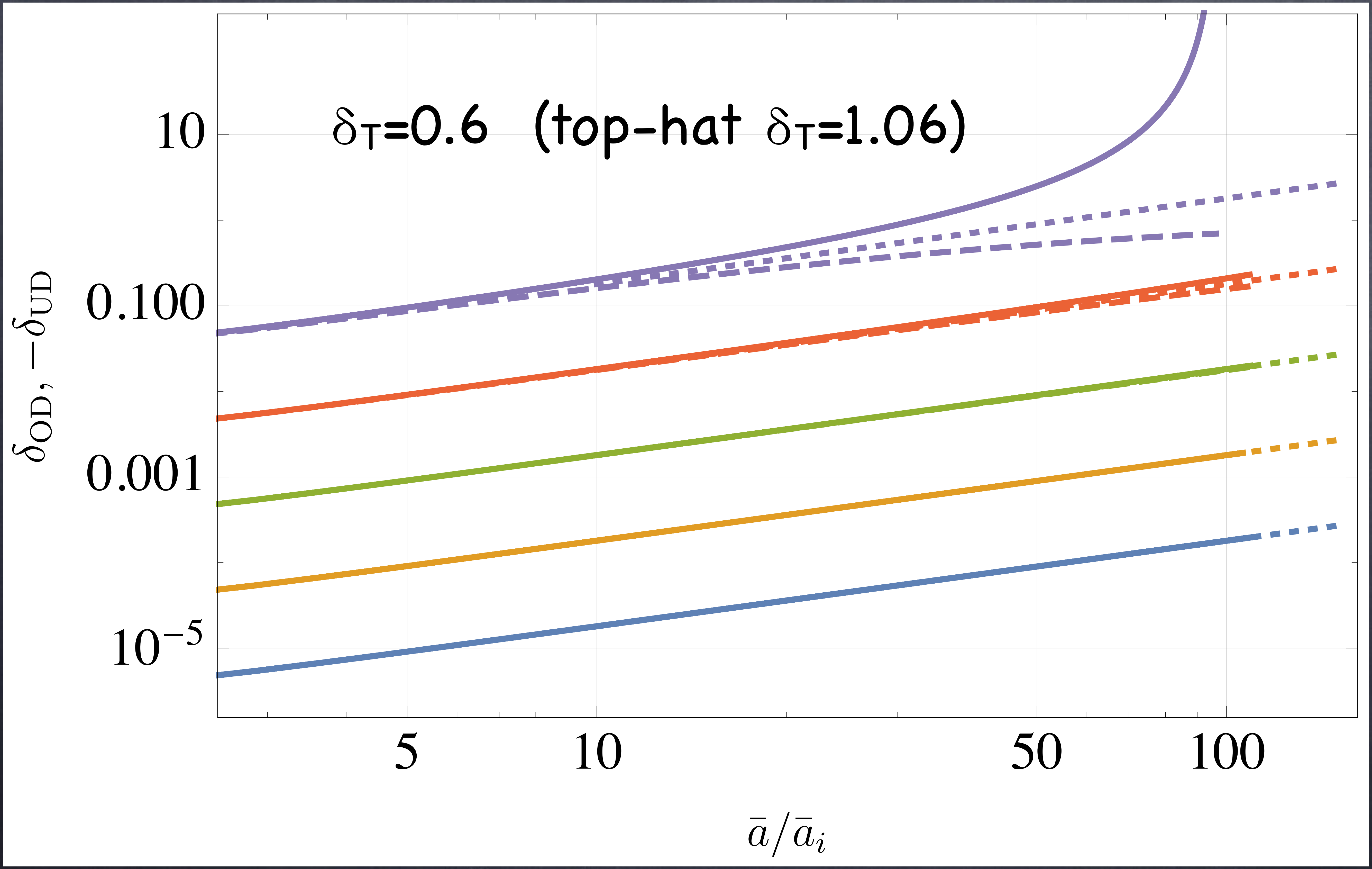
$$\left(\frac{\ddot a_{\mathcal{D}}}{a_{\mathcal{D}}}\right) = -\frac{\langle\rho\rangle_{\mathcal{D}}}{6} - \frac{\langle p\rangle}{2} w\rho_b + \frac{1}{3}\mathcal{Q}_{\mathcal{D}} ,$$

back-reaction in action and a no-go theorem

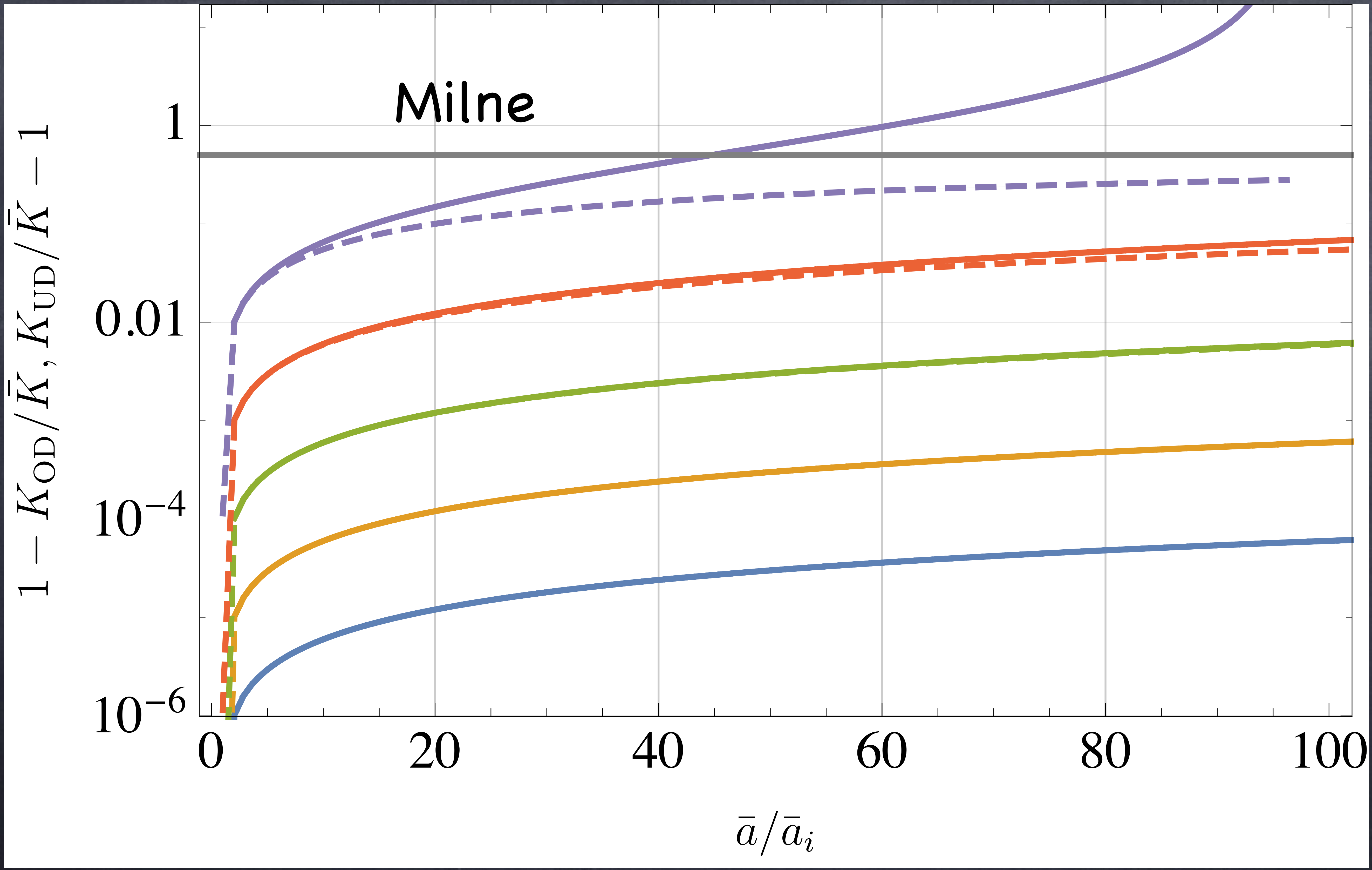
- we find that, asymptotically,
- $$a_{\mathcal{D}} \propto S(f_+)^{1/3} \propto S^{\frac{4+3w}{3}} \qquad a_{\mathcal{D}} \propto t^{\frac{2}{9}\frac{4+3w}{1+w}}$$
- back-reaction no-go theorem for Szkeres-Szafron models:
  - in a Szkeres-Szafron model with decelerating background with w>-1/3 back-reaction is effective in speeding up the expansion. However, the average scale factor never accelerate
- it is the negative average curvature term and not Q that produces the extra speed-up $a_{\mathcal{D}} \sim t^{\alpha} \quad \frac{8}{9} \leq \alpha \leq 1$



$1 - a_{\mathcal{D}}/\bar{a}$
10^{-5}
10^{-8}
10^{-11}
0
20
40
60
80
100
$\bar{a}/\bar{a}_i$



δ_T=0.6 (top-hat δ_T=1.06)
10
0.100
0.001
10^{-5}
$\delta_{\rm OD}, -\delta_{\rm UD}$
5
10
50
100
$\bar{a}/\bar{a}_i$



Milne
1
0.01
10^{-4}
10^{-6}
0
20
40
60
80
100
$\bar{a}/\bar{a}_i$
$1 - K_{\rm OD}/\bar{K}, K_{\rm UD}/\bar{K} - 1$