Rapid particle acceleration at perpendicular shocks

John Kirk\textsuperscript{1}, Makoto Takamoto\textsuperscript{2}

\textsuperscript{1}Max-Planck-Institut für Kernphysik
Heidelberg
\textsuperscript{2}Department of Earth and Planetary Science
University of Tokyo

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Galactic Cosmic Rays

- \( r_g < \text{galactic disk} \)
  \[ \Rightarrow \quad E_p < 10^{18} \text{ eV} \]
- Single mechanism
- SNR shocks?

\[
\begin{array}{c|cccccc}
\text{Energy (eV/particle)} & 10^{13} & 10^{14} & 10^{15} & 10^{16} & 10^{17} & 10^{18} \\
\text{Scaled flux } E^{-2.5} J(E) (m^2 s^{-1} sr^{-1} eV^{-5}) & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{Equivalent c.m. energy } \sqrt{s_{pp}} (\text{GeV}) & 10^2 & 10^3 & 10^4 & 10^5 & 10^6 \\
\end{array}
\]

- RHIC (p-p)
- HERA (\( \gamma \)-p)
- Tevatron (p-p)
- 7 TeV 14 TeV
- LHC (p-p)

\[
\begin{array}{c|c|c|c|c}
\text{HiRes-MIA} & \text{HiRes I} & \text{HiRes II} & \text{Auger 2009} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\text{ATIC} & \text{PROTON} & \text{RUNJOB} & \text{KASCADE (QGS/JET 01)} & \text{KASCADE (SIBYLL 2.1)} & \text{KASCADE-Grande 2009} & \text{Tibet ASg (SIBYLL 2.1)} \\
\end{array}
\]

**Galactic Cosmic Rays**

- $r_g < \text{galactic disk}$
  \[ \Rightarrow \quad E_p < 10^{18} \text{ eV} \]
- Single mechanism
- SNR shocks?
- DSA:
  \[ E_p < \text{few} \times 10^{15} \text{ eV} \]

![Graph showing scaling flux and equivalent c.m. energy](image)

For test particles, DSA too slow: \( \epsilon = \frac{u_s}{v_{cr}} \ll 1 \), a first order Fermi process, \( (\Delta p/p \sim \epsilon) \) but \( t_{\text{cycle}}^{-1} \sim u_s v_{cr}/\kappa \sim \epsilon \omega_g \),

\[ \Rightarrow t_{\text{acc}}^{-1} \sim \epsilon^2 \omega_g \]
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$\Rightarrow t_{\text{acc}}^{-1} \sim \epsilon^2 \omega_g$

The problem applies to parallel shocks, where $\kappa = \kappa_{\parallel} \approx \eta \omega_g$ and $\eta \gtrsim 1$. *Particles spend a long time wandering up and down field lines, without crossing the shock*
An old story

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Not so at perpendicular shocks (Jokipii 1987):
\( \kappa = \kappa_\perp \approx \omega_g / \eta \), so that \( t_{\text{acc}}^{-1} \sim \eta \epsilon^2 \omega_g \), which is rapid for \( \eta \gg 1 \).
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Fundamental objection: diffusion approximation valid only for small anisotropy. At a parallel shock, anisotropy \( \sim \epsilon \). But, for \( \eta \gg 1 \), a perpendicular shock should drive a strong anisotropy.
Transport model

Diffusion in direction of motion:

\[ \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \omega_g \frac{\partial f}{\partial \phi} = \nu_{\text{coll}} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right] \]

Valid up/downstream. No deflection by the shock itself.
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Previous work:

  Nonrelativistic case expensive, only stationary solutions.
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- Expansion in spherical harmonics and finite difference solution: Bell, Schure & Reville (2011), only stationary solutions.
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- Analytic approach using eigenfunctions (stationary case).
- SDE solution (≈ MC), stationary and time-dependent cases.
Stationary solution

Separation of variables:

\[ f(z, \vec{p}) = p^{-s} \sum_i c_i e^{\Lambda_i z \omega_g/\nu} Q_i(\mu, \phi) \]

\[ \Lambda_i (\hat{v}_z - u) Q_i = \left\{ -\frac{\partial}{\partial \phi} + \frac{1}{2\eta} \left[ \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2}{\partial \phi^2} \right] \right\} Q_i \]

\( (\hat{v}_z = \sqrt{1 - \mu^2} \sin \phi, \ \eta = \omega_g/\nu_{\text{coll}}. ) \)
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- But two-parameter \((\eta, u)\) problem in two-dimensions \((\mu, \phi)\).
- Approximate by retaining only the ‘leading’ upstream eigenfunction.
Approximate analytic solution for $u_s \sim 1/\eta \sim \epsilon \ll 1$

- $Q = e^{\Lambda \nu} \sqrt{1-\mu^2} \cos \phi P_{S_0}^0 (\mu, -\Lambda^2/2)$
- $P_{S_n}^m$: angular, oblate, spheroidal wave function.

Leading eigenfunction, $\eta u_s = 2$,

Anisotropic at order $\epsilon^0$, as suggested by Schatzman (1963).
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- \( \lambda_0^0 (-\Lambda^2/2) = \Lambda (\Lambda + 2\eta u) \)
  
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  (Fisch & Kruskal 1980).
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- Series in $\eta u$:

  $$s = \frac{3r}{r-1} + \frac{9(r+1)}{20r(r-1)} \eta^2 u_s^2 + O(\eta^4 u_s^4)$$

  ($r =$ compression ratio)

  Leading eigenfunction, $\eta u_s = 2$,

  Anisotropic at order $\epsilon^0$, as suggested by Schatzman (1963).
Numerical solution of SDE’s

Angular distribution

- $\eta = 22, \ u_s = 0.012$.

- Upper: $\int_{-1}^{1} d\mu f(\mu, \phi)$

- Lower: $\int_{0}^{2\pi} d\phi f(\mu, \phi)$
Numerical solution of SDE’s

- \( \eta = 1–100 \) (10 values)
- \( u_s = 0.01–0.2 \), (30 values)

- Top: spectral index
- Bottom: acceleration rate/DSA (Jokipii) prediction

\[
\frac{-d \ln (f)}{d \ln (p)} = s - 0.7 \eta u_s + 4
\]

\[
t_{acc}^{-1} = t_{DSA}^{-1} \left/ (1.1 \eta u_s + 1.0) \right.
\]
Conclusions

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- For \(\eta u_s \sim 1\), \(s\) softens by \(\sim 1\), acceleration rate slows by factor \(\sim 2\)
Conclusions

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- For weak collisionality: fan beam, opening angle \(\approx (\eta u_s)^{-1}\)
- For \(\eta u_s \sim 1\), \(s\) softens by \(\sim 1\), acceleration rate slows by factor \(\sim 2\)
- Maximum CR energy for SN in WR-star wind (DSA):

\[
E_{\text{max}} = 3 \frac{3}{8} \eta u_s \left( \frac{R_* \Omega}{\nu_w} \right) eB_* R_*
\]

\[
= 1.7 \times 10^{16} \eta u_s \left( \frac{R_* \Omega}{\nu_w} \right) \left( \frac{B_*}{50 \text{ G}} \right) \left( \frac{R_*}{3.10^{12} \text{ cm}} \right) \text{ eV}
\]

not changed much provided \(\eta u_s \sim 1\).