Neutrinos beyond the linear regime: a new theoretical approach

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Figure 6.4 Matter power spectrum at redshift zero for a $\Lambda$CDM model with three degenerate massive neutrino species ($m_\nu = 0.3$ eV), compared to the individual power spectrum of CDM, baryon and neutrino density perturbations. In this model $k_{nr}$ is equal to $5.1 \times 10^{-3}$ $h$/Mpc (see Eq. (5.94)). For wavenumbers $k > k_{nr}$, neutrino perturbations remain smaller than CDM and baryon perturbations, because of their low growth rate after the nonrelativistic transition.

Source: Neutrino Cosmology (book)
(J.Lesgourgues, G. Mangano, G. Miele and S. Pastor, 2013)
Numerical simulations showing the effect of massive neutrinos on the nonlinear matter power spectrum ($m = 0.6$ eV)

Authors: S. Bird *et al.* (arXiv: 1109.4416)
a) The standard description of the nonlinear growth of structure

Review on the subject ➔ arXiv: 1311.2724 (F. Bernardeau)
In cold fluids, the velocity dispersion is negligible.

No curl mode can be generated in the velocity field $u_i$.

The velocity field is entirely characterized by its divergence:

$$\theta(x^i, t) = \frac{1}{aH} \frac{\partial u_i(x^i, t)}{\partial x^i}.$$ 

In reciprocal space, the system can be rewritten compactly with the help of the variable

$$\Psi_a(k, \eta) \equiv (\delta(k, \eta), -\theta(k, \eta))^T.$$

The resulting equation is:

$$a \frac{\partial \Psi_a}{\partial a}(k, \eta) + \Omega_a^b(\eta) \Psi_b(k, \eta) = \gamma_a^{bc}(k_1, k_2) \Psi_b(k_1, \eta) \Psi_c(k_2, \eta).$$
b) The standard description of massive neutrinos

All details can be found e.g. in *Neutrino Cosmology* (J. Lesgourgues, G. Mangano, G. Miele and S. Pastor, 2013).
• Neutrinos are described via their **phase-space** distribution function.

• The quantities useful for cosmology are obtained by computing the moments of the distribution function:

\[
T_{\mu\nu}(\eta, x^i) = \int d^3p_i \, (-g)^{-1/2} \frac{p_{\mu}p_{\nu}}{p^0} \, f(\eta, x^i, p_i).
\]

• The distribution function satisfies the **Vlasov equation**, 

\[
\frac{df(x, q, \eta)}{d\eta} = 0.
\]

• In **linear** perturbation theory, \( f = f_0(1 + \Psi) \).

• Explicit calculations are done with the help of **Boltzmann codes** (integration of the Boltzmann hierarchy).
c) Proposition of a new approach
NEUTRINOS AS A COLLECTION OF FLOWS

• Total distribution function: $f^{\text{tot}}(\eta, x, p) = \sum f^\tau(\eta, x, p)$.

• One density field per flow: $n_c(\eta, x; \bar{\tau}) = \int d^3 p_i f^\tau(\eta, x^i, p_i)$.

• In each flow, $f^\tau(\eta, x^i, p_i) = n_c(\eta, x; \bar{\tau}) \delta_D(p_i - P_i(\eta, x; \bar{\tau}))$. 

initial time

later time

initial momentum (label of the flow)

momentum field of the flow
• In each flow, \( P_i(\eta, x; \tau) = \frac{\int d^3p_i \ f_\tau(\eta, x^i, p_i)p_i}{\int d^3p_i \ f_\tau(\eta, x^i, p_i)} \).

• More generally, \( F[P_i(\eta, x)] \ n_c(\eta, x) = \int d^3p_i \ f(\eta, x^i, p_i) \ F[p_i] \).

• The physical quantities of interest can be expressed in terms of our fields:

\[
\int d^3p_i \ f^{\text{tot}}(\eta, x^i, p_i) \ F(p_i) = \int d^3\tau_i \ n_c(\eta, x; \tau_i) \ F(P_i(\eta, x; \tau_i)).
\]

Boltzmann approach \hspace{2cm} our approach
• In each flow, the equation of motion of the density field is

\[
\frac{\partial}{\partial \eta} n_c + \frac{\partial}{\partial x^i} \left( \frac{P^i}{P^0} n_c \right) = 0,
\]

where \( P^i = g^{ij} P_j \) and \( P^0 \) is defined so that \( P^\mu P_\mu = -m^2 \).

• In each flow,

\[ T^{\mu\nu} = -P^\mu J^\nu. \]

energy-momentum tensor particle four-current

Combined conservation laws impose

\[
P^\nu \partial_\nu P_i = \frac{1}{2} P^\sigma P^\nu \partial_i g_{\sigma\nu}.
\]

Relative differences (in units of $10^{-4}$) in comparison with the Boltzmann approach.

More details can be found in arXiv: 1311.5487 (H. Dupuy and F. Bernardeau).

$(l_{\text{max}} = 6, N_\mu = 12, N_q = N_\tau = 40, k = k_{\text{eq}} = 0.01 h/\text{Mpc}, m = 0.3 \text{ eV})$. 
USEFUL PROPERTIES ON SUBHORIZON SCALES

• In a perturbed Friedmann-Lemaître metric, the equations read:

\[ \frac{\partial}{\partial \eta} n_c + \frac{\partial}{\partial x^i} \left( \frac{P^i}{P^0} n_c \right) = 0, \]

\[ \frac{\partial P_i}{\partial \eta} + \frac{P^j}{P^0} \frac{\partial P_i}{\partial x^j} = a^2(\eta) \left[ -P^0 \partial_i A + P^j \partial_i B_j + \frac{1}{2} \frac{P^j P^k}{P^0} \partial_i h_{jk} \right]. \]

• In the subhorizon limit, they become:

\[ D_\eta n_c + \partial_i (V_i n_c) = 0, \]

\[ D_\eta P_i + V_j \partial_j P_i = \tau_0 \partial_i A + \tau_j \partial_i B_j - \frac{1}{2} \frac{\tau_j \tau_k}{\tau_0} \partial_i h_{jk}, \]

with \( \tau_0 = -\sqrt{m^2 a^2 + \tau_i^2} \), \( D_\eta = \frac{\partial}{\partial \eta} - \frac{\tau_i}{\tau_0} \frac{\partial}{\partial x^i} \)

and \( V_i = -\frac{P_i - \tau_i}{\tau_0} + \frac{\tau_i}{\tau_0} \frac{\tau_j (P_j - \tau_j)}{(\tau_0)^2}. \)
GENERALIZATION 1: NO CURL MODES IN THE MOMENTUM FIELD

• On subhorizon scales the curl field, defined as

\[ \Omega_i = \epsilon_{ijk} \partial_k P_j, \]

obeys the equation \( \mathcal{D}_\eta \Omega_k + V_i \partial_i \Omega_k + \partial_i V_i \Omega_k - \partial_i V_{k} \Omega_i = 0. \)

The curl field is only sourced by itself.

For adiabatic initial conditions, the comoving momentum field can be written as a gradient.

As the velocity field of cold dark matter, it is entirely characterized by its divergence.
By analogy with cold dark matter, we introduce

\[
\theta_{\tau_i}(\eta, x^i) = - \frac{P_{i,i}(\eta, x^i; \tau_i)}{ma\mathcal{H}}, \quad \delta_{\tau_i}(\eta, x^i) = \frac{n_c(\eta, x^i; \tau_i)}{n_c^{(0)}(\tau_i)} - 1.
\]

In reciprocal space, it gives

\[
\left( a \partial_a - i \frac{\mu k \tau}{\mathcal{H} \tau_0} \right) \delta_\tau(k) - \frac{ma}{\tau_0} \left( 1 - \frac{\mu^2 \tau^2}{\tau^2_0} \right) \theta_\tau(k) =
\]

\[
\frac{ma}{\tau_0} \int d^3k_1 d^3k_2 \alpha_R(k_1, k_2; \bar{\tau}) \delta_\tau(k_1) \theta_\tau(k_2),
\]

\[
\left( 1 + a \frac{\partial_a \mathcal{H}}{\mathcal{H}} + a \partial_a - i \frac{\mu k \tau}{\mathcal{H} \tau_0} \right) \theta_\tau(k) + \frac{k^2}{ma\mathcal{H}^2} S_\tau(k) =
\]

\[
\frac{ma}{\tau_0} \int d^3k_1 d^3k_2 \beta_R(k_1, k_2; \bar{\tau}) \theta_\tau(k_1) \theta_\tau(k_2).
\]
GENERALIZATION 2: COMPACT FORM OF THE EQUATIONS

• Considering $N$ flows, it is useful to introduce the $2N$-uplet

$$\Psi_a(k) = (\delta_{\tau_1}(k), \theta_{\tau_1}(k), \ldots, \delta_{\tau_n}(k), \theta_{\tau_n}(k))^T.$$ 

• The resulting equations is

$$a\frac{\partial \Psi_a}{\partial a}(k, \eta) + \Omega_a^b(k, \eta) \Psi_b(k, \eta) = \gamma_a^{bc}(k_1, k_2, \eta) \Psi_b(k_1, \eta) \Psi_c(k_2, \eta).$$

The relativistic equation of motion is formally the same as the equation describing cold dark matter.

• This study is presented in arXiv: 1411.0428 (H. Dupuy and F. Bernardeau).
CONCLUSIONS AND PERSPECTIVES

• We proposed a new approach to study massive neutrinos beyond the linear regime.

• Principle of the method: describing neutrinos as a collection of flows.

• How could we make the explicit computation of the nonlinear matter power spectrum feasible?

• What are the scales at which both nonlinearities and relativistic effects are relevant?

• What is the number of flows necessary for the method to be satisfactory with a given precision?

• What is the most efficient way of discretizing momenta?