Evolution of a self-gravitating spherical massless scalar field on compactified constant mean curvature hypersurfaces

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Motivation

Why care about CMC hypersurfaces?

- To "track" the radiation, one needs foliations which reach ($\mathscr{J}^+$). With this scheme it is possible, using smooth spatial asymptotically null slices, without the need of implementing larger grids and longer run times.

Why care about null-infinity ($\mathscr{J}^+$) compactification?

- Artificial boundary conditions (BCs), and truncated spatial domains generally introduce errors which are very difficult to deal with. Why not work with a formulation which can totally remove this difficult?

Conformal methods are a good way to numerically study isolated systems with strong gravity in the far field regime: Gravitational waves.

Some previous works

- Friedrich, 1981
  Regular conformal Einstein's field equations in vacuum. Newman & Penrose's spin-frame formalism of General Relativity.
- Hübner, 1993; Frauendiener, 1998
  Numerical implementations of Friedrich's scheme. The first one, applied to a spherically symmetric scalar field -as a companion of Christodolou's analytic work in 1991-; and the second one, to Einstein's vacuum field equations in 2D.
- Moncrief & Rinne, 2009; Rinne 2010
  Einstein's conformal equations in the ADM formulation. CMC hypersurfaces, spatial harmonic gauge condition and regularization of constraints at $\mathscr{J}^+$. Spherically symmetric and axisymmetric codes.
- Vañó, Husa & Hilditch, 2015
  Einstein's conformal equation in the GBSSN formulation and conformal Z4 equations. Unconstrained evolution for a spherically symmetric scalar field.

This work in brief

Numerical implementation:

$$\left.\begin{array}{c}\text{Tetrad formalism of General Relativity}\\ +\\ \text{Hypersurfaces with constant mean curvature (CMC)}\end{array}\right\}\ \text{Bardeen, Sarbach \& Buchman, 2011}$$

$+$

Self-gravitating massless spherical scalar field

- No potential
- Partially constrained evolution
- Black hole surrounded by the scalar field

Tetrad formalism on CMC hypersurfaces

- Tetrads adapted to a CMC foliation

$$\mathbf{e}_0 = \frac{1}{\alpha}\left(\partial_t - \beta^i\partial_i\right) \ ,$$

$$\mathbf{e}_a = B_a{}^i\partial_i \ \ ,\text{with } a = 1,2,3 \ .$$

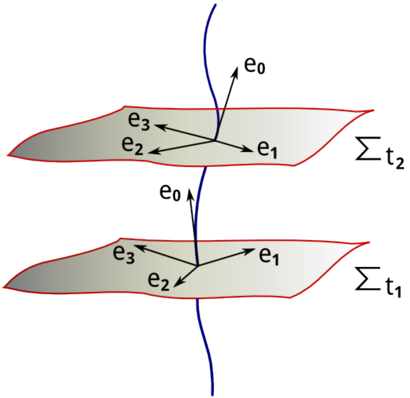
- Generalized connection coefficients

$$\Gamma_{\alpha\beta\gamma} := \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{e}_\gamma}\mathbf{e}_\beta) = -\Gamma_{\beta\alpha\gamma} \ .$$

$$a_b := \Gamma_{b00}, \ , \ \omega_b := -\frac{1}{2}\varepsilon_b{}^{cd}\Gamma_{cd0} \ ,$$

$$K_{ab} := \Gamma_{b0a} \ , \ N_{ab} := \frac{1}{2}\varepsilon_b{}^{cd}\Gamma_{cda} \ .$$

- $D_\alpha = e_\alpha$: Directional derivative along e_α, with $\alpha = 0,1,2,3$.
- In addition $a_b = D_b\left(\log\alpha\right)$.



Evolution of the tetrads on CMC hypersurfaces $\sum_t$,
i.e. $K^a{}_a = C :=$ constant.

Conformal transformations

- Penrose, 1965: Conformal compactification.
- We define a conformal factor to reach $\mathcal{J}^+$.

$$\Omega = \begin{cases} \text{positive} & \text{, inside the domain} \\ 0 & \text{, at null-infinity} \end{cases}$$

- Under the present formalism,

$$\mathbf{e}_\alpha = \Omega \tilde{\mathbf{e}}_\alpha \ , \ \ B_a{}^i = \Omega \tilde{B}_a{}^i \ , \ \ \alpha = \frac{1}{\Omega} \tilde{\alpha} \ , \ \ \omega_b = \Omega \tilde{\omega}_b \ ,$$
$$K_{ab} = \Omega \tilde{K}_{ab} - \delta_{ab} \tilde{D}_0 \Omega \ , \ \ N_{ab} = \Omega \tilde{N}_{ab} + \varepsilon_{abc} \tilde{D}_c \Omega \ .$$

- The next step: Write Einstein's equations, without symmetries, in terms of these rescaled quantities. Already made in vacuum in the BSB scheme. So we focus on the spherically symmetric case, with the scalar field.

Scalar Field Matter Sources

- Wave equation: $\Box\Phi = 0 \ , \ \Box := -g^{\mu\nu}\nabla_\mu\nabla_\nu$

  with $T_{\mu\nu} = (\nabla_\mu\Phi)(\nabla_\nu\Phi) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\nabla_\alpha\Phi)(\nabla_\beta\Phi)$.

- Conformal transformation: $\Phi = \Omega\tilde{\phi} \ , \ g^{\mu\nu} = \Omega^2\tilde{g}^{\mu\nu}$.

  $$\Rightarrow \tilde{\Box}\tilde{\phi} + \frac{1}{6}\tilde{R}^{(4)}\tilde{\phi} = \frac{1}{6\Omega^2}R^{(4)}\tilde{\phi} \ .$$

  The factor $\frac{R^{(4)}}{\Omega^2}$ actually is regular at $\mathscr{J}^+$.

- Defining $\tilde{\pi} := \tilde{D}_0\tilde{\phi}$ and $\tilde{\chi} := \tilde{D}_a\tilde{\phi}$, the wave equation can be rewritten as a symmetric hyperbollic system of coupled PDEs.

  $$\tilde{D}_0\tilde{\phi} = \ldots \ , \quad \tilde{D}_0\tilde{\chi} = \ldots \ , \quad \tilde{D}_0\tilde{\pi} = \ldots \ .$$

Choice of the Gauge

- Spherical symmetry: We introduce the compactified coordinate $R = r\,\Omega$
- We choose a gauge such that $\tilde{N}_{ab} = 0$, and the conformal spatial metric is flat, with the form $\tilde{\mathbf{h}} = dR^2 + R^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$.
- So in this gauge, the only variables which survive are:

$$\tilde{a}_b = \frac{1}{\tilde{\alpha}}\partial_R\tilde{\alpha}\hat{x}_b \quad , \quad \beta^i = b(t,R)\hat{x}^i \quad ,$$

$$\tilde{K}_{ab} = -\frac{\partial_R b}{\tilde{\alpha}}\frac{x_a x_b}{R^2} - \frac{b}{2\tilde{\alpha}R}\left(\delta_{ab} - x_a x_b\right) \; .$$

(in the practice, we define the quantity $\tilde{\nu}$ which parametrizes the rescaled extrinsic curvature such that $\hat{\tilde{K}}_{ab} = \tilde{\nu}\left[\frac{x_a x_b}{R^2} - \frac{1}{2}\left(\delta_{ab} - x_a x_b\right)\right]$) .

The equations of the system

Evolution equations

$$\tilde{D}_0\tilde{\nu} = -\frac{\tilde{K}}{3}\tilde{\nu} + \frac{2}{3\tilde{\alpha}}\left(\tilde{\alpha}'' - \frac{\tilde{\alpha}'}{R}\right) - \frac{4}{3\Omega}\left(\Omega'' - \frac{\Omega'}{R} + \frac{K}{2}\tilde{\nu}\right) + 8\pi G\Omega^2\tilde{\sigma}_R \,, \qquad \textbf{Extrinsic curvature}$$

$$\left.\begin{aligned}
\tilde{D}_0\tilde{\phi} &= \hat{\pi} - \tfrac{\tilde{K}}{3}\tilde{\phi}\,,\\
\tilde{D}_0\tilde{\chi} &= \tfrac{1}{\tilde{\alpha}}(\tilde{\alpha}\hat{\pi})' - \left(\tilde{\nu} + \tfrac{2\tilde{K}}{3}\right)\tilde{\chi} - \tfrac{3}{2R}\tilde{\phi}\tilde{\nu} - \tfrac{1}{2\tilde{\alpha}}(\tilde{\alpha}\tilde{\nu})'\tilde{\phi}\,,\\
\tilde{D}_0\hat{\pi} &= \tfrac{1}{\tilde{\alpha}R^2}(\tilde{\alpha}R^2\tilde{\chi})' - \tfrac{2\tilde{K}}{3}\hat{\pi} - \left(\tfrac{1}{4}\tilde{\nu}^2 - \tfrac{1}{3}\tfrac{\tilde{\alpha}''}{\tilde{\alpha}} - \tfrac{1}{6}\tfrac{R^{(4)}}{\Omega^2}\right)\tilde{\phi}\,,
\end{aligned}\right\} \qquad \textbf{Scalar field}$$

where $\tilde{\pi} = D_0\tilde{\phi} = \hat{\pi} + \frac{\tilde{K}\tilde{\phi}}{3}$, $\tilde{\chi} = \partial_R\tilde{\phi}$. Also $\tilde{\sigma}_R$, $\tilde{\rho}$, $\tilde{j}_R$, $\tilde{\sigma}^c c$ are the source terms and $R^{(4)}$ the Ricci scalar, depending on the scalar field $\tilde{\phi}$,$\tilde{\chi}$,$\hat{\pi}$ and the conformal factor Ω.

Constraint equations

$$\Omega\left[\Omega'' + \frac{2}{R}\Omega'\right] = \frac{3}{2}\left[\Omega'^2 - \left(\frac{K}{3}\right)^2\right] + \frac{3}{8}\Omega^2\tilde{\nu}^2 + 4\pi G\Omega^4\tilde{\rho}, \qquad \textbf{Hamiltonian constraint}$$

$$\Omega\left[\tilde{\alpha}'' + \frac{2}{R}\tilde{\alpha}'\right] = -3\Omega'\tilde{\alpha}' + \left[\Omega'' + \frac{2}{R}\Omega' - \frac{9}{4}\Omega\tilde{\nu}^2\right]\tilde{\alpha} + 4\pi G\Omega^3(3\tilde{\rho} + \tilde{\sigma}^c{}_c)\tilde{\alpha}\,, \qquad \textbf{CMC slicing}$$

$$\left(\frac{2}{3}\tilde{\alpha}\tilde{K}\right)' = (\tilde{\alpha}\tilde{\nu})' + \frac{3}{R}\tilde{\alpha}\tilde{\nu}\,, \qquad \textbf{Choice of the conformal factor}$$

$$\tilde{\nu}' + \frac{3}{R}\tilde{\nu} - \frac{2}{\Omega}\Omega'\tilde{\nu} = -8\pi G\Omega^2\tilde{j}_R\,, \qquad \textbf{Momentum constraint}$$

Treatment at null-infinity

The quantities Ω and $\tilde{\alpha}$ require a special treatment at $R_{\mathcal{J}^+}$

- Elliptic constraints are singular at $R_{\mathcal{J}^+}$.
- We approximate the solutions as polyhomogeneous truncated series:

$$\Omega(R)|_{R\to R_{\mathcal{J}^+}} \approx \sum_{\substack{i=1\\ j=0}}^{n_i,n_j} \Omega_{ij} R^i \log^j(R) \quad , \quad \tilde{\alpha}(R)|_{R\to R_{\mathcal{J}^+}} \approx \sum_{\substack{i=1\\ j=0}}^{n_i,n_j} \tilde{\alpha}_{ij} R^i \log^j(R)$$

- In vacuum, without symmetries, logarithmic terms are due to gravitational radiation (Andersson & Chrusciel, 1994; Chrusciel et.al. 1995). In this case they are due to scalar radiation.
- Logarithmic terms arise in the series only from Ω_{41} and $\tilde{\alpha}_{31}$ on.
- The coeff. Ω_{40} is related to the total mass of the system.

Some specific configurations

- Spatial domain: $R_{ini} = \frac{1}{4}$ to $R_{\mathcal{J}^+} = 1$
- Boundary conditions for the integration of Ham. and CMC constraints:

  Initially set $R_{ini} = R_{AH}$ (apparent horizon)

  For the evolution, $R_{ini} \neq R_{AH}$, the inner boundary conditions are:
  - The value of Ω is determined from $\tilde{D}_0\Omega = \frac{1}{3}\left(\Omega\tilde{K} - K\right)$.
  - The value of $\tilde{\alpha}$ is frozen to its initial value.
- The Misner-Sharp mass $m(r(R), t)$, according to this scheme:

$$1 - \frac{2m}{r} = 1 - \frac{2m}{R/\Omega} = -\left[\frac{b}{\tilde{\alpha}} + \frac{R}{\Omega}\left(\frac{\Omega\tilde{K}}{3} - \frac{K}{3}\right)\right]^2 + \left[1 - \frac{R}{\Omega}\Omega'\right]^2 ,$$

  which is useful for the monitoring of trapped surfaces.

Numerical Methods

Evolution equations *Hyperbolic eqs. for* $\tilde{\nu}, \tilde{\phi}, \tilde{\chi}$ *and* $\hat{\pi}$

- Evolve all equations as a system of four coupled PDEs
- SBP differential operators D_{63} (Diener, Dorband, Schnetter, Tiglio, 2007)
- Time integration: Runge-Kutta algorithm of 4th order

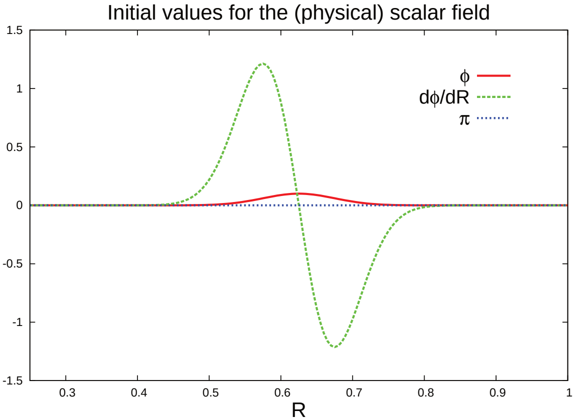
Ham. and CMC slicing constraints *Elliptic eqs. for* Ω *and* $\tilde{\alpha}$

- Rewrite each constraint as a system of two coupled PDEs
- Shooting method: from $R_{\mathcal{J}^+}$ to the fitting point R_{mid}
  from R_{ini} to the fitting point R_{mid}
- Matching: Newton-Raphson algorithm to find suitable BCs
- Spatial integration: Runge-Kutta algorithm of 4th order

Conformal factor's choice constraint *ODE for* $\tilde{K}$

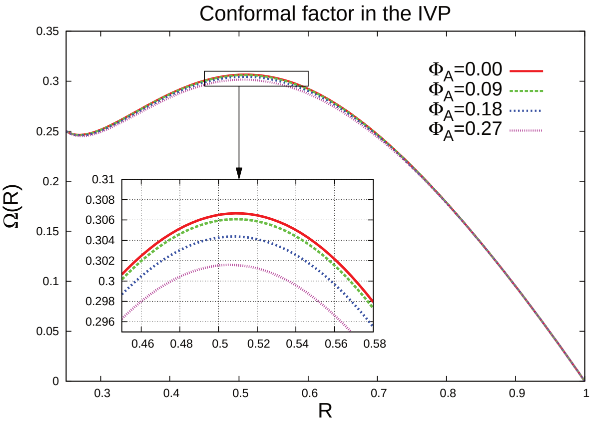
- Solve by a simple shoot from $R_{\mathcal{J}^+}$ to R_{ini}
  Asymptotic value $\tilde{K}_{\mathcal{J}^+}$ is known (Bardeen, Sarbach & Buchman, 2011)

The initial data



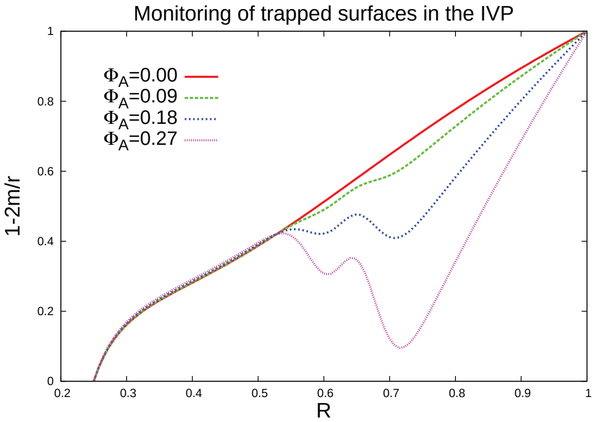
A typical Gaussian pulse as an example of initial data. In the graph: amplitude $\Phi_A = 0.1$, width $\Phi_W = 0.05$, gridpoints $N_R = 150$ and resolution $\Delta R = 0.005$.

Conformal factor in the IVP



The conformal factor Ω for different amplitudes of the physical scalar field.

Trapped surfaces in the IVP

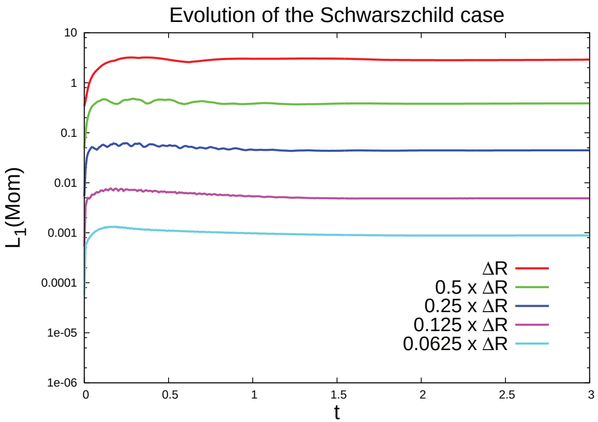


Quantity $1 - 2m/r(R)$ as a function of the amplitude of the physical scalar field.

Φ_0	$m_{\mathcal{J}^+}$
0.00	0.5002029
0.09	0.6094814
0.18	0.9366961
0.27	1.4799783

Total mass in the Initial Value Problem

Evolution in the Schwarszchild case



Convergence test for the momentum constraint. Schwarszchild case using 5 different resolutions,

Code tested for the evolution in the case $\Phi_0 = 0$ (Schwarszchild), with results that converge between 3rd and 4th order in the L_1 norm.

$Nx = 50$,
$\Delta R = 0.015$,
$CFL = 0.1$.

Final comments

Conclusions

- We have taken important steps towards the first implementation of the BSB scheme on CMC hypersurfaces.
- We also have paid special attention to subtle details such as:
  - Analytical work which allow us to partially decouple constraints,
  - Singularities at $R_{\mathcal{J}^+}$ in the Elliptic constraints and application of the polyhomogenous series.
- We have obtained good results in the convergence tests for the IVP and the evolution in the Schwarzschild case.

Prospects

- Consider the evolution for the case when $\Phi_0 \neq 0$ –almost ready, currently in process of debugging to reach a reasonable convergence.
- Apply this code to study some particular things:
  - Quasinormal modes, tail decays, etc.
  - Critical collapse (setting a strong scalar field),
  - Changing the physical scenario: AdS space-time, black hole with hair, etc.