

Saturation of the f -mode instability in neutron stars

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in collab. with K. D. Kokkotas

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Outline

1 Oscillation modes

2 The CFS instability

- The instability window

3 Instability saturation

- Non-linear mode coupling
- Parametric resonance
- Saturation conditions

4 Results

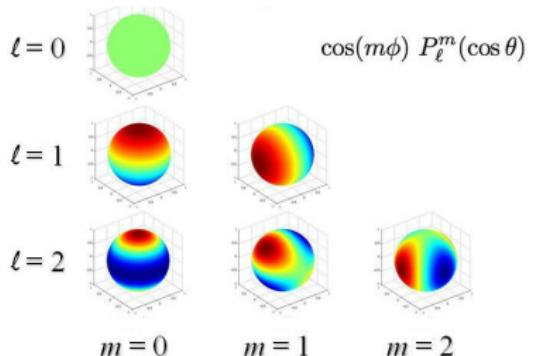
- Supernova-derived neutron stars
- Merger-derived neutron stars
- Stochastic background

Oscillation modes

$$\xi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l [U_l^m(r)Y_l^m(\theta, \phi)\hat{e}_r + V_l^m(r)\nabla Y_l^m(\theta, \phi) + W_l^m(r)\hat{e}_r \times \nabla Y_l^m(\theta, \phi)]$$

- Polar modes: $W_l^m = 0$
- Axial modes: $U_l^m = V_l^m = 0$ as $\Omega \rightarrow 0$

l :	degree
m :	order
n :	overtone



Mode name	Mode class	Mode type	Restoring force
p -mode	Polar	Sound wave ($\omega \rightarrow \infty$ as $n \rightarrow \infty$)	Pressure gradient
f -mode	Polar	Low- ω sound wave High- ω gravity wave	$n = 0$
g -mode	Polar	Gravity wave ($\omega \rightarrow 0$ as $n \rightarrow \infty$)	Buoyancy
r -mode	Axial	Inertial wave	Coriolis
<i>Hybrid mode</i>	Combination	Zero-buoyancy limit or r - and g -modes	

- Only for non-zero rotation

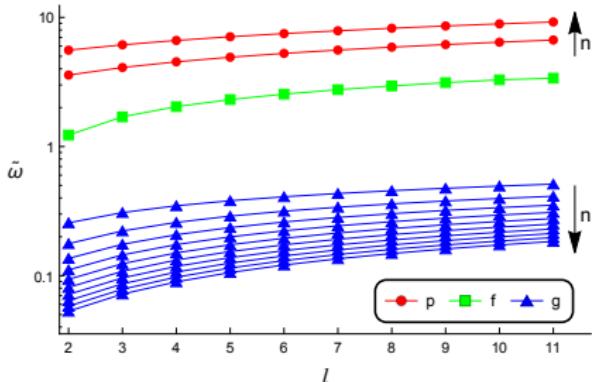
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The CFS instability

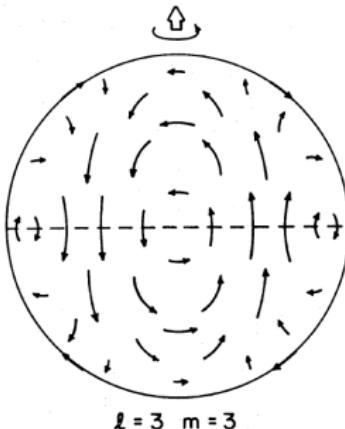
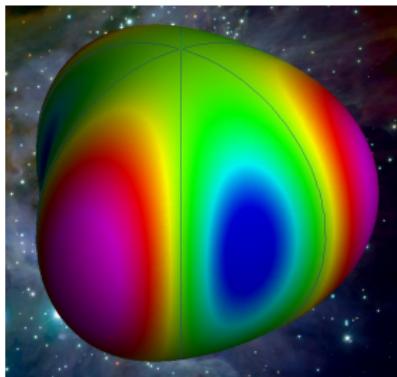
- Chandrasekhar (1970) first realised its existence for Maclaurin spheroids
- Friedman and Schutz (1978) proved that the instability is *generic*

For *any* angular velocity Ω there is always a mode driven unstable by GW emission

- Multipole expansion of power radiated in GWs (Thorne, 1980):

$$\left(\frac{dE}{dt} \right)_{\text{GW}} = - \sum_{l \geq 2}^{\infty} N_l \omega (\omega - m\Omega)^{2l+1} \left(|\delta D_l^m|^2 + |\delta J_l^m|^2 \right)$$

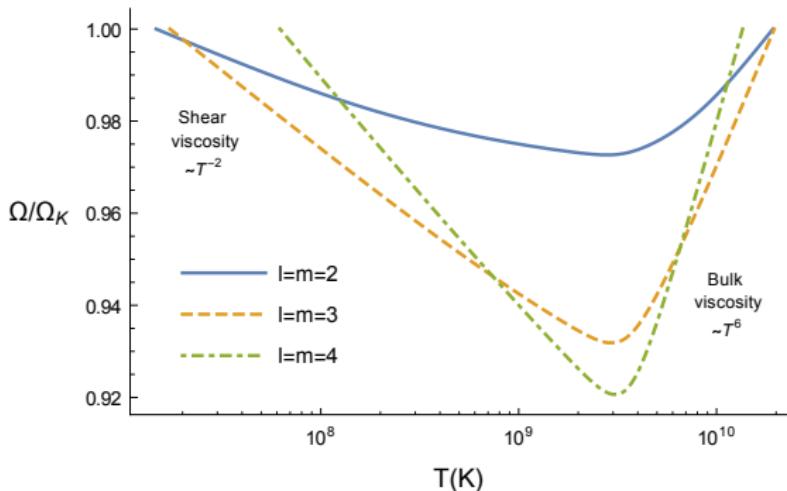
- If $\omega (\omega - m\Omega) < 0$, then $(dE/dt)_{\text{GW}} > 0$
- Polar (axial) modes emit through the mass (current) multipoles
- Most susceptible to the CFS instability are the *f-modes* and *r-modes*



Octupole ($l = m = 3$)
f-mode [left, credit: Wolfgang Köhler, GFZ Potsdam] and
r-mode [right, credit: Saio (1982)]; **density** and **velocity**
perturbations dominate, respectively.

The CFS instability – The instability window

- The instability is suppressed by **viscosity** (Ipser and Lindblom, 1991)
- **Instability window:** $\left(\frac{dE}{dt}\right)_{GW} + \left(\frac{dE}{dt}\right)_v > 0$



Instability windows of the quadrupole, octupole, and hexadecapole **f-modes**, for a Newtonian star with $p \propto \rho^3$.

Shear viscosity, due to particle scattering, and **bulk viscosity**, due to disturbance of β -equilibrium by the perturbation, dominate at low and high temperatures, respectively.

- The **r-mode** instability is favoured, because of i) much larger window, and ii) shorter growth time scales
- **Significance:**
 - Neutron star evolution [nascent (Bondarescu et al., 2009; Passamonti et al., 2013), LMXBs (Levin, 1999; Bondarescu et al., 2007)]
 - Gravitational wave asteroseismology (Andersson and Kokkotas, 1996, 1998)

Instability saturation – Mode coupling

- Non-linear *mode coupling* stops instability's growth (Dziembowski, 1982)
- *Linear* amplitude evolution:

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha$$

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- *Quadratic* amplitude evolution:

Modes couple in *triplets*

$$\begin{cases} \dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H} Q_\beta Q_\gamma e^{-i\Delta\omega t} \\ \dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \mathcal{H} Q_\gamma^* Q_\alpha e^{i\Delta\omega t} \\ \dot{Q}_\gamma = \gamma_\gamma Q_\gamma + i\omega_\gamma \mathcal{H} Q_\alpha Q_\beta^* e^{i\Delta\omega t} \end{cases}$$

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- Detuning $\Delta\omega \equiv \omega_\alpha - \omega_\beta - \omega_\gamma \approx 0$ resonance condition

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- Detuning $\Delta\omega \equiv \omega_\alpha - \omega_\beta - \omega_\gamma \approx 0$ resonance condition
- Coupling coefficient $\mathcal{H} \neq 0$ if

$$\left. \begin{array}{l} m_\alpha = m_\beta + m_\gamma \\ l_\alpha + l_\beta + l_\gamma = \text{even number} \\ |l_\beta - l_\gamma| \leq l_\alpha \leq l_\beta + l_\gamma \end{array} \right\} \text{coupling selection rules}$$

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- Growth/damping rates $\gamma_i = \frac{1}{2E_i} \frac{dE_i}{dt} \gtrless 0$

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_{GW} + \left(\frac{dE}{dt} \right)_V \gtrless 0$$

Instability saturation – Parametric resonance

$$\begin{array}{l|l} \dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H} Q_\beta Q_\gamma e^{-i\Delta\omega t} & \text{Detuning } \Delta\omega \\ \dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \mathcal{H} Q_\gamma^* Q_\alpha e^{i\Delta\omega t} & \text{Coupling coefficient } \mathcal{H} \\ \dot{Q}_\gamma = \gamma_\gamma Q_\gamma + i\omega_\gamma \mathcal{H} Q_\alpha Q_\beta^* e^{i\Delta\omega t} & \text{Growth/damping rates } \gamma_i \end{array}$$

- *Parent mode:* unstable r -mode ($\gamma_\alpha > 0$)
- *Daughter modes:* other (stable) axial modes ($\gamma_{\beta,\gamma} < 0$)

(Schenk et al., 2001;
Morsink, 2002; Arras
et al., 2003; Brink et al.,
2004, 2005)

Instability saturation – Parametric resonance

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- Parent mode: unstable f -mode ($\gamma_\alpha > 0$)
- Daughter modes: other (stable) polar modes ($\gamma_{\beta,\gamma} < 0$)

(PP and Kokkotas, 2015)

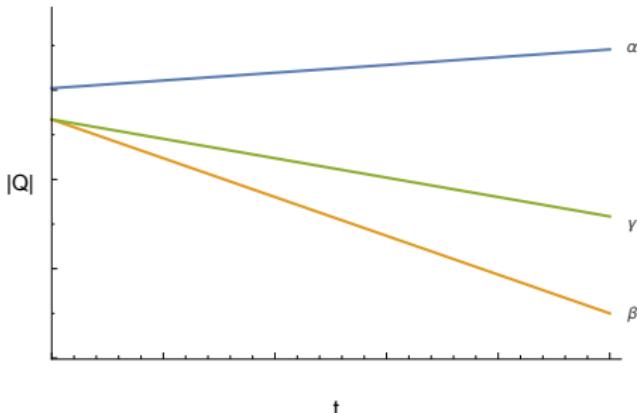
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No mode coupling: $\mathcal{H} = 0$ or $\Delta\omega \gg 0$



- Modes evolve independently
- No non-linear interaction

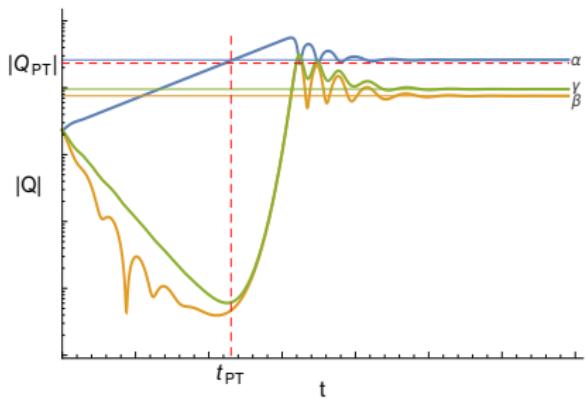
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- Parent mode: unstable f -mode ($\gamma_\alpha > 0$)
 - Daughter modes: other (stable) polar modes ($\gamma_{\beta, \gamma} < 0$)
- (PP and Kokkotas, 2015)

Parametric resonance: $\mathcal{H} \neq 0$ and $\Delta\omega \approx 0$



- Parent feeds daughters and makes them grow
- *Parametric threshold:* daughters grow when

$$|Q_\alpha|^2 > |Q_{PT}|^2 \equiv \frac{\gamma_\beta \gamma_\gamma}{\omega_\beta \omega_\gamma \mathcal{H}^2} \left[1 + \left(\frac{\Delta\omega}{\gamma_\beta + \gamma_\gamma} \right)^2 \right]$$

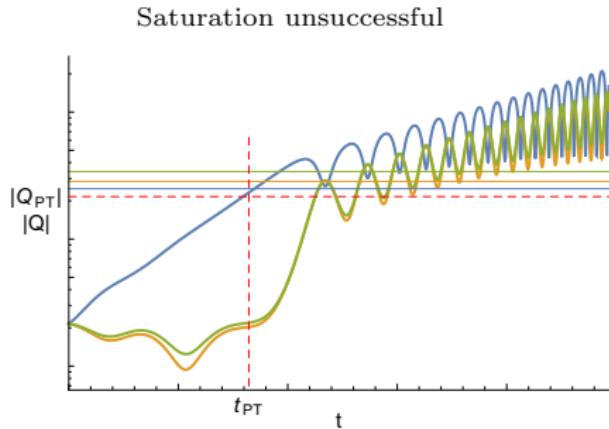
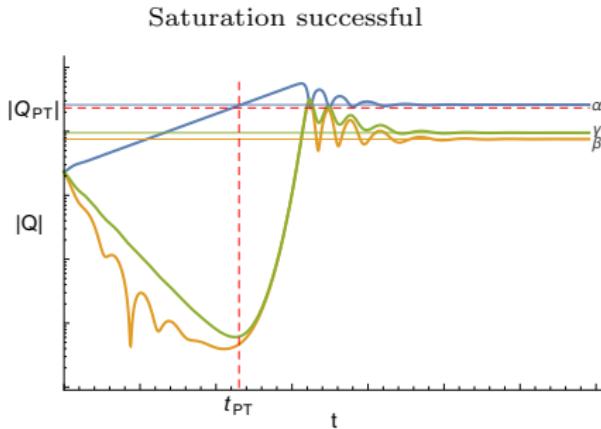
- $|Q_\alpha^{\text{sat}}| \approx |Q_{PT}|$

Instability saturation – Saturation conditions

$$\begin{array}{l} \dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H} Q_\beta Q_\gamma e^{-i\Delta\omega t} \\ \dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \mathcal{H} Q_\gamma^* Q_\alpha e^{i\Delta\omega t} \\ \dot{Q}_\gamma = \gamma_\gamma Q_\gamma + i\omega_\gamma \mathcal{H} Q_\alpha Q_\beta^* e^{i\Delta\omega t} \end{array} \quad \left| \begin{array}{c} \gamma_\alpha > 0, \gamma_{\beta,\gamma} < 0 \\ |Q_{PT}|^2 \equiv \frac{\gamma_\beta \gamma_\gamma}{\omega_\beta \omega_\gamma \mathcal{H}^2} \left[1 + \left(\frac{\Delta\omega}{\gamma_\beta + \gamma_\gamma} \right)^2 \right] \end{array} \right| \quad \begin{array}{c} \text{Detuning } \Delta\omega \\ \text{Coupling coefficient } \mathcal{H} \\ \text{Growth/damping rates } \gamma_i \end{array}$$

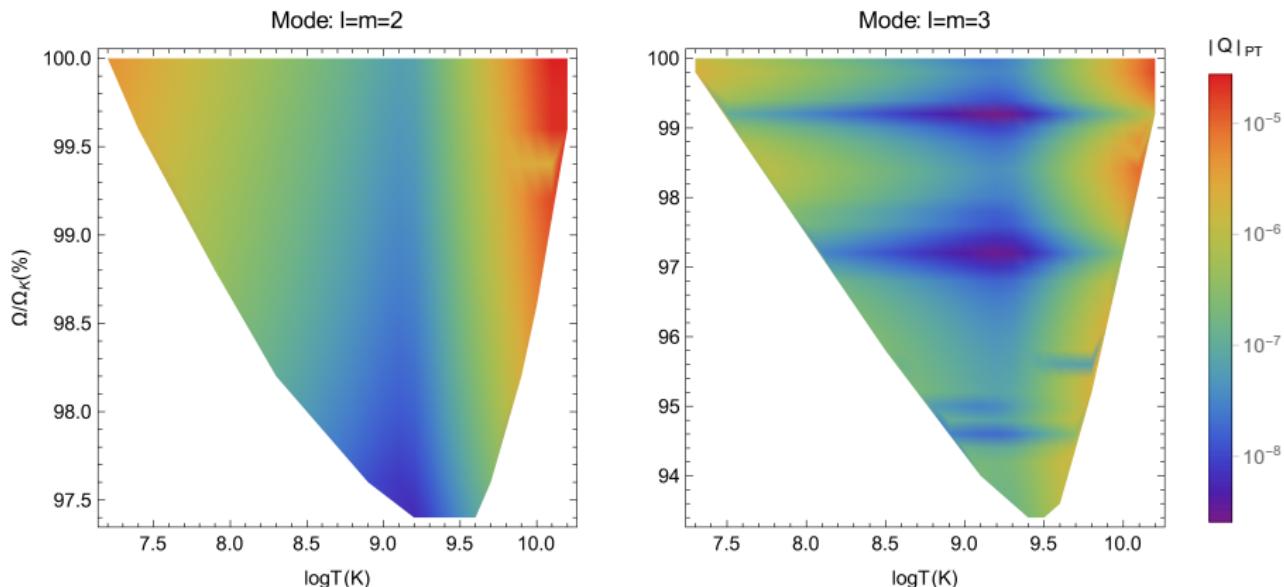
- Saturation successful if:

$$|\gamma_\beta + \gamma_\gamma| \gtrsim |\gamma_\alpha| \quad \text{and} \quad |\Delta\omega| \gtrsim |\gamma_\alpha + \gamma_\beta + \gamma_\gamma|$$



Results – Supernova-derived neutron stars

Saturation amplitude throughout the instability window
(PP and Kokkotas, in prep.)

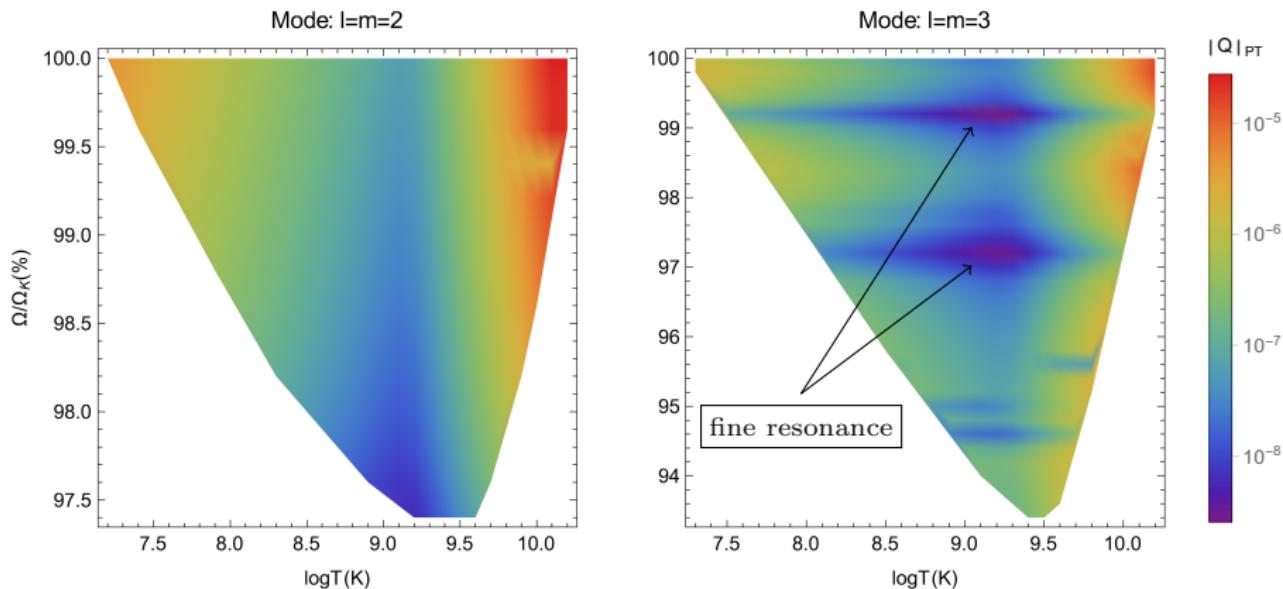


Model: $M = 1.4 M_\odot$, $R = 10 \text{ km}$, $p \propto \rho^3$
Units: $E_{\text{mode}} = |Q|^2 Mc^2$

$$|Q_{\text{sat}}| \propto \begin{cases} T^{-1}, & T \lesssim 10^9 \text{ K} \\ T^3, & T \gtrsim 10^9 \text{ K} \end{cases} \quad \text{for } \Omega = \text{const.}$$

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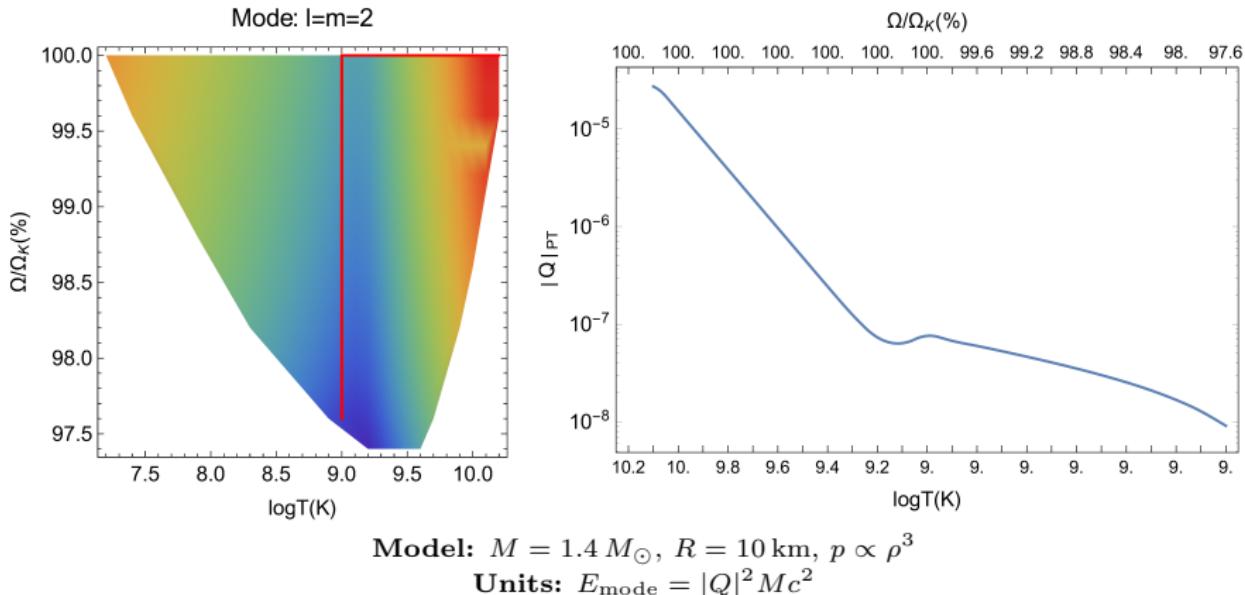


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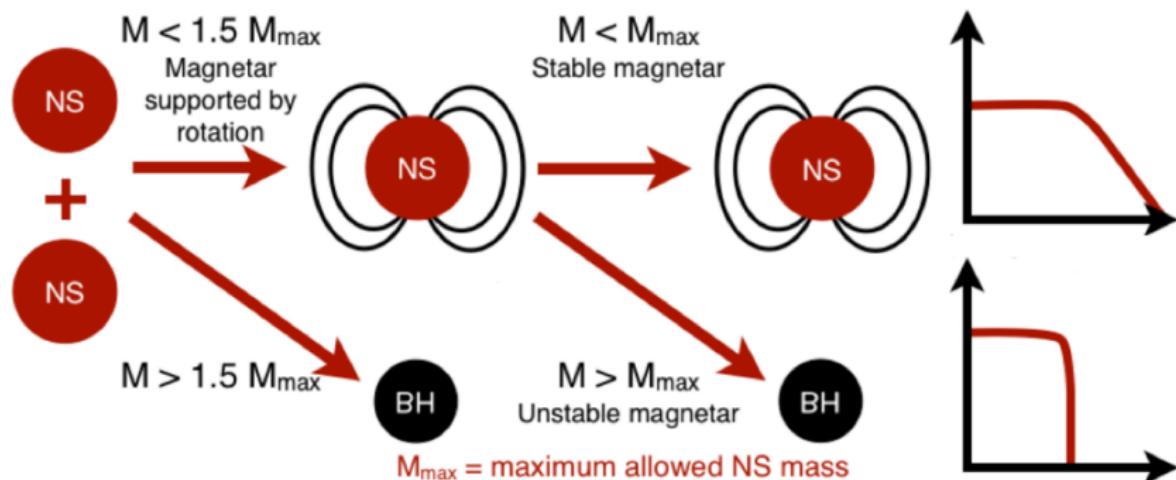
Saturation amplitude during hypothetical neutron star evolution



- **Competing mechanisms:** magnetic braking, r -mode instability
- Signal detectable with **ET** (**Adv. LIGO?**) from local galactic group (Passamonti et al., 2013, rescaled results)
- **Event rate:** \sim local group supernova event rate

Results – Merger-derived neutron stars

- Short γ -ray bursts associated with compact binary mergers
- Persistent X-ray afterglow suggests ongoing central engine activity (Rowlinson et al., 2013)

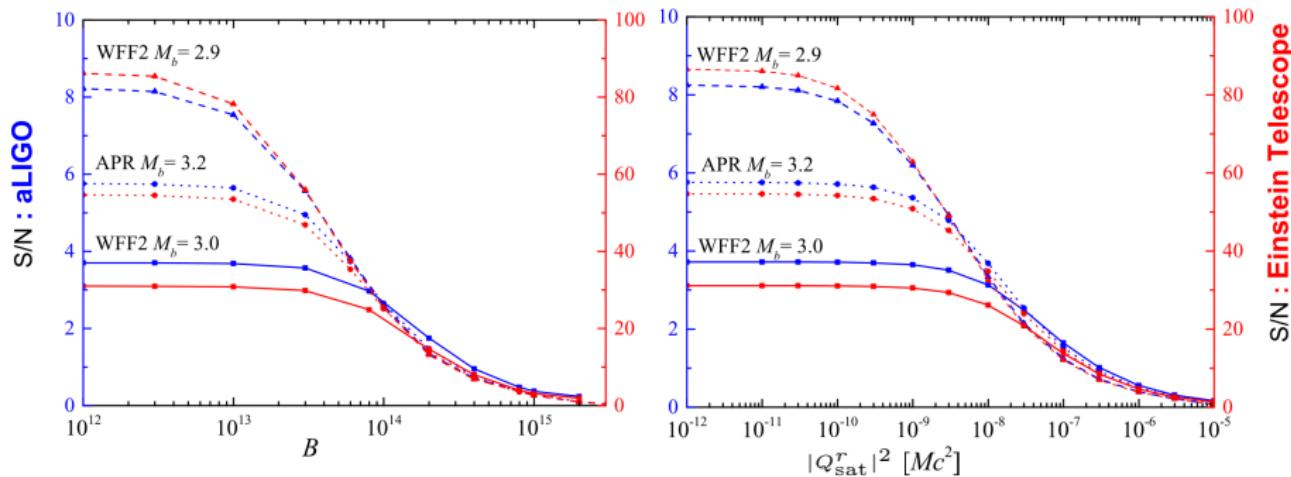


Credit: Rowlinson (2013)

- Rotationally supported, *supramassive* neutron stars can form after the binary merger
- **Lifetimes:** up to $\sim 10^4$ s (Ravi and Lasky, 2014)

Results – Merger-derived neutron stars

- f -mode instability develops rapidly ($\sim 10 - 100$ s) in supramassive stars (Doneva et al., 2015)

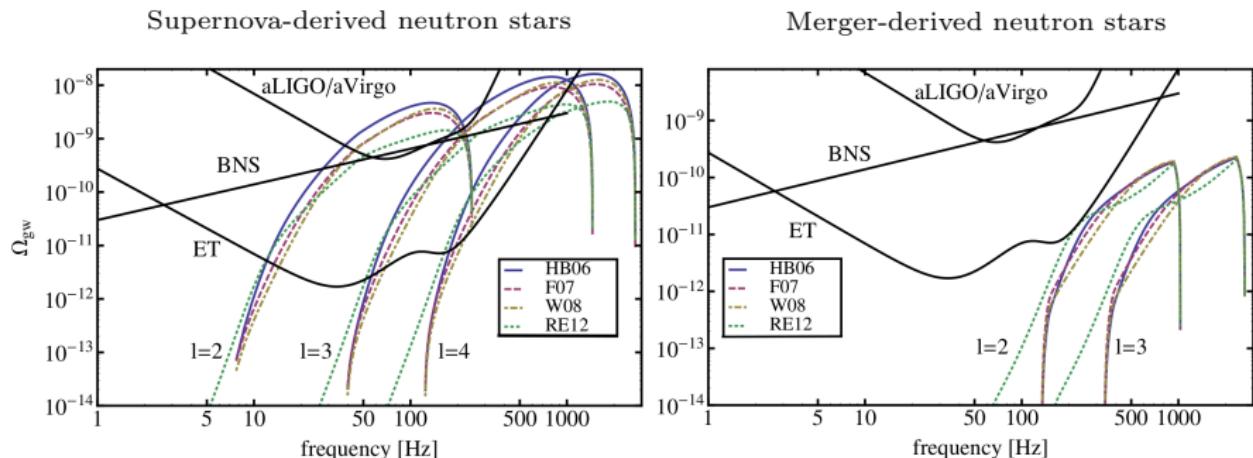


Signal-to-noise ratio from an unstable quadrupole f -mode
 $|Q_{\text{sat}}|^2 = 10^{-6}$, $d = 20$ Mpc

- **Competing mechanisms:** magnetic braking (left), r -mode instability (right)
- Signal detectable with **Adv. LIGO (ET)** at **20 Mpc (200 Mpc)**
- **Event rate:** \sim binary neutron star merger event rate (**few/yr** at **200 Mpc**)

Results – Stochastic background

- Superposition of unresolved GW signals from f -mode instabilities throughout the universe (Surace et al., 2015)



Dimensionless energy density Ω of the stochastic GW background, from $l = m = 2, 3, 4$ f -modes, for different cosmic star formation rate models

- Supernova-derived stochastic background optimally *detectable* with **Adv. LIGO** for the quadrupole f -mode
- Merger-derived stochastic background *undetectable* even with **ET**

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