Backreaction of voids in a Friedman background with constant w equation of state

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Newtonian Cosmology

1. Newtonian self-gravitating fluid: described by the continuity, Euler and Poisson equations

2. rescale physical coordinates to comoving coordinates

\[ r = \dot{H} r + \ddot{a} v \]

\[ \frac{d\delta}{dt} + \frac{\vec{\nabla} \cdot \vec{v}}{a} (1 + \delta) \]

\[ \frac{d\vec{v}}{dt} + \dot{a} \vec{v} = -\vec{\nabla} \phi \]

\[ \nabla^2 \phi = 4\pi G \rho_b \delta \]

**note:** convective time derivative

**dust:** \( p = 0 \)
Linear perturbations

- for dust, linearise, combine continuity and Euler, substitute from Poisson, to get

\[ \delta'' + \frac{3}{2a} \delta' - \frac{3}{2a^2} \delta = 0, \]

- In GR, for a \( w \)=constant fluid, use energy and momentum conservation equations, and the Energy constraint, to get (\( \Delta \) gauge-invariant)

\[ \Delta'' + \frac{3}{2S} (1 - 3w) \Delta' + \frac{3}{2S^2} (3w^2 - 2w - 1) \Delta - \frac{wD^2 \Delta}{H_0^2 \Omega_0} S^{1+3w} = 0 \]
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Solution in EdS and top-hat

top-hat turnaround and collapse time: characterized by the value of $\delta$ at these events:

$$a(t) = a_i \left( \frac{t}{t_i} \right)^{2/3},$$

$$\delta(t) = \delta_+ a(t) + \delta_- a(t)^{-3/2}.$$

$\delta_T = 1.06$  $\delta_c = 1.696$
the Averaging, BR & Fitting program

- Strictly speaking, Einstein Field Equations (EFE) describe the fundamental interaction, gravity.

- Only the truly inhomogeneous universe obeys EFE, precisely in the same way that in the Newtonian N-body problem each particle interact will all others

- Thus, in principle we should simulate inhomogeneous models and extract an average expansion a-posteriori

- Instead, we first *assume* the existence of a *fitting* homogeneous isotropic metric, *then* solve EFE for this.

- We should instead average EFE, obtaining an effective homogeneous limit that satisfies EFE with effective back-reaction terms.
Buchert’s approach to the averaging problem

consider an irrotational dust spacetime \([(-,+,+,+)\) and \(c=1\)] and adopt synchronous comoving coordinates, so that the line element reads

\[
ds^2 = -dt^2 + h_{ab}(\vec{x}, t)dx^a dx^b,
\]

where \(h_{ab}\) is the spatial metric of the constant \(t\) hypersurfaces, with determinant \(h\).

then we define the average of a scalar \(\Psi\) on a compact coordinate domain \(\mathcal{D}\) and the proper volume \(V_D\) as

\[
\langle \Psi \rangle_D = \frac{1}{V_D} \int_{\mathcal{D}} d^3x \sqrt{h} \Psi.
\]

\[
V_D := \int_{\mathcal{D}} d^3x \sqrt{h}
\]

(*) see e.g.: Buchert (2008), GRG 40(2), pp.467–527
Buchert’s averaging

From $V$, we can then define the average scale factor:

$$V_D := \int_D d^3x \sqrt{h}$$

$$a_D \equiv \left( \frac{V_D}{V_{Dini}} \right)^{1/3}$$

then, the key to getting BR through averaging is the non-commutativity of the time derivative and the spatial averaging:

$$\partial_t \langle \Psi \rangle_D - \langle \partial_t \Psi \rangle_D = \langle \Theta \Psi \rangle_D - \langle \Theta \rangle_D \langle \Psi \rangle_D$$

then, averaging the continuity equation, Hamiltonian constraints and the Raychaudhuri equation gives effective Friedmann equations:

$$\langle \dot{\rho} \rangle_D = -3 \frac{\ddot{a}_D}{a_D} \langle \rho \rangle_D$$

$$\left( \frac{\dot{a}_D}{a_D} \right)^2 = \frac{8\pi G}{3} \langle \rho \rangle_D - \frac{1}{6} (Q_D + \langle R \rangle_D)$$

$$\left( \frac{\ddot{a}_D}{a_D} \right) = -\frac{4\pi G}{3} \langle \rho \rangle_D + \frac{1}{3} Q_D.$$
Buchert's averaging

in the effective Friedmann equations

\[
\left( \frac{\dot{a}_D}{a_D} \right)^2 = \frac{8\pi G}{3} \langle \rho \rangle_D - \frac{1}{6} (Q_D + \langle R \rangle_D)
\]

\[
\left( \frac{\ddot{a}_D}{a_D} \right) = -\frac{4\pi G}{3} \langle \rho \rangle_D + \frac{1}{3} Q_D,
\]

the term \( \langle R \rangle_D \) represents the average of the spatial Ricci scalar, while

\[
Q_D \equiv \frac{2}{3} \left( \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 \right) - 2 \langle \sigma^2 \rangle_D.
\]

is the back-reaction term, which can be positive. If this term satisfies

\[
Q_D > 4\pi G \langle \rho \rangle_D
\]

then clearly it can act as Dark Energy.
So, we can get an accelerated expansion of the averaged volume if
\( Q_D > \frac{2}{3} (\langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2) - 2\langle \sigma^2 \rangle_D \)

Even if the local expansion rate is slowing down, this non-local effects
may cause acceleration.

This non-local effect is in essence the main argument of those
supporting the idea that back-reaction can be important against the
argument – used by detractors – that local perturbations are always
very small.

Big bonus: there is no coincidence problem. Not only because there isn’t
a real additional DE, but really because the effective BR DE, the
variance of \( \Theta \), grows naturally as structure grows.
Szekeres-Szafron models
We reconsider Szekeres (1975) models including $\Lambda$, first considered by Barrow & Stein-Schabes (1984) - Cosmic No-Hair

Rewriting and classifying
Goode and Wainwright 1982 - splitting in background and deviations possible

\[
ds^2 = -dt^2 + S^2 \left[ e^{2\alpha(x)} \left( dx^2 + dy^2 \right) + Z(x, t)^2 dz^2 \right]
\]

Class 1
- $S = S(z, t)$
- e.g. LTB
- can get spherical symmetry

Class 2
- $S = S(t)$
- e.g. FRW
- can get axial symmetry

$\alpha = 0$

we will focus on this subclass
We find
\[ e^\alpha = \frac{1}{1 + \frac{1}{4} k(x^2 + y^2)}. \]

Restricting ourselves to \( k = 0 \) implies
\[ ds^2 = -dt^2 + S(t)^2 \left[ dx^2 + dy^2 + Z(x, t)^2 dz^2 \right]. \]

Note:
- For \( Z = 1 \) this is flat FRW
- How much choice in \( Z(x, t) \)?

EFE dictate \( Z(x,t) = A(x) + F(t,z) \)
Szekeres models have been generalised by Szafron to include a homogeneous pressure $p=p(t)$, a free function.

We are free to choose $p(t)$ as $p=w_0S^{-3(1+w)}$, i.e. as the pressure of a FLRW model with $p=w\rho_b$ equation of state, with $w=\text{const}$, where $S=S(t)$ is the scale factor.

It then turns out that again we have that $S(t)$ can be interpreted as the scale factor of the FLRW background, and $Z=A+F$ the exact perturbation.

We are interested in the late time behaviour; therefore we neglect a possible extra $p=0$ component and we restrict to the case of $-1/3<w\leq0$, so that the $w$-fluid component dominates at late times but is not a DE.
gauge-invariant density perturbations in a $w=\text{const}$ FLRW universe obey the equation

$$\Delta'' + \frac{3}{2S} (1 - 3w)\Delta' + \frac{3}{2S^2} (3w^2 - 2w - 1)\Delta - \frac{wD^2}{H_0^2\Omega_0} S^{1+3w} = 0$$

it turns out that with our choice of $p(t)$, the $F$ function in the Szekereses metric satisfies

$$F'' + \frac{3}{2S} (1 - 3w)F' + \frac{3}{2S^2} (3w^2 - 2w - 1)F = 0 ,$$

we may therefore interpret $F$ as a large-scale perturbation on top of the FLRW background
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BackReaction in Szekeres-Szafron models

Indeed

\[ F = \beta_+ S^{1+3w} + \beta_- S^{-\frac{3}{2}(1-w)} \]

and

\[ \delta = -\frac{F}{A+F}(w+1)(3w+1) \]

so that, as in the w=0 case, for the growing mode \( \delta \propto -F \) at early times.

focusing on the case of voids, for \( F>0 \), at late times \( \delta \propto \) constant
Back Reaction in Szekeres-Szafron models

In this model, we now focus on voids and we consider a comoving coordinate volume $\mathcal{D}$.

We can now perform a Buchert averaging and, crucially, compute the back-reaction term in Buchert's equations due to the growing mode.

\[
Q_{\mathcal{D}} \equiv \frac{2}{3} \left( \langle \Theta^2 \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}}^2 \right) - 2\langle \sigma^2 \rangle_{\mathcal{D}} .
\]

We therefore average the expansion and shear, which are

\[
\Theta = 3 \frac{\dot{S}}{S} + \frac{\dot{Z}}{Z},
\]

\[
\sigma^2 = \frac{1}{3} \left( \frac{\dot{Z}}{Z} \right)^2 .
\]
Strong BR in Szekeres-Szafron models

we get

\[ \langle \Theta \rangle_D = \frac{\dot{S}}{S} + \frac{\langle \dot{Z} \rangle}{Z} = \frac{\dot{V}_D}{V_D} \]

so that the back-reaction term is

\[ Q_D = -\frac{2}{3} \frac{\langle \dot{Z} \rangle^2}{\langle Z \rangle} \]

and the average curvature is

\[ \langle R \rangle_D = -\frac{2}{3} (5 + 3w)(1 + 3w)\rho_0 \langle \frac{\beta_+}{ZS^2} \rangle_D \]

to find the averaged scale factor, we have to insert these into Buchert's equations:

\[ \left( \frac{a_D}{a_D} \right)^2 = \frac{\langle \rho \rangle_D}{3} - \frac{1}{6} (Q_D + \langle R \rangle_D) \]
\[ \left( \frac{\dot{a}_D}{a_D} \right) = -\frac{\langle \rho \rangle_D}{6} - \frac{\langle p \rangle}{2} w \rho_b + \frac{1}{3} Q_D \]
back-reaction in action and a no-go theorem

we find that, asymptotically,

\[
\begin{align*}
    a_D &\propto S(f_+)^{1/3} \propto S^{\frac{4+3w}{3}} \\
    a_D &\propto t^{\frac{2}{9} \frac{4+3w}{1+w}}
\end{align*}
\]

back-reaction no-go theorem for Szkeres-Szafron models:

in a Szkeres-Szafron model with decelerating background with \( w > -1/3 \) back-reaction is effective in speeding up the expansion. However, the average scale factor never accelerate

it is the negative average curvature term and not \( Q \) that produces the extra speed-up

\[ a_{\text{up}} \sim t^{\frac{8}{9}} \leq \alpha \leq 1 \]
\( \delta_T = 0.6 \) (top-hat \( \delta_T = 1.06 \))

\[ \delta_{OD}, -\delta_{UD} \]

\[ \frac{\bar{a}}{\bar{a}_i} \]