Modelling inhomogeneous cosmologies with numerical relativity

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Abstract

Building accurate, multi-scale models of the Universe is a complex but necessary task in the era of precision cosmology, when observational data demands a thorough understanding of all effects which are expected to contribute at the 1% level, among which the full role of general relativity. This task has recently been tackled with a variety of approaches, which range from the study of toy models, to analytical expansions and hybrid analytical-numerical methods where relativistic effects are superimposed on classical, Newtonian N-body systems.

In this contribution, I will describe recent work carried out in numerical relativity to describe the relativistic Universe exactly, integrating Einstein's equation in three dimensions. This approach is the only one that can account for the full extent of the theory, and has already yielded significant results in several scenarios.

1 Introduction

The standard technique to model cosmological inhomogeneities is based on a three-fold approach: inhomogeneous spacetimes are assumed to be accurately described, on the largest scales, by the perturbation of a model from the Friedmann-Lemaître-Robertson-Walker (FLRW) class; linearized general relativity around an exactly homogeneous and isotropic spacetime is therefore used to study the evolution of density perturbations throughout the history of the Universe. On the smaller scales, once gravitational collapse prevails, nonlinear effects become important, but relativistic contributions are deemed to be negligible and Newtonian gravity is used to simulate these processes. This picture leads to the so-called Λ Cold Dark Matter (ΛCDM) model of cosmology, a ubiquituous reference tool in the analysis and interpretation of current cosmological datasets.

However, we lack a method to assess the quality of this approximation: are nonlinear relativistic effects really negligible in all sectors of ΛCDM? Evaluating the systematic error involved in the use of this framework is particularly relevant in the age of precision cosmology, when observations are expected to gain a picture of the Universe which is accurate down to at least the 1% level [1].

Unfortunately, a full account of these errors requires the direct integration of Einstein's equation, a coupled, nonlinear system for the components of the metric tensor. This is the task of numerical relativity[2, 3, 4, 5], a field originally developed to model gravitational-wave sources, but which has also found countless cosmological applications throughout its history [6]: from singularities studies to inflation and phase transitions, as well as gravitational-wave generation in the early universe and large-scale structure formation studies. In this contribution I will review some of the recent progress.

2 The fitting problem in cosmology

The problem of interpreting observations carried out in an inhomogeneous universe through a homogeneous class of theoretical models was formulated explicitly, for the first time, by Ellis and Stoeger in 1987 [7]. The authors analyzed, in particular, the different mappings that can be established between a generic universe and the FLRW class, showing how the systematic, "fitting" errors due to this procedure are not only technically difficult to compute, but also complex to define.

Since that study, many directions have been explored, which range from averaging schemes to semirelativistic approaches in numerical simulations. Recently, exact numerical-relativity studies have also been carried out, albeit in simplified scenarios. Hopes, however, are that further development will lead to usable, realistic cosmological models which can provide robust tools for the treatment of the fitting problem.

3 Building cosmological models with numerical relativity

Numerical relativity, or the numerical integration of Einstein's equation, is based on the so-called 3+1 decomposition, entailing a choice of time coordinate t, and the projection of the equation on directions parallel and perpendicular to the time gradient $n_a = -\nabla_a t$. Within this decomposition, the line element takes on the form:

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t^2)(\mathrm{d}x^j + \beta^j \mathrm{d}t^2) \tag{1}$$

and Einstein's equation $G_{ab} = 8\pi T_{ab}$ turns into the so-called Arnowitt-Deser-Misner (ADM) system for the spatial metric γ_{ij} and its first time derivative K_{ij} (known as the *extrinsic curvature*):

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{2}$$

$$D_j K^j_i - D_i K = 8\pi j_i \tag{3}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + 2D_{(i}\beta_{j)} \tag{4}$$

$$\begin{aligned}\partial_t K_{ij} = &-D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k_j + K K_{ij}) \\ &+\beta^k D_k K_{ij} + 2K_{k(i}D_{j)}\beta^k - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho))\end{aligned} \tag{5}$$

This system, cast into an appropriate form, is amenable to numerical integration [8, 9, 10]. The first two equations constrain the instantaneous values of γ_{ij} and K_{ij}, and must be solved first to obtain valid initial data. The latter two govern its subsequent time evolution.

Over the years, a number of codes have been developed to tackle this problem, some of which are now part of the Einstein Toolkit [11] (ET), a free-software collection of tools that integrate Einstein's equation, analyze the resulting spacetimes (looking for gravitational radiation, for instance, or apparent horizons) and provide the infrastructure to deploy the corresponding simulations on worldwide supercomputers. The ET was developed in the compact-object community, and is geared towards simulations of binary systems of black holes and neutron stars, as well as of gravitational collapse and accretion phenomena. Whilst some of the tools can be directly used to construct cosmological solutions, others have to be extended or replaced by appropriate counterparts.

For instance, a multigrid solver was recently developed [12], and has recently generated the initial data for a few cosmological models, enabling the study of these systems [13, 14]. In the remaining of this section, I will describe this construction and the physical properties of the resulting spacetimes.

3.1 Black-hole lattices

The first three-dimensional simulation of a cosmological space did not need a numerical solution of the initial-data problem. In 1957, for the first time, Lindquist and Wheeler posed the question of what a universe, filled with a regular black-hole lattice (BHL), would look like. Their approach was approximate: a number of Schwarzschild regions were joined together to form a cosmology spatially

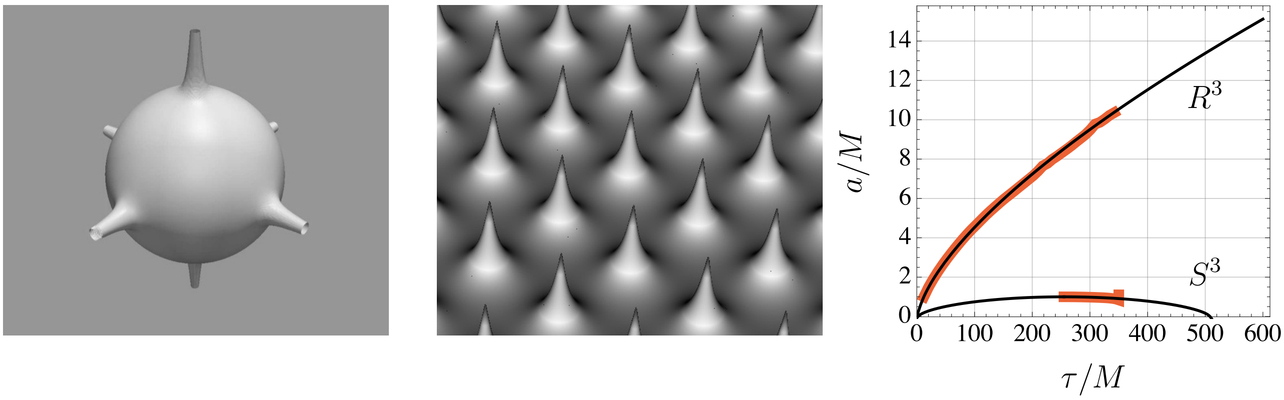

Figure 1: Two-dimensional sections of spatial hypersurfaces from an S^3 (left) and an R^3 (middle) BHL. On the right, length scaling for an R^3 (top red curve) and an S^3 (bottom red curve) BHL, as a function of proper time τ, along with the best-fit FLRW models (thin black lines). The BHL length scale has been normalized to its initial value; all curves have also been shifted in time so that the corresponding best-fit FLRW have $a(0) = 0$. M is a characteristic length scale that depends on the BH masses. For details on the definition of a scale factor a in a BHL or on the fitting procedure, as well as a discussion of the departures of BHLs from FLRW models, see [15, 13].

conformal to S^3, and the evolution of the lattice edge lengths was compared to the scale factor of a FLRW model. In that work, the authors pointed out that an exact solution of the problem could be obtained by considering a special class of regular BHLs, and the subsequent direct numerical integration. This was first achieved in [15], showing results not dissimilar from those of Lindquist and Wheeler: a conformally-S^3 BHL remains, at least for one third of the total recollapse time, fairly close to its FLRW counterpart. Around the same time, similar systems were also studied under various approximations [16, 17], allowing one to work out their properties and characterize their departure from the FLRW class under fairly generic circumstances.

Such exact initial data, however, is only available at instants of time symmetry, and subsequently leads to a volume contraction analogous to that of closed FLRW models. From the observational standpoint, it is more interesting to study expanding cosmologies; in this case, however, numerics are necessary even to construct the initial hypersurface. In [18], Yoo and collaborators have proposed an initial-data construction capable of generating expanding black-hole lattices, spatially conformal to R^3, for a range of mass-over-spacing values.

The evolution of these systems was subsequently presented in [19, 13] (and in [20] for a non-vanishing cosmological constant). Here, again, there is at least one possible mapping choice according to which length scaling in a BHL remains close to that in the FLRW class. It is already possible, however, to notice how equally legitimate choices of the mapping can lead to poor agreement. Furthermore, as in the contracting models, it is possible to calculate some effective parameters from the mapping, which show that mass gets a significant dressing when it is distributed discretely, so that fitting a model through its curvature leads to a substantial misestimation of the matter content.

The results for both classes of BHLs are summarized in Fig. 1. Overall, BHLs have proven to be interesting systems in which the effects of inhomogeneity can be studied in the full non-linear regime. Many questions, however, remain open: above all, why does the FLRW-behaviour in the scaling of lengths appear to remain dominant, even though these spaces are by no means perturbative?

3.2 Inhomogeneous dust cosmologies

Ideally, a numerical study of inhomogeneous cosmologies should have a continuous parameter controlling the strength of the inhomogeneous features, thereby allowing one to observe the non-perturbative effects as they emerge out of the perturbative regime, where we have solid analysis tools and physical expectations based on cosmological perturbation theory. In this sense, BHLs are not optimal systems;

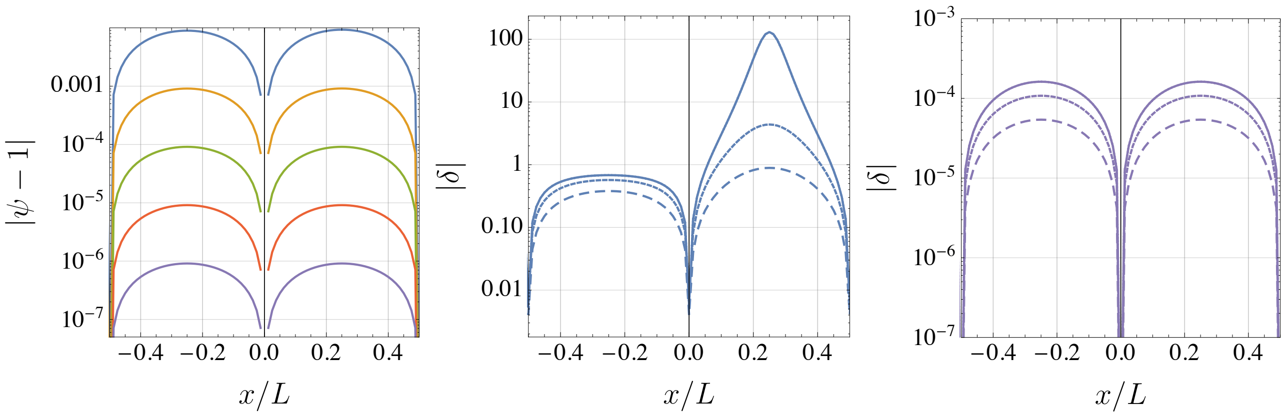

Figure 2: Left: solution of equation (7) along the cube diagonal, for $\delta_i = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and 10^{-6} (top to bottom curve). Middle and right: density contrast at $a/a_i = 30, 60$ and 90 for the $\delta_i = 10^{-2}$ and 10^{-6}, respectively, along the cube diagonal. Unlike the latter, the former clearly leaves the perturbative regime (as soon as $|\delta| \gtrsim 1$) during the evolution.

the cosmologies that are best known perturbatively are the FLRW models.

Possibly the simplest example is a cosmological model with the stress-energy of a perfect fluid with zero pressure (usually referred to as *dust*), a setting that had recently been investigated in [21, 22] . Using the Einstein Toolkit coupled to a new module for the evolution of a perfect fluid, I calculated the evolution of initial data describing a universe filled with perturbed dust [14]. The dust has the following initial profile:

$$\rho_i = \bar{\rho}_i(1 + \delta_i \sum_{j=1}^{3} \sin \frac{2\pi x^j}{L}) \tag{6}$$

where $\bar{\rho}_i$ is the density of an Einstein-de Sitter model at a chosen initial time t_i and δ_i is the initial magnitude of the density contrast (defined as $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$). We studied a cube of this model of comoving edge L, with periodic boundary conditions at the faces.

We needed, first, to construct a metric tensor γ_{ij} compatible with this density profile. To do so, we chose a constant, diagonal extrinsic curvature $K_{ij} = \frac{K}{3}\delta_{ij}$, set $\gamma_{ij} = \psi^4\delta_{ij}$, and solved the Hamiltonian constraint for ψ

$$\Delta\psi - \left(\frac{K_i^2}{12} - 2\pi\rho_i\right)\psi^5 = 0\,, \tag{7}$$

where $K_i \equiv -\sqrt{24\pi\bar{\rho}_i}$, obtaining the profiles in the left panel of Fig. 2 for $\delta_i = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and 10^{-6}. For this choice of K_{ij}, the momentum constraint is identically satisfied.

Having constructed the initial data, we evolved it in time and followed some observables throughout the expansion (which we ended when the final scale factor, defined as $a = V_{\mathcal{D}}^{1/3}$, $V_{\mathcal{D}}$ being the cube volume, was roughly one hundred times its initial value). We noticed, first, that our initial data led to an initial transient phase where the volume expansion proceeded at a faster rate than the background EdS solution. Furthermore, during the same initial phase the backreaction function $\mathcal{Q}_{\mathcal{D}}$ [23]:

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{3}(\langle K^2\rangle_{\mathcal{D}} - \langle K\rangle_{\mathcal{D}}^2) - 2\langle A^2\rangle_{\mathcal{D}} \tag{8}$$

(where $A^2 \equiv \frac{1}{2}A_{ij}A^{ij}$, and A_{ij} is the traceless part of K_{ij}) rapidly grew from its zero initial value. This phase had roughly the same duration for all values of δ_i, ending around $a/a_i \sim 12$ (a being the same-time value of the scale factor in the background EdS model, and a_i its initial value). After this phase, $\mathcal{Q}_{\mathcal{D}}$ became negative but remained small, contributing a small deceleration of the overall expansion with respect to the background EdS model. Furthermore, $\mathcal{Q}_{\mathcal{D}}$ appeared (at least for some time) to stay close to the second-order perturbative result for small values of the initial density contrast, just like expected. To the best of our knowledge, this was the first exact calculation of this quantity in a three-dimensional, fully relativistic model.

The expansion is therefore not very affected by the inhomogeneities, a conclusion similar to that obtained in [21, 22]; this happens also for the largest value of the initial density perturbation, where the overdensities have exited the perturbative regime and collapsed by the time $a/a_i \sim 90$, as shown in Fig. 2. A number of other interesting properties are discussed in [14], including a comparison with the top-hat collapse model. Other features, such as proper lengths, are investigated in [21, 22], and many more will surely be addressed in future works.

4 Conclusions

The experiments described in this contribution show how certain non-perturbative, relativistic effects in cosmological models can only be reproduced and analyzed via numerical models. Building fully relativistic models of the Universe, a feat traditionally considered beyond our technical capabilities, is now starting to become possible in simple settings, and has already led to surprising answers. Numerical relativity applied to cosmology promises to develop into an interesting and useful field, provided we manage to adapt its tools and algorithms to the new problems; this is certainly one of the major challenges in this field for the upcoming years.

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