A unifying description of Dark Energy (& Modified Gravity)

David Langlois
(APC, Paris)
Introduction & motivations

• Plethora of models of dark energy & modified gravity:
  – Cosmological constant
  – quintessence, K-essence
  – f(R) gravity
  – Horndeski & beyond Horndeski
  – Massive gravity
  – ...

• Large amount of data from future cosmological surveys (DES, LSST, eBOSS, DESI, Euclid, …)

• General framework to confront models with data
Theories

Effective description (unified language)

Observational constraints
Introduction & motivations

• Plethora of models of dark energy & modified gravity:
  – Cosmological constant
  – quintessence, K-essence
  – f(R) gravity
  – Horndeski & beyond Horndeski
  – Massive gravity
  – …

• Large amount of data from future cosmological surveys (DES, LSST, eBOSS, DESI, Euclid, …)

• General framework to confront models with data:
  – Parametrized modified Einstein equations
  – Effective action
Uniform scalar field slicing

[Inflation: Creminelli et al. ’06; Cheung et al. ’07]

• Restriction: **single scalar field** models

• The scalar field defines a **preferred slicing**
  
  Constant time hypersurfaces = uniform field hypersurfaces

  \[
  \phi = \phi_1, \quad \phi = \phi_2, \quad \phi = \phi_3
  \]

• All perturbations embodied by the metric only
Uniform scalar field slicing

• 3+1 decomposition based on this preferred slicing

• Basic ingredients
  – Unit vector normal to the hypersurfaces
    \[ n^\mu = -\frac{\nabla^\mu \phi}{\sqrt{-\nabla^2 \phi}} \]

  – Projection on the hypersurfaces:
    \[ h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \]
ADM formulation

- ADM decomposition of spacetime

\[ ds^2 = -N^2 dt^2 + h_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right) \]

Extrinsic curvature:

\[ K_{ij} = \frac{1}{2N} \left( \dot{h}_{ij} - D_i N_j - D_j N_i \right) \]

Intrinsic curvature:

\[ X \equiv g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = -\frac{\dot{\phi}^2(t)}{N^2} \]

- Generic Lagrangians of the form

\[ S_g = \int d^4x \; N\sqrt{h} \; L(N, K_{ij}, R_{ij}; t) \]
Example: GR + quintessence

• Consider a quintessence model

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} (4) R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \]

• In the uniform \( \phi \) slicing, this leads to the Lagrangian

\[ L = \frac{M_{Pl}^2}{2} \left[ K_{ij} K^{ij} - K^2 + R \right] + \frac{\dot{\phi}^2(t)}{2N^2} - V(\phi(t)) \]
Homogeneous evolution

- FLRW metric: \( ds^2 = -\bar{N}^2(t) \, dt^2 + a^2(t) \, \delta_{ij} dx^i dx^j \)

- Extrinsic curvature: \( K^i_j = \frac{\dot{a}}{\bar{N} a} \delta^i_j \equiv H \, \delta^i_j \)

- Homogeneous Lagrangian
  \[ \bar{L}(a, \dot{a}, \bar{N}) \equiv L \left[ K^i_j = \frac{\dot{a}}{\bar{N} a} \delta^i_j, R^i_j = 0, N = \bar{N}(t) \right] \]

- One can include **matter** by adding the Lagrangian for matter (assumed to be minimally coupled to the metric).
Friedmann equations

• Variation of the action

\[ \bar{S}_g = \int dt \, d^3 x \, \bar{N} a^3 \, \bar{L}(a, \dot{a}, \bar{N}) \]

Using \( \left( \frac{\partial L}{\partial K^j_i} \right)_{\text{bgd}} \equiv \mathcal{F} \delta^i_j \), one finds

\[ a^{-3} \frac{\delta \bar{S}_g}{\delta \bar{N}} = \bar{L} + \bar{N} L_N - 3H \mathcal{F} = \rho_m \]

\[ \frac{1}{3a^2 \bar{N}} \frac{\delta \bar{S}_g}{\delta a} = \bar{L} - 3H \mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} = -p_m \]

• For GR:

\[ \bar{L}_{\text{GR}} = -3M_P^2 H^2, \quad \mathcal{F}_{\text{GR}} = -2M_P^2 H \]
Linear perturbations

• Perturbations

\[ \delta N \equiv N - \bar{N}, \quad \delta K^i_j \equiv K^i_j - H\delta^i_j, \quad \delta R^i_j \equiv R^i_j \]

• Expand the Lagrangian

\[ L(q_A) \quad \text{with} \quad q_A \equiv \{N, K^i_j, R^i_j\} \]

yields

\[ L(q_A) = \bar{L} + \frac{\partial L}{\partial q_A} \delta q^A + \frac{1}{2} \frac{\partial^2 L}{\partial q_A \partial q_B} \delta q_A \delta q_B + \ldots \]

• The quadratic action describes the \textbf{dynamics of linear perturbations}
Linear perturbations

- The coefficients are evaluated on the homogeneous background, e.g.

\[
\frac{\partial^2 L}{\partial K_i^j \partial K_k^l} \equiv \hat{A}_K \delta_j^i \delta^k_l + A_K \left( \delta_i^j \delta_k^l + \delta^{ik} \delta_{jl} \right)
\]

\[
\frac{\partial^2 L}{\partial R_i^j \partial R_k^l} \rightarrow (\hat{A}_R, A_R) \quad \frac{\partial^2 L}{\partial K_i^j \partial R_k^l} \rightarrow (\hat{C}, C) \quad \ldots
\]

- For simplicity, we assume the three conditions

\[
\hat{A}_K + 2A_K = 0, \quad \hat{C} + \frac{1}{2} C = 0, \quad 4\hat{A}_R + 3A_R = 0
\]

so that the EOM are 2nd order in spatial gradients.
Linear perturbations

- Quadratic action in terms of 5 functions of time

\[ S^{(2)} = \int dx^3 dt a^3 \frac{M^2}{2} \left[ \delta K^i_j \delta K^j_i - \delta K^2 + \alpha_K H^2 \delta N^2 + 4 \alpha_B H \delta K \delta N \right. \]
\[ \left. + (1 + \alpha_T) \delta_2 \left( \frac{\sqrt{\hbar}}{a^3 R} \right) + (1 + \alpha_H) R \delta N \right] \]

- Includes many models
  - GR: \( M = M_P, \quad \alpha_i = 0 \)
  - Quintessence, K-essence: \( \alpha_K \neq 0 \)
  - Kinetic braiding, DGP: \( \alpha_B \neq 0 \)
  - Brans-Dicke, F(R): \( M = M(t) \)
  - Horndeski: \( \alpha_T \neq 0 \)
  - beyond Horndeski: \( \alpha_H \neq 0 \)

Gleyzes, DL, Piazza & Vernizzi ’13, [notation from Bellini & Sawicki ’14]
Scalar degree of freedom

• Scalar perturbations: $\delta N, \quad N_i \equiv \partial_i \psi, \quad h_{ij} = a^2(t)e^{2\zeta}\delta_{ij}$

• Quadratic action for the physical degree of freedom:

$$S^{(2)} = \frac{1}{2} \int dx^3 dt \, a^3 \left[ \mathcal{K}_t \zeta'^2 + \mathcal{K}_s \frac{(\partial_i \zeta)^2}{a^2} \right]$$

$$\mathcal{K}_t \equiv \frac{\alpha_K + 6\alpha_B^2}{(1 + \alpha_B)^2}, \quad \mathcal{K}_s \equiv 2M^2 \left\{ 1 + \alpha_T - \frac{1 + \alpha_H}{1 + \alpha_B} \left( 1 + \alpha_M - \frac{\dot{H}}{H^2} \right) - \frac{1}{H} \frac{dt}{dt} \left( 1 + \alpha_H \right) \right\}$$

• Stability
  – No ghost: $\mathcal{K}_t > 0$

  – No gradient instability: $c_s^2 \equiv -\frac{\mathcal{K}_s}{\mathcal{K}_s} > 0$
Tensor degrees of freedom

• Quadratic action for the tensor modes:

\[ S^{(2)}_\gamma = \frac{1}{2} \int dt\,d^3x\,a^3 \left[ \frac{M^2}{4} \dot{\gamma}_{ij}^2 - \frac{M^2}{4} (1 + \alpha_T) \left( \frac{\partial_k \gamma_{ij}}{a^2} \right)^2 \right] \]

• Stability
  – No ghost: \( M^2 > 0 \)
  – No gradient instability: \( c_T^2 \equiv 1 + \alpha_T > 0 \)
Example: Horndeski theories

- Most general scalar-tensor action leading to at most second order equations of motion for the scalar field and metric. Horndeski 74

- **Generalized galileons** coupled to gravity Nicolis et al. 08; Deffayet et al. 09 & 11

- Combination of the following four Lagrangians

\[
L^H_2 = G_2(\phi, X) \\
L^H_3 = G_3(\phi, X) \Box \phi \\
L^H_4 = G_4(\phi, X)^{(4)}R - 2G_4X(\phi, X) (\Box \phi^2 - \phi^{\mu\nu} \phi_{\mu\nu}) \\
L^H_5 = G_5(\phi, X)^{(4)}G_{\mu\nu} \phi^{\mu\nu} + \frac{1}{3} G_5X(\phi, X) (\Box \phi^3 - 3 \Box \phi \phi_{\mu\nu} \phi^{\mu\nu} + 2 \phi_{\mu\nu} \phi^{\mu\sigma} \phi^\nu_{\sigma})
\]

- **Higher order** derivatives in the Lagrangian
Beyond Horndeski

• 2\textsuperscript{nd} order time derivatives in the \textit{Lagrangian} usually lead to an extra DOF, which is unstable (\textit{Ostrogradski})
  
  \textit{e.g.} \( L(q, \dot{q}, \ddot{q}) \)

• 2\textsuperscript{nd} order \textbf{EOMs} were believed to be necessary to avoid Ostrogradski’s ghost but higher order equations of motion are in fact possible.

• Two extensions beyond Horndeski \[ \text{[Gleyzes, DL, Piazza & Vernizzi '14]} \]
  
  \[
  \begin{align*}
  L_4^{bH} & \equiv F_4(\phi, X) \epsilon^{\mu\nu\rho} \sigma \epsilon^{\mu'\nu'\rho'\sigma} \phi_{\mu\phi_{\mu'}} \phi_{\nu\nu'} \phi_{\rho\rho'} \\
  L_5^{bH} & \equiv F_5(\phi, X) \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma'} \phi_{\mu\phi_{\mu'}} \phi_{\nu\nu'} \phi_{\rho\rho'} \phi_{\sigma\sigma'}
  \end{align*}
  \]

• Crucial ingredient: \( \mathcal{L}[\phi, g_{\mu\nu}] \) must be \textbf{degenerate} \[ \text{[DL & Noui '15]} \]
Horndeski & beyond in ADM form

- One obtains combinations of the Lagrangians

\[ L_2 = A_2 \quad L_3 = A_3 K \]
\[ L_4 = A_4 \left( K^2 - K_{ij} K^{ij} \right) + B_4 R \]
\[ L_5 = A_5 \left( K^3 - 3 K K_{ij} K^{ij} + 2 K_{ij} K^{ik} K^{j}_{\ k} \right) + B_5 K^{ij} \left[ R_{ij} - h_{ij} R/2 \right] \]

where the A’s and B’s depend on the functions G’s & F’s.

- Horndeski theories (only four G’s) satisfy the relations

\[ A_4 = - B_4 + 2X B_4 X \]
\[ A_5 = - X B_5 X / 3 \]

- One can then use the results of the general formalism.
Generalized couplings to matter

Gleyzes, DL, Mancarella & Vernizzi ’15

- Minimal coupling: \[ S_m = S_m[\psi_m, g_{\mu\nu}] \]

- Conformal coupling: \[ S_m = S_m[\psi_m, C(\phi) \, g_{\mu\nu}] \]

- Conformal-disformal couplings

\[
S_{m}^{(I)} = S_{m}^{(I)}[\psi_{m}, \tilde{g}_{\mu\nu}^{(I)}] \\
\text{with} \quad \tilde{g}_{\mu\nu}^{(I)} = C_{I}(\phi) \, g_{\mu\nu} + D_{I}(\phi) \, \partial_{\mu} \phi \partial_{\nu} \phi \quad [\text{Bekenstein '93}]
\]

\[
\alpha_{C,I} \equiv \frac{1}{2} \frac{d \ln C_{I}}{d \ln a}, \quad \alpha_{D,I} \equiv \frac{D_{I}}{C_{I} - D_{I}}
\]
“Frame” transformation

• Gravity can be described by a different metric

\[ g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu} + D(\phi)\partial_\mu \phi \partial_\nu \phi \]

• Horndeski’s structure is invariant \[ \text{[Bettoni & Liberati ’12]} \]

The quadratic action (with \( \alpha_H = 0 \) here)

\[
S^{(2)} = \int dx^3 dt a^3 \frac{M^2}{2} \left[ \delta K^i_j \delta K^j_i - \delta K^2 + \alpha_K H^2 \delta N^2 + 4 \alpha_B H \delta K \delta N \right. \\
\left. + (1 + \alpha_T) \delta_2 \left( \frac{\sqrt{h}}{a^3} R \right) + R \delta N \right]
\]

gets transformed into a similar action

\[
\{ M, \alpha_K, \alpha_B, \alpha_T \} \rightarrow \{ \tilde{M}, \tilde{\alpha}_K, \tilde{\alpha}_B, \tilde{\alpha}_T \}
\]
“Frame” transformation

- Metric transformation
  \[ \alpha_C \equiv \frac{1}{2} \frac{d \ln C}{d \ln a}, \quad \alpha_D \equiv \frac{D}{C - D} \]

- New gravitational coefficients
  \[ \tilde{M}^2 = \frac{M^2}{C \sqrt{1 + \alpha_D}}, \quad \tilde{\alpha}_T = (1 + \alpha_T)(1 + \alpha_D) - 1 \quad \tilde{\alpha}_B = \frac{1 + \alpha_B}{(1 + \alpha_C)(1 + \alpha_D)} - 1 \]
  \[ \tilde{\alpha}_K = \frac{\alpha_K + 12 \alpha_B [\alpha_C + (1 + \alpha_D) \alpha_D] - 6[\alpha_C + (1 + \alpha_D) \alpha_D]^2 + 3\Omega_m \alpha_D}{(1 + \alpha_C)^2(1 + \alpha_D)^2} \]

- New matter couplings
  \[ \{\alpha_{C,I}, \alpha_{D,I}\} \rightarrow \{\tilde{\alpha}_{C,I}, \tilde{\alpha}_{D,I}\} \]
  \[ \tilde{\alpha}_{C,I} = \frac{\alpha_{C,I} - \alpha_C}{1 + \alpha_C} \]
  \[ \tilde{\alpha}_{D,I} = \frac{\alpha_{D,I} - \alpha_D}{1 + \alpha_D} \]

- \( N_S \) species: \( 4 + 2N_S - 2 = 2(N_S + 1) \) independent parameters
Confrontation with observations

• Use a traditional gauge, e.g. Newtonian gauge

\[ ds^2 = -(1 + 2\Phi)dt^2 + a^2(t) (1 - 2\Psi) \delta_{ij} dx^i dx^j \]

• Description in an arbitrary slicing?

\[ \phi = \text{const} \]

• Coordinate change

\[ t \rightarrow t + \pi(t, \vec{x}) \]

• Perturbations: \( \Phi, \Psi, \pi, \delta_m, \vec{v}_m \)
Cosmological perturbations

• Standard equations (in GR)

\[ \nabla^2 \Psi = 4\pi G a^2 \rho_m \delta_m \]  
(relativistic Poisson)

\[ \dot{\delta}_m + \nabla \cdot \vec{v}_m = 0 \]  
(continuity)

\[ \dot{\Phi} \]

\[ \hat{\Psi} = \Phi \]

\[ \dot{\vec{v}}_m + H \vec{v}_m = -\nabla \Phi \]  
(Euler)
Cosmological perturbations

- **Modified equations** (minimal coupling)

\[ \nabla^2 \Psi = 4\pi G (1 + \gamma_G) a^2 \rho_m \delta_m \]

\[ G_{\text{eff}} = G_{\text{eff}}(\alpha_i), \quad \eta = \eta(\alpha_i) \]

which can be confronted to observations (RSD, weak lensing, ...).
Cosmological perturbations

• **Modified equations** (non minimal coupling)

\[
\nabla^2 (\Phi + \Psi) = 4\pi G (2 + \gamma_{\text{lens}}) a^2 \rho_m \delta_m
\]

Quasi-static approximation

\[
\nabla^2 \Psi = 4\pi G (1 + \gamma_G) a^2 \rho_m \delta_m
\]

Gleyzes, DL, Mancarella & Vernizzi ’15

\[
\dot{\vec{v}}_I + (H + 3\gamma_I) \vec{v}_m = -\vec{\nabla} \Phi - 3H \gamma_I \vec{\nabla} \pi
\]

• Generalization of coupled quintessence  [Amendola ‘00]
On smaller scales

- Deviations from GR on cosmological scales should be compatible with small-scale observations (solar system, binary systems)

- **Screening mechanism**

  \[ Z(\phi_0) \nabla^2 \delta \phi - m^2(\phi_0) \delta \phi = -\beta(\phi_0) \frac{\delta T}{M_P} \]

  - Chameleon: \( m(\phi_0) \) is large
  - Dilaton & symmetron: \( \beta(\phi_0) \ll 1 \)
  - Vainshtein: \( Z(\phi_0) \gg \beta^2(\phi_0) \)
Example: beyond Horndeski

Saito, Yamauchi, Mizuno, Gleyzes & DL ‘15
(see also Koyama & Sakstein ‘15)

• Partial breaking of Vainshtein mechanism inside matter
  Kobayashi, Watanabe & Yamauchi ‘14

• Spherical symmetry & nonrelativistic limit:

\[
\frac{d\Phi}{dr} = G_N \left( \frac{\mathcal{M}}{r^2} - \epsilon \frac{d^2\mathcal{M}}{dr^2} \right), \quad \mathcal{M}(r) = 4\pi \int_0^r r'^2 \rho(r')dr'
\]

• Modified Lane-Emden equation
  (for \( P = K \rho^{1+\frac{1}{n}} \))
  – Universal bound \( \epsilon < 1/6 \)
  – Astrophysical constraints [Sakstein ‘15]

\[ \epsilon > -0.0068 \]
Conclusions

• **Unifying description** of dark energy and modified gravity models
  – Easy comparisons between models
  – Identification of degeneracies
  – Observational data can constrain many models simultaneously
  – Explore unchartered territories (e.g. theories beyond Horndeski)

• Very general and efficient way to describe linear perturbations in scalar-tensor theories with **only five time-dependent functions**.

• Extension to include non-universal couplings