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Erike Cazaroto

The City University of New York – (USA)

University of Sao Paulo – (Brazil)

The small-x evolution equation for the 2n Wilson line operator and its weak field limit

Co-authors:

Jamal Jalilian-Marian – The City University of New York (USA)

Alejandro Ayala – Universidad Nacional Autónoma de México (Mexico)

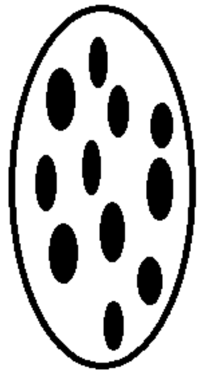
Maria Elena Tejeda-Yeomans – Universidad de Sonora (Mexico)

Luis Alberto Hernández – University of Cape Town (South Africa)

Introduction

Saturation effect

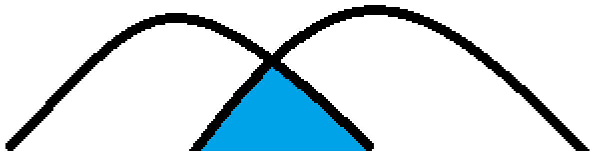
The linear evolution equations (e.g. DGLAP equation) predict a very fast growth of the gluonic distribution with the energy (or decreasing x)



Nonetheless, it is expected that there is a kinematical region where the recombination processes

g g → g

become important



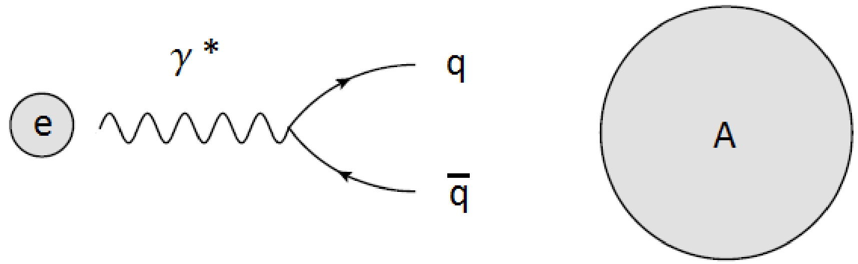
This happens when the overlap of the gluonic wave functions cannot be neglected

This kinematical region is determined by the saturation scale Q_s

Introduction

<u>The color dipole formalism</u>

In the color dipole formalism
γ^* or g
opens in a quark-antiquark pair
(color dipole)
before scattering with the target



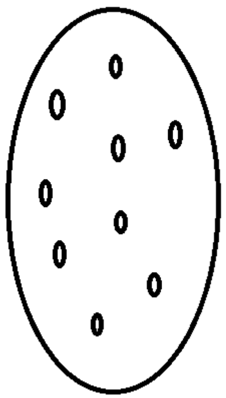
In the dilute regime, the dipole evolution with the energy (or rapidity) can be described by the BFKL equation

The BFKL is a linear evolution equation
(does not include saturation effects)

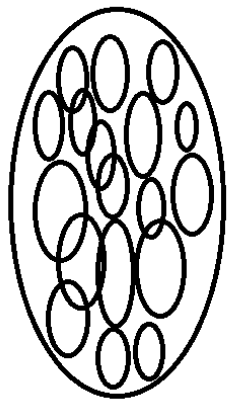
However, one of the main reasons to use the dipole approach is that
it allows to easily include saturation effects

Introduction

Saturation effects
in the
color dipole formalism



Linear Regime



Saturation Regime

The evolution equation for the dipole which includes saturation effects is the Balitsky-Kovchegov (BK) equation

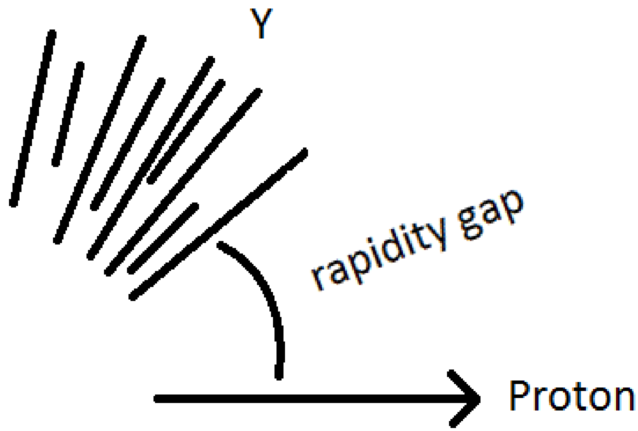
$$\frac{\partial \mathcal{N}(\rho, Y)}{\partial Y} = \int \frac{d^2\vec{z}}{2\pi} K(\vec{\rho}, \vec{\rho_1}, \vec{\rho_2}) \left[\mathcal{N}(\rho_1, Y) + \mathcal{N}(\rho_2, Y) - \mathcal{N}(\rho, Y) - \mathcal{N}(\rho_1, Y)\mathcal{N}(\rho_2, Y)\right]$$

BK eq. is obtained from the JIMWLK equation at large N_c limit and mean field approximation

The saturation effect is related to the non-linear term of this equation

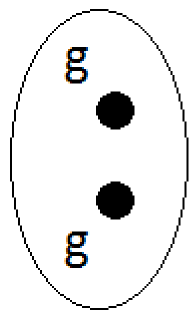
Motivation

Our final aim is to construct a saturation based model for diffractive processes



The most robust aspect of such model is that the rapidity gap will depend on the saturation scale Q_S, which is well determined

The pomeron is a Reggeized state of 2 gluons

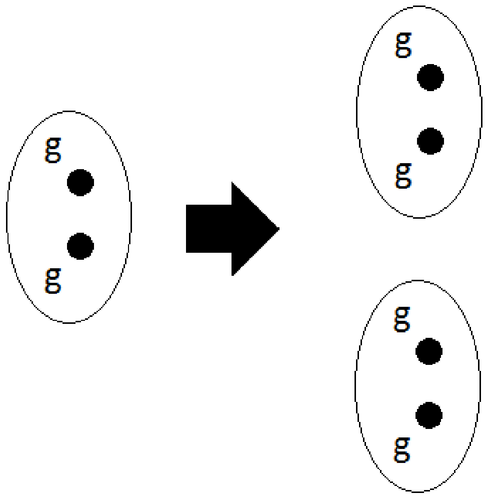


One theoretical explanation for this kind of process is that the hadrons exchange a hypothetical particle called “Pomeron”

Motivation

The vertex
1 → 2 pomeron
is known in the literature

This vertex can be obtained from the non-linear term of the BK equation by expanding the Wilson lines in powers of gluon field



Our aim is to use the same method to extract the
n → m
Pomeron vertex

For this we must write the JIMWLK equation describing the evolution of the 2n-Wilson line operator

But, before extracting the vertices, we want to verify if the JIMWLK equation reduces to the BJKP equation in the weak field limit

The BJKP (Bartels-Jaroszewicz-Kwiecinski-Praszalowicz--1980) describes the evolution in energy of a state of 2n-Reggeized gluons

The small-x evolution equation for 2n-Wilson line operator

The 2n-Wilson line operator

Color dipole =
normalized trace of 2 Wilson lines

$$\hat{S}^{(2)}_{x_1\,x_2} = \frac{1}{N_c} Tr\left(V_{x_1} V^{\dagger}_{x_2}\right)$$

Wilson line
(fundamental representation)

$$V(x) \equiv e^{-ig\,\alpha^a(x)\,t^a}$$

Quadrupole:

$$\hat{S}^{(4)}_{x_1\,x_2\,x_3\,x_4} = \frac{1}{N_c} Tr\left(V_{x_1} V^{\dagger}_{x_2}\, V_{x_3} V^{\dagger}_{x_4}\right)$$

2n-pole:

$$\hat{S}^{(2n)}_{\left(\prod_{k=1}^{2n} x_k\right)} = \frac{1}{N_c} Tr\left(V_{x_1} V^{\dagger}_{x_2}\, V_{x_3} V^{\dagger}_{x_4} \cdots V_{x_{2n-1}} V^{\dagger}_{x_{2n}}\right)$$

Small-x evolution equation for the 2n-Wilson line operator

JIMWLK Equation

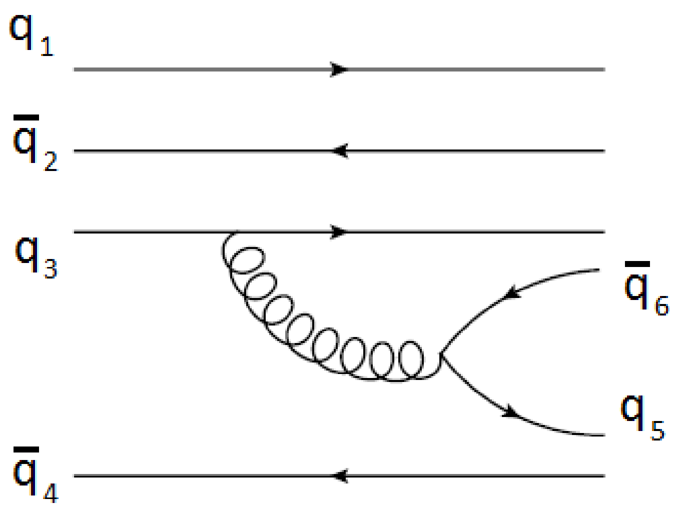
$$\frac{\partial < O >}{\partial Y} = < H\, O >$$

$$H = -\frac{1}{16\pi^3}\int d^2x\, d^2y\, d^2z\, M_{xyz}\left(1 + U_x^\dagger U_y - U_x^\dagger U_z - U_z^\dagger U_y\right)^{ab} \frac{\delta}{\delta \alpha_x^a}\frac{\delta}{\delta \alpha_y^b}$$

$$M_{xyz} \equiv \frac{(x-y)^2}{(x-z)^2(z-y)^2}$$

Example:

The evolution of a Quadrupole



Small-x evolution equation for the 2n-Wilson line operator

Hamiltonian →

$$H \;=\; -\frac{1}{16\pi^3}\int d^2x\, d^2y\, d^2z\, M_{xyz}\left(1+U_x^\dagger U_y - U_x^\dagger U_z - U_z^\dagger U_y\right)^{ab}\frac{\delta}{\delta\alpha_x^a}\frac{\delta}{\delta\alpha_y^b}$$

2n Correlator →

$$\hat{S}^{(2n)}_{(\prod_{k=1}^{2n} x_k)} = \frac{1}{N_c} Tr\left(V_{x_1} V^\dagger_{x_2}\, V_{x_3} V^\dagger_{x_4} \cdots V_{x_{2n-1}} V^\dagger_{x_{2n}}\right)$$

1st step:
Functional derivatives

$$\frac{\delta}{\delta\alpha^a(x)} V^\dagger_{x_i} \;=\; ig\,\delta^2(x_i - x)\, t^a\, V^\dagger_{x_i}$$

$$\frac{\delta}{\delta\alpha^a(x)} V_{x_i} \;=\; -\,ig\,\delta^2(x_i - x)\, V_{x_i}\, t^a$$

2nd step:
Color contractions

$$(U^\dagger_x)^{ac}\, t^a \;=\; V^\dagger_x\, t^c\, V_x$$

$$(U_y)^{cb}\, t^b \;=\; V^\dagger_y\, t^c\, V_y$$

Small-x evolution equation for the 2n-Wilson line operator

Hamiltonian →

$$H = -\frac{1}{16\pi^3}\int d^2x\, d^2y\, d^2z\, M_{xyz}\left(1 + U_x^\dagger U_y - U_x^\dagger U_z - U_z^\dagger U_y\right)^{ab}\frac{\delta}{\delta\alpha_x^a}\frac{\delta}{\delta\alpha_y^b}$$

2n Correlator →

$$\hat{S}^{(2n)}_{(\prod_{k=1}^{2n} x_k)} = \frac{1}{N_c} Tr\left(V_{x_1}V_{x_2}^\dagger\, V_{x_3}V_{x_4}^\dagger \cdots V_{x_{2n-1}}V_{x_{2n}}^\dagger\right)$$

3rd step:

Integrate in x and y (eliminate the delta functions)

4th step:

Fierz identity to eliminate t matrices

$$Tr[t^c B\, t^c C A] = \frac{1}{2}Tr[B]\, Tr[AC] - \frac{1}{2N_c}Tr[ABC]$$

Final Result

$$H\,\hat{S}^{(2n)} = \frac{\bar{\alpha}}{4\pi}\int_z \left(H_1\,\hat{S}^{(2n)} + H_2\,\hat{S}^{(2n)} - H_3\,\hat{S}^{(2n)} - H_4\,\hat{S}^{(2n)} \right)$$

$$\bar{\alpha} = \alpha_s N_c/\pi$$

$$\begin{aligned} H_1\,\hat{S}^{(2n)} &= -\sum_{i=1}^{n} \left[M_{x_{2i-1}x_{2i}z}\right] \hat{S}^{(2n)}_{(\prod_{k=1}^{2n} x_k)} \\ &+ \sum_{j=2\,;\,j>i}^{n} \sum_{i=1}^{n-1} \left[M_{x_{2i-1}x_{2j-1}z} - M_{x_{2i-1}x_{2j}z} - M_{x_{2i}x_{2j-1}z} + M_{x_{2i}x_{2j}z} \right] \hat{S}^{(2j-2i)}_{(\prod_{k=2i}^{2j-1} x_k)} \hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i-1} x_k)(\prod_{k=2j}^{2n} x_k)} \end{aligned}$$

$$\begin{aligned} H_2\,\hat{S}^{(2n)} &= -\sum_{i=1}^{n} \left[M_{x_{2i-1}x_{2i}z}\right] \hat{S}^{(2)}_{(x_{2i-1}x_{2i})} \hat{S}^{(2n-2)}_{(\prod_{k=1}^{2i-2} x_k)(\prod_{k=2i+1}^{2n} x_k)} \\ &+ \sum_{j=2\,;\,j>i}^{n} \sum_{i=1}^{n-1} \left[M_{x_{2i-1}x_{2j-1}z}\right] \hat{S}^{(2j-2i)}_{(\prod_{k=2i-1}^{2j-2} x_k)} \hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i-2} x_k)(\prod_{k=2j-1}^{2n} x_k)} \\ &- \sum_{j=2\,;\,j>i}^{n} \sum_{i=1}^{n-1} \left[M_{x_{2i-1}x_{2j}z}\right] \hat{S}^{(2j-2i+2)}_{(\prod_{k=2i-1}^{2j} x_k)} \hat{S}^{(2n-2j+2i-2)}_{(\prod_{k=1}^{2i-2} x_k)(\prod_{k=2j+1}^{2n} x_k)} \\ &- \sum_{j=2\,;\,j>i}^{n} \sum_{i=1}^{n-1} \left[M_{x_{2i}x_{2j-1}z}\right] \hat{S}^{(2j-2i-2)}_{(\prod_{k=2i+1}^{2j-2} x_k)} \hat{S}^{(2n-2j+2i+2)}_{(\prod_{k=1}^{2i} x_k)(\prod_{k=2j-1}^{2n} x_k)} \\ &+ \sum_{j=2\,;\,j>i}^{n} \sum_{i=1}^{n-1} \left[M_{x_{2i}x_{2j}z}\right] \hat{S}^{(2j-2i)}_{(\prod_{k=2i+1}^{2j} x_k)} \hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i} x_k)(\prod_{k=2j+1}^{2n} x_k)} \end{aligned}$$

$$
\begin{aligned}
H_3\,\hat{S}^{(2n)} = & -\sum_{i=1}^{n}\left[M_{x_{2i-1}x_{2i}z}\right]\,\frac{1}{2}\hat{S}^{(2)}_{z\,x_{2i-1}}\,\hat{S}^{(2n)}_{(\prod_{k=1}^{2i-2}x_k)\,z\,(\prod_{k=2i}^{2n}x_k)} \\
& -\sum_{i=1}^{n}\left[M_{x_{2i-1}x_{2i}z}\right]\,\frac{1}{2}\hat{S}^{(2)}_{z\,x_{2i}}\,\hat{S}^{(2n)}_{(\prod_{k=1}^{2i-1}x_k)\,z\,(\prod_{k=2i+1}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[M_{x_{2i-1}x_{2j-1}z}-M_{x_{2i-1}x_{2j}z}\right]\frac{1}{2}\hat{S}^{(2j-2i+2)}_{z\,(\prod_{k=2i-1}^{2j-1}x_k)}\,\hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i-2}x_k)\,z\,(\prod_{k=2j}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[-M_{x_{2i}x_{2j-1}z}+M_{x_{2i}x_{2j}z}\right]\frac{1}{2}\hat{S}^{(2j-2i)}_{z\,(\prod_{k=2i+1}^{2j-1}x_k)}\,\hat{S}^{(2n-2j+2i+2)}_{(\prod_{k=1}^{2i}x_k)\,z\,(\prod_{k=2j}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[M_{x_{2j-1}x_{2i-1}z}-M_{x_{2j-1}x_{2i}z}\right]\frac{1}{2}\hat{S}^{(2j-2i)}_{z\,(\prod_{k=2i}^{2j-2}x_k)}\,\hat{S}^{(2n-2j+2i+2)}_{(\prod_{k=1}^{2i-1}x_k)\,z\,(\prod_{k=2j-1}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[-M_{x_{2j}x_{2i-1}z}+M_{x_{2j}x_{2i}z}\right]\frac{1}{2}\hat{S}^{(2j-2i+2)}_{z\,(\prod_{k=2i}^{2j}x_k)}\,\hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i-1}x_k)\,z\,(\prod_{k=2j+1}^{2n}x_k)}
\end{aligned}
$$

Mathematica program available for download

$$
\begin{aligned}
H_4\,\hat{S}^{(2n)} = & -\sum_{i=1}^{n}\left[M_{x_{2i-1}x_{2i}z}\right]\,\frac{1}{2}\hat{S}^{(2)}_{z\,x_{2i}}\,\hat{S}^{(2n)}_{(\prod_{k=1}^{2i-1}x_k)\,z\,(\prod_{k=2i+1}^{2n}x_k)} \\
& -\sum_{i=1}^{n}\left[M_{x_{2i-1}x_{2i}z}\right]\,\frac{1}{2}\hat{S}^{(2)}_{z\,x_{2i-1}}\,\hat{S}^{(2n)}_{(\prod_{k=1}^{2i-2}x_k)\,z\,(\prod_{k=2i}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[M_{x_{2i-1}x_{2j-1}z}-M_{x_{2i}x_{2j-1}z}\right]\frac{1}{2}\hat{S}^{(2j-2i)}_{z\,(\prod_{k=2i}^{2j-2}x_k)}\,\hat{S}^{(2n-2j+2i+2)}_{(\prod_{k=1}^{2i-1}x_k)\,z\,(\prod_{k=2j-1}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[-M_{x_{2i-1}x_{2j}z}+M_{x_{2i}x_{2j}z}\right]\frac{1}{2}\hat{S}^{(2j-2i+2)}_{z\,(\prod_{k=2i}^{2j}x_k)}\,\hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i-1}x_k)\,z\,(\prod_{k=2j+1}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[M_{x_{2j-1}x_{2i-1}z}-M_{x_{2j}x_{2i-1}z}\right]\frac{1}{2}\hat{S}^{(2j-2i+2)}_{z\,(\prod_{k=2i-1}^{2j-1}x_k)}\,\hat{S}^{(2n-2j+2i)}_{(\prod_{k=1}^{2i-2}x_k)\,z\,(\prod_{k=2j}^{2n}x_k)} \\
& +\sum_{j=2\,;\,j>i}^{n}\sum_{i=1}^{n-1}\left[-M_{x_{2j-1}x_{2i}z}+M_{x_{2j}x_{2i}z}\right]\frac{1}{2}\hat{S}^{(2j-2i)}_{z\,(\prod_{k=2i+1}^{2j-1}x_k)}\,\hat{S}^{(2n-2j+2i+2)}_{(\prod_{k=1}^{2i}x_k)\,z\,(\prod_{k=2j}^{2n}x_k)}
\end{aligned}
$$

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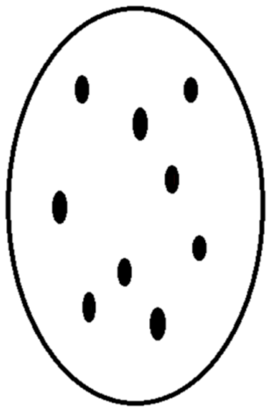
The weak field limit

The weak field limit

In the weak field limit (dilute regime) saturation effects are negligible

Non-linear terms are neglected and the equation becomes linear



Example: In the weak field limit the BK equation reduces to the BFKL equation

The BJKP equation describes the evolution of 2n Reggeized gluons with energy in the dilute regime (BJKP is a linear equation)

Our aim in this work is to establish a formal equivalence between the JIMWLK equation in the weak field limit and the BJKP equation

The weak field limit

If we define the **T matrix** as $\hat{T} = 1 - \hat{S}$

Then the **weak field limit** is defined as the regime $\hat{T} \ll 1$

So we substitute T for S in our equation for 2n-Wilson line and neglect terms quadratic in T

We also expand the Wilson lines in T to 1st order in the gluon field α

$$V_{x_i} = 1 - ig\alpha_{x_i} + \cdots$$
$$V^{\dagger}_{x_i} = 1 + ig\alpha_{x_i} + \cdots$$

$$\hat{T}^{(2n)}_{(\prod_{k=1}^{2n} x_k)} \simeq g^{2n} \frac{1}{N_c} Tr[\alpha_{x_1}\alpha_{x_2}\cdots\alpha_{x_{2n-1}}\alpha_{x_{2n}}]$$

The weak field limit

We also Fourier transform the equation for T to the momentum space

(BJKP is written in the momentum space)

$$\tilde{T}^{(2n)}_{\left(\prod_{k=1}^{2n} l_k\right)} = \int \left[\prod_{k=1}^{2n} d^2 x_k\right] e^{i\left(\sum_{k=1}^{2n} l_k \cdot x_k\right)} \hat{T}^{(2n)}_{\left(\prod_{k=1}^{2n} x_k\right)}$$

And substitute the gluon fields with the color charge density

(BJKP is written in terms of ρ)

$$\alpha(p_t) \sim \frac{\rho(p_t)}{p_t^2}$$

The weak field limit

After some algebraic manipulations we finally arrive at our final result

$$\frac{d}{dY}T^{(2n)}_{(\prod_{k=1}^{2n} l_k)} = -\frac{\bar{\alpha}}{2\pi}\sum_{j=1}^{2n}\int d^2p_t\left[\frac{l_j^2}{p_t^2\,[p_t^2+(p_t-l_j)^2]}\right]T^{(2n)}_{(\prod_{k=1}^{2n} l_k)}$$
$$+\ \frac{\bar{\alpha}}{4\pi}\int d^2p_t\left[\frac{l_1^2}{p_t^2\,(p_t+l_1)^2}+\frac{l_{2n}^2}{p_t^2\,(p_t-l_{2n})^2}-\frac{(l_1+l_{2n})^2}{(p_t+l_1)^2\,(p_t-l_{2n})^2}\right]T^{(2n)}_{(l_1+p_t)\,(\prod_{k=2}^{2n-1} l_k)\,(l_{2n}-p_t)}$$
$$+\ \frac{\bar{\alpha}}{4\pi}\sum_{j=2}^{2n}\int d^2p_t\left[\frac{l_{j-1}^2}{p_t^2\,(p_t+l_{j-1})^2}+\frac{l_j^2}{p_t^2\,(p_t-l_j)^2}-\frac{(l_{j-1}+l_j)^2}{(p_t+l_{j-1})^2\,(p_t-l_j)^2}\right]T^{(2n)}_{(\prod_{k=1}^{j-2} l_k)(l_{j-1}+p_t)(l_j-p_t)(\prod_{k=j+1}^{2n} l_k)}$$

This result formally establishes the equivalence of the JIMWLK equation for 2n Correlator in the dilute regime and the BJKP equation for 2n Reggeized gluons

With

$$T^{(2n)}_{(\prod_{k=1}^{2n} l_k)} = \frac{1}{N_c}Tr\left[\prod_{k=1}^{2n}\rho(l_k)\right]$$

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Conclusion and future works

We showed that the expansion of Wilson lines in the gluon field is the systematic way of recovering the physics of BJKP equation

This suggests that one can use this expansion to derive other more interesting results

For example, this method has already been used to derive the 1 ➔ 2 pomeron vertex from the JIMWLK equation

We expect that similar expressions can be derived for all other n ➔ m vertices

Work in this direction is in progress

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