Accretion of a relativistic kinetic gas into a black hole

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1. Motivation and summary

Our goal is to provide rigorous results regarding the accretion of a relativistic kinetic gas into a black hole. To this purpose, in this work we start with the simple case which is based on the following assumptions: (i) collisions between the gas particles may be neglected, (ii) the self-gravity of the gas is unimportant, (iii) the black hole is non-rotating. As a consequence of these assumptions, each gas particle follows a future-directed timelike geodesic in the Schwarzschild geometry.

In the first part of this work we derive the most general collisionless distribution function on a Schwarzschild spacetime describing accretion. This is achieved by exploiting the natural Hamiltonian structure of the cotangent bundle T^*M associated with the spacetime manifold and by constructing suitable symplectic coordinates on T^*M which trivialize the Liouville vector field.

In the second part we assume, in addition, that the gas flow is spherically symmetric and stationary, and show that these assumptions lead to a one-particle distribution function which depends only on the mass m, the energy E and the total angular momentum L of the particle. The value of L plays an important role in distinguishing those particle that fall into the black hole from those that are reflected at the centrifugal barrier. As we show, the former particles contribute to the accretion rate but not to the particle density n_∞ at infinity, while the reflected particles yield a positive n_∞ but do not contribute to the accretion rate.

To provide an explicit example, we consider the steady-state spherical accretion of a simple, collisionless gas which is described by an equilibrium distribution function at infinity. We compute the particle current density and the stress-energy tensor as a function of inverse temperature β. In the limit $\beta \to \infty$ we reproduce the results in [1] for the accretion rate that were obtained mostly with Newtonian calculations. Furthermore, we also compute the energy density and radial and tangential pressures p_{rad} and p_{tan} at the horizon. When $\beta \to \infty$ we show that p_{tan} is much larger than p_{rad}, which provides a partial explanation for the fact that the accretion is much less intense than in the perfect, polytropic fluid case where $p_{tan} = p_{rad}$.

We use geometrized units where the gravitational constant and speed of light are set equal to one.

2. Kinetic theory on the cotangent bundle

In the general relativistic description, the one-particle phase space is a suitable subset of the tangent or cotangent bundle associated with the spacetime manifold M, see for instance [2, 3]. Since this work is based on the Hamiltonian framework, it is more natural to work on the cotangent bundle, defined as

$$T^*M := \{(x,p) : x \in M, p \in T_x^*M\}. \tag{1}$$

Hence, T^*M consists of points (x,p), where $x \in M$ is a spacetime event representing the location of the particle and $p \in T_x^*M$ is a co-vector at this event, representing the canonical momentum of the particle. Therefore, the phase space should be restricted to those p that are future-directed timelike.

The cotangent bundle T^*M admits a natural symplectic structure which is constructed from the Poincaré one-form Θ on T^*M, given by

$$\Theta_{(x,p)}(X) := p\left(d\pi_{(x,p)}(X)\right), \tag{2}$$

for a vector field X on T^*M, where here $\pi : T^*M \to M$, $(x,p) \mapsto x$ is the natural projection map and $d\pi_{(x,p)} : T_{(x,p)}(T^*M) \to T_xM$ its differential at (x,p). In local coordinates (x^μ, p_μ), we have

$$\Theta = p_\mu dx^\mu, \tag{3}$$

and thus the symplectic structure is given by the closed, non-degenerated two-form

$$\Omega := d\Theta = dp_\mu \wedge dx^\mu \tag{4}$$

on T^*M. The symplectic form Ω allows us to assign to any function H on T^*M a unique vector field X_H on T^*M, called the associated Hamiltonian vector field, which is defined by

$$dH = \Omega(\cdot, X_H). \tag{5}$$

In local coordinates (x^μ, p_μ) one has

$$X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu}\frac{\partial}{\partial p_\mu}, \tag{6}$$

and the integral curves of X_H are described by Hamilton's equations

$$\frac{dx^\mu}{d\lambda} = +\frac{\partial H}{\partial p_\mu}(x,p), \qquad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}(x,p). \tag{7}$$

For the following, we consider the free-particle Hamiltonian

$$H(x,p) := \frac{1}{2}g_x^{-1}(p,p) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu. \tag{8}$$

The associated Hamiltonian vector field $L := X_H$ is called the Liouville vector field; its local coordinate expression reads

$$L = g^{\mu\nu}(x)p_\nu\frac{\partial}{\partial x^\mu} - \frac{1}{2}p_\alpha p_\beta\frac{\partial g^{\alpha\beta}(x)}{\partial x^\mu}\frac{\partial}{\partial p_\mu}. \tag{9}$$

The corresponding integral curves, when projected onto the spacetime manifold (M, g), describe geodesics.

The Liouville (or collisionless Boltzmann) equation, which describes the evolution of a collisionless distribution function f on the cotangent bundle is simply

$$L[f] = 0. \tag{10}$$

In the following, we derive the most general solution of this equation for the case where (M, g) is a Schwarzschild black hole. This will be achieved by introducing new symplectic coordinates on T^*M which trivialize the Liouville vector field L.

3. Accretion on a Schwarzschild background

From now on, we consider a Schwarzschild black hole of mass $M_b > 0$. Since we are interested in computing observables in the exterior region and on the horizon, it is convenient to describe the metric in terms of horizon-penetrating Eddington-Finkelstein coordinates $(t, r, \vartheta, \varphi)$, for which

$$g = -\left(1 - \frac{2M_b}{r}\right)dt^2 + \frac{4M_b}{r}dtdr + \left(1 + \frac{2M_b}{r}\right)dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d^2\varphi\right).$$

This spacetime is stationary and spherically symmetric, hence the following quantities are conserved along the particle trajectories:

$$E = -p_t \quad \text{(energy)}, \tag{11}$$

$$\ell_z = p_\varphi \quad \text{(azimutal angular momentum)}, \tag{12}$$

$$L = \sqrt{p_\vartheta^2 + \frac{\ell_z^2}{\sin^2\vartheta}} \quad \text{(total angular momentum)}, \tag{13}$$

$$m = \sqrt{-2H} \quad \text{(rest mass)}. \tag{14}$$

Given E, ℓ_z, L, m, the canonical momentum of the particle can be reconstructed from these equations which yield the relation

$$\left[p_r\left(1 - \frac{2M_b}{r}\right) - \frac{2M_b}{r}E\right]^2 + V(r) = E^2 \tag{15}$$

for p_r, with the effective potential

$$V(r) = \left(1 - \frac{2M_b}{r}\right)\left(m^2 + \frac{L^2}{r^2}\right). \tag{16}$$

Since we are only interested in particles coming in from the asymptotic region $r \to \infty$, we restrict ourselves to the energy range $m < E < \infty$. For each fixed value of E in this range, the incoming particle either falls into the black hole, or it is reflected at the potential barrier and returns to the asymptotic region. Which of the two cases occurs depends on the value of the total angular momentum L:

* *absorbed particles*: These particles have total angular momentum L in the range $0 \leq L < L_c(E)$, where $L_c(E)$ corresponds to the value of L for which the maximum of the effective potential V is equal to E^2. In this case, the sign of the expression inside the square parenthesis in Eq. (15) is negative, corresponding to a negative radial velocity. As it turns out [4], these particles do not contribute to the particle density n_∞ at infinity.
* *reflected particles*: These particles have angular momentum lying in the range $L_c(E) < L < L_{max}(r)$, where $L_{max}(r)$ is the maximum possible angular momentum at position r. Since these particles are reflected at the potential, both signs in Eq. (15) need to be considered. These particles do contribute to n_∞; however they do not contribute to the accretion rate.

The most general collisionless distribution function

Since the motion possess four conserved quantities, the Hamiltonian system describing geodesic motion on the Schwarzschild spacetime is integrable. Using the method of Hamilton-Jacobi [5], we find a symplectic transformation $(x^\mu, p_\mu) \mapsto (Q^\alpha, P_\alpha)$ which trivializes the Liouville vector field L, and in these new coordinates the Liouville equation (10) is simply

$$\frac{\partial}{\partial Q^0}f = 0,$$

whose general solution is

$$f(x,p) = F(Q^1, Q^2, Q^3, P_0, P_1, P_2, P_3), \tag{17}$$

for some function F. Here, the new P_α-variables are the conserved quantities $P_0 = m^2/2$, $P_1 = E$, $P_2 = \ell_z$, $P_3 = L$, and the relevant Q^α-variables are given by the following integrals:

$$Q^1 = -t + \int^r\left(\frac{E}{Np_r - \frac{2M_b}{r}E} + \frac{2M_b}{r}\right)\frac{dr}{N}, \qquad N := 1 - \frac{2M_b}{r}, \tag{18}$$

$$Q^2 = \varphi - \ell_z\int^\vartheta\frac{d\vartheta}{p_\vartheta\sin^2\vartheta}, \tag{19}$$

$$Q^3 = -L\int^r\frac{dr}{r^2\left(Np_r - \frac{2M_b}{r}E\right)} + L\int^\vartheta\frac{d\vartheta}{p_\vartheta}. \tag{20}$$

Note that in general, the quantities Q^1, Q^2 and Q^3 are multivalued since p_r and p_ϑ are only determined up to a sign. However, in our case, the integrals over r are well-defined since the projection of the trajectories onto the (r, p_r)-plane are open. In contrast to this, the projection of the trajectories onto the (ϑ, p_ϑ)-plane describe closed curves, and one can show that the variables Q^2 and Q^3 change by a factor of 2π under a complete revolution about these curves. Therefore, one needs to require the function F to be 2π-periodic in the variables Q^2 and Q^3 for the distribution function f to be well-defined.

4. Observables and spherical steady-state accretion

From now on, we assume that the gas flow is spherically symmetric and stationary. One can show [4] that this implies that the function F in Eq. (17) is independent of the Q^α's and P_2. Furthermore, we consider a simple gas consisting of identical particles with positive rest mass m. Consequently,

$$f(x,p) = F_m(E, L),$$

with a function F_m which depends on E and L only. In order to focus on a specific example, we assume the distribution function to represent a relativistic gas in thermodynamic equilibrium at $r \to \infty$, such that [6]

$$F_m(E, L) = \alpha e^{-\beta E}, \tag{21}$$

with a normalization constant $\alpha > 0$, and where $\beta = (k_BT)^{-1}$ with k_B Boltzmann's constant and T the temperature of the gas at infinity.

In order to understand the properties of the gas, we compute the particle current density J_μ and the stress-energy tensor $T_{\mu\nu}$ on M, which in local coordinates (x^μ, p_μ) are given by

$$J_\mu = \int_{\pi^{-1}(x)} f(x,p)p_\mu\pi_x, \qquad T_{\mu\nu} = \int_{\pi^{-1}(x)} f(x,p)p_\mu p_\nu\pi_x, \tag{22}$$

where $\pi_x = \sqrt{-\det g^{\mu\nu}(x)}d^4p$ is the induced volume element on the fibre $\pi^{-1}(x) = \{(x,p) : p \in T_x^*M\}$ over the event $x \in M$. In the following, we compute these observables in the asymptotic region $r \to \infty$ and at the event horizon $r = 2M_b$.

Observables at $r \to \infty$ for arbitrary temperature

In the asymptotic region, the gas behaves as a perfect fluid [6, 7],

$$J^\mu|_{r\to\infty} = n_\infty U^\mu, \qquad T^{\mu\nu}|_{r\to\infty} = (\epsilon_\infty + p_\infty)U^\mu U^\nu + p_\infty g^{\mu\nu},$$

where the four-velocity is $U = U^\mu\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}$, and the particle density n_∞, energy density ε_∞ and pressure p_∞ are given by

$$n_\infty = 4\pi m^4\alpha\frac{K_2(z)}{z}, \tag{23}$$

$$\varepsilon_\infty = 4\pi m^5\alpha\left[\frac{K_1(z)}{z} + 3\frac{K_2(z)}{z^2}\right], \qquad p_\infty = 4\pi m^5\alpha\frac{K_2(z)}{z^2}, \tag{24}$$

where

$$z := m\beta = \frac{mc^2}{k_BT}, \tag{25}$$

and $K_n(z)$ refer to the modified Bessel functions of the second kind.

Observables at $r = 2M_b$ for $\beta \to \infty$

At the horizon, the particle and energy densities n_H and ε_H and the radial and tangential pressures p_{rad} and p_{tan} are determined by the decomposition (see Ref. [8] and references therein)

$$J^\mu|_{r=2M_b} = n_H u^\mu, \tag{26}$$

$$T^\mu{}_\nu|_{r=2M_b} = \varepsilon_H e_0^\mu e_{0\nu} + p_{rad}e_1^\mu e_{1\nu} + p_{tan}e_2^\mu e_{2\nu} + p_{tan}e_3^\mu e_{3\nu}, \tag{27}$$

where $u^\mu u_\mu = -1$ and e_0, e_1, e_2, e_3 is an orthonormal frame of eigenvectors of $T^\mu{}_\nu$, e_0 being timelike and e_1 being radial.

In the limit $\beta \to \infty$ we obtain

$$\frac{\varepsilon}{n_H} = \frac{mc^2}{2\sqrt{3}}\left(3 + \sqrt{\frac{31}{3}}\right) \approx 1.79398mc^2, \tag{28}$$

$$\frac{p_{rad}}{n_H} = \frac{mc^2}{2\sqrt{3}}\left(-3 + \sqrt{\frac{31}{3}}\right) \approx 0.06193mc^2, \tag{29}$$

$$\frac{p_{tan}}{n_H} = \frac{mc^2}{4\sqrt{3}} \approx 0.1443375mc^2. \tag{30}$$

We note that the tangential pressure is more than twice as large as the radial one, showing that the collisionless kinetic gas behaves very differently than a perfect fluid near the horizon. This difference is probably due to the fact that most gas particles have nonvanishing angular momenta and do not collide.

Finally, we compute the accretion rate $\mu := 4\pi J^r/n_\infty$ and compression ratio n_H/n_∞ of the gas. For large z we find

$$\mu \approx -16M_b^2\sqrt{2\pi z}, \qquad \frac{n_H}{n_\infty} \approx \sqrt{\frac{6z}{\pi}}. \tag{31}$$

For the typical situation of gas being accreted by the interstellar medium, $z \approx 10^9$. These quantities are smaller by a factor of z compared to the corresponding quantities in the Michel model [9, 1, 10], describing the spherical steady-state accretion of a polytropic perfect fluid.

5. Conclusions

We have provided a systematic discussion for the properties of a collisionless, relativistic kinetic gas which is accreted by a Schwarzschild black hole. In particular, we have derived the most general collisionless distribution function describing such a gas. Additionally, we have applied our results to the spherical steady-state accretion of a gas into a nonrotating black hole, assuming that the gas is in thermodynamic equilibrium at infinity. For low temperatures, we recover the results described in [1] and showed that in this case the tangential pressure is much larger than the radial one and thus diminishes the accretion process. This provides a partial explanation for the fact that the accretion of a collisionless gas is much less intense than the accretion of a perfect, polytropic fluid in the Michel model.

It should be interesting to generalize our study to include collisions, and to check whether collisions are able to "channel the flow effectively in the radial direction" [1].

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