Thermalization: Results, Issues, Perspectives
The surprising success of hydrodynamics

What is Hydrodynamics?
I) Macroscopic theory
II) Few field variables: $P_L, P_T, \epsilon, \vec{u}$
III) Conservation law: $\partial_\mu T^{\mu\nu} = 0$
IV) Need input:

1) Equation of state $f(P) = \epsilon$
2) Small anisotropy
3) Short isotropization time
4) Initialization: $\epsilon(\tau_0), P_L(\tau_0)$? ...
5) Viscous coefficients: shear viscosity $\eta,...$
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5) Viscous coefficients: shear viscosity $\eta$, ...

None of this is easy to get from QCD
The Color Glass Condensate [McLerran, Venugopalan (1993)]

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu \quad \text{(no quarks)} \]
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The Color Glass Condensate [McLerran, Venugopalan (1993)]

\[ \mathcal{L} = -\frac{1}{4} \mathcal{F}_{\mu \nu} \mathcal{F}^{\mu \nu} + J_{\mu} A_{\mu} \] (no quarks)

\[ [\mathcal{D}_\mu, \mathcal{F}^{\mu \nu}] = J^\nu \sim \frac{Q_s^3}{g} \]

\[ \epsilon = \frac{1}{2} \left( \mathcal{E}^2 + \mathcal{B}^2 \right) \sim \frac{Q_s^4}{g^2} \]

Color sources on the light cone

Classical color fields

[Krasnitz, Venugopalan (1998)]
THE COLOR GLASS CONDENSATE AT ITS LO

classical time evolution

classical initial condition

FS_1
THE COLOR GLASS CONDENSATE AT ITS LO
Strong anisotropy at early time

\[ \text{Thermalization: Results, Issues, Perspectives} \]
The Color Glass Condensate at NLO

- Because of instabilities, the NLO correction eventually becomes as large as the LO ⇒ Important effect, should be included

Instabilities [Mrowczynski (1988)...

- NLO alone grows forever ⇒ unphysical effect, should be taken care of

Secular divergences

- Such growing contributions are present at all orders of the perturbative expansion

How to deal with them?
THE CLASSICAL STATISTICAL APPROXIMATION (CSA)

- At the initial time $\tau = \tau_0$, take:

$$A_0^\mu(\tau_0, x) = A_0^\mu(\tau, x) + \int c_k a_k^\mu(\tau_0, x)$$

where $c_k$ are random coefficients: $\langle c_k c_{k'} \rangle \sim \delta_{kk'}$

- Solve the **classical** Equation Of Motion (EOM) $[D_\mu, F^{\mu\nu}] = J^\nu$

- Compute $\langle \vec{E}^2(\tau, \vec{x}) \rangle$, where $\langle \rangle$ is the average on the $c_k$ (Monte-Carlo)

- One can show that this resums all the fastest growing terms at each order, leading to a result that remain bounded when $\tau \to \infty$

[GELIS, LAPPI, VENUGOPALAN (2008)]

Provided the $a_k^\mu$ are chosen correctly, gives

$\Rightarrow$ LO+NLO+Subset of higher orders
The Classical Statistical Approximation (CSA)

Classical time evolution
Quantum initial condition

FS$_2$

classical time evolution

Quantum initial condition
The classical Lagrangean reads

\[ \mathcal{L}_{\text{clas}} [\phi_1, \phi_2] = \frac{1}{2} \left( \partial_\mu \phi_1 \right) \left( \partial^\mu \phi_2 \right) - \frac{g^2}{3!} \phi_1 \phi_2^3 \]

Differs from the full Lagrangean

\[ \mathcal{L}_{\text{quant}} [\phi_1, \phi_2] = \mathcal{L}_{\text{clas}} [\phi_1, \phi_2] - \frac{g^2}{4!} \phi_1 \phi_2^3 \]
\[ \Gamma_{1122} = \Gamma_{\text{CSA}}_{1122} + \Gamma_{\text{non-CSA}}_{1122} \]

Divergences? (ingoing momentum \( P = (p^0, p), P^2 = (p^0)^2 - |p|^2 \))

\[ \Gamma_{\text{CSA}}_{1122} = \frac{g^4 \theta(-P^2)}{16\pi|p|} \Lambda g^4 + \text{finite} \]

Linearly divergent, non analytic in \( P \! \Rightarrow \) non-renormalizable.

In the full theory:

\[ \Gamma_{\text{non-CSA}}_{1122} = \frac{g^4 \theta(-P^2)}{16\pi|p|} \Lambda g^4 + \text{finite} \]

So in the full theory \( \Gamma_{1122} \) is finite.

No big \( \Lambda \) effect if \( \Lambda \ll \frac{16\pi}{g^4} Q_{\text{phys}} \)

Effects at late times no matter what...
**The NLO Spectrum**

- Need to know $a_k^\mu(\tau_0, x)$ at the time $\tau_0$ we start the numerical simulation.
- For practical reasons, we must start in the forward light cone ($\tau_0 > 0$).

This can be done analytically [TE, GELIS (2013)]
**Application of the CSA to the QGP**

Initial condition

\[ A_0^{\mu a}(\tau_0, x_\perp, \eta) = A_0^{\mu a}(\tau_0, x_\perp) + \sum_{\lambda c} \int_{\nu k_\perp} c_{\nu k_\perp}^{c \lambda} a_{\mu a}^{k_\perp \nu \lambda c}(\tau_0, x_\perp, \eta) \]

Time evolution \((I = x, y, \eta)\) for each configuration

\[ D_\mu F^{\mu I} = 0 \quad \Rightarrow \quad T^{\mu \nu} = \frac{1}{4} g^{\mu \nu} F^{\rho \sigma} F_{\rho \sigma} - F^{\mu \rho} F^{\nu \rho} \]

Cross checks: Gauss’s law and Bjorken’s law

\[ D_\mu E^\mu = 0 \quad \tau \partial_\tau \epsilon = - \epsilon - P_L \]
Numerical results [TE, Gelis (2013)]

\[ \alpha_s = 2 \times 10^{-2} \ (g = 0.5) \]

\( \tau \) [fm/c]
ANOMALOUSLY SMALL VISCOSITY

Assuming simple first order viscous hydrodynamics

\[ \epsilon \approx \epsilon_0 \tau^{-\frac{4}{3}} - 2\eta_0 \tau^{-2} \]

\(\epsilon\) ideal hydro \(\epsilon_0\) first order correction

we can compute the dimensionless ratio \((\eta = \eta_0 \tau^{-1})\)

\[ \eta \epsilon^{-\frac{3}{4}} \lesssim 1 \]

In contrast, perturbation theory at LO gives \(\eta \epsilon^{-\frac{3}{4}} \sim 300\).

If the system is nearly thermal

\[ \epsilon^{\frac{3}{4}} \sim s \implies \frac{\eta}{s} \text{ close to } \frac{1}{4\pi} \text{ (ideal fluid)} \]
TG VERSUS BBSV

Initial occupancy: $n_0 = 1$

Initial anisotropy: $\xi_0 = 1$ (isotropic)

Bulk Anisotropy: $P_L / P_T$

Time: $Q_\tau$

$\xi_0 = 1$

$\xi_0 = 3/2$

$\xi_0 = 2$

$\xi_0 = 4$

$\xi_0 = 6$

$\xi_0 = 1$ (isotropic)

Free streaming

$n_0 = 2$

$n_0 = 1$

$n_0 = 1/\sqrt{8}$

$n_0 = 1/4$
BBSV Scenario

- Start at $Q\tau \sim 100$ with $A = 0$ and $a \sim \frac{1}{g} \to g$ scales out
- at $Q\tau \gtrsim 300$: $f(p_\perp, p_z) = (Q_s\tau)^\alpha f_0((Q_s\tau)^\beta p_\perp, (Q_s\tau)^\gamma p_z)$
- $\alpha, \beta, \gamma = (-\frac{2}{3}, 0, \frac{1}{3})$ "universals". deduced from

$$
\epsilon = \text{cst} \times \tau^{-1} \quad n = \text{cst} \times \tau^{-1} \quad \partial_\tau f = \hat{q} \partial_z^2 f
$$

Other possible explanation? [TE, MOORE (Work in progress)]
TG versus BBSV
BBSV Scenario

- Start at $Q\tau \sim 100$ with $A = 0$ and $a \sim \frac{1}{g} \to g$ scales out
- at $Q\tau \gtrsim 300$: $f(p_\perp, p_z) = (Q_s\tau)^\alpha f_0((Q_s\tau)^\beta p_\perp, (Q_s\tau)^\gamma p_z)$
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  \[
  \epsilon = \text{cst} \times \tau^{-1} \quad n = \text{cst} \times \tau^{-1} \quad \partial_\tau f = \hat{q} \partial^2_\tau f
  \]

<table>
<thead>
<tr>
<th>TG</th>
<th>LO+NLO fully</th>
<th>BBSV</th>
<th>not clear LO, not NLO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_s\tau_{\text{init}} \ll 1$</td>
<td>$g \lesssim 0.5$</td>
<td>$Q_s\tau_{\text{init}} \gg 1$</td>
<td>$g \lesssim 10^{-6}$</td>
</tr>
</tbody>
</table>

upside: close to real situation

downside: $\Lambda$ effects? $Q_s\tau_{\text{init}} \gg 1$ not accessible

upside: Almost no $\Lambda$ effects

downside: Phenomenological relevance? Fixed point IC dependent?
Kinetic treatment with elastic scattering

CSA non renormalizable: How to really see the $\Lambda$ effects?\(^1\)

Prerequisites
- $g \ll 1$
- $f \ll g^{-2} (Qt \gg 1)$
- $f$ isotropic: $f(p) \rightarrow f(|p|) = f_p$

\(^1\)Gauge case: [ABRAAO, KURKELA, LU, MOORE (2014)]
Kinetic treatment with elastic scattering

CSA non renormalizable: How to really see the $\Lambda$ effects?\(^1\)

Prerequisites

\[ g \ll 1 \]
\[ f \ll g^{-2} (Qt \gg 1) \]
\[ f \text{ isotropic: } f(p) \to f(|p|) = f_p \]

\[
\partial_t f_p = \frac{(2\pi)^4}{4E_p} \int_{p',k,k'} \delta^4 (P + K - P' - K') \left[ (1 + f_p)(1 + f_k)f_{p'}f_{k'} - f_p f_k (1 + f_{p'})(1 + f_{k'}) \right] F[f]
\]

\(^1\)Gauge case: [Abraao, Kurkela, Lu, Moore (2014)]
CSA non renormalizable: How to really see the $\Lambda$ effects?\(^1\)

Prerequisites
\(g \ll 1\)
\(f \ll g^{-2} (Qt \gg 1)\)
\(f\) isotropic: \(f(p) \rightarrow f(|p|) = f_p\)

Quantum theory, \(\mathcal{Q}\): keep everything

\[
F_{\mathcal{Q}}[f] = (f_p + f_k)f_{p'}f_{k'} - f_pf_k(f_p' + f_k') + f_p'f_k' - f_pf_k
\]
\[\sim f^3\]
\[\sim f^2\]

\(^1\)Gauge case: [Abraao, Kurkela, Lu, Moore (2014)]
CSA non renormalizable: How to really see the $\Lambda$ effects?\(^1\)

Prerequisites

\[ g \ll 1 \]
\[ f \ll g^{-2} (Qt \gg 1) \]
\[ f \text{ isotropic: } f(p) \rightarrow f(|p|) = f_p \]

Classical approximation, $C^0$: $f \gg 1$, keep the dominant term in $F_Q[f]$

\[
F_{e0}[f] = (f_p + f_k) f'_p f'_k - f_p f_k (f'_p + f'_k) \sim f^3
\]

\(^1\)Gauge case: [Abraao, Kurkela, Lu, Moore (2014)]
CSA non renormalizable: How to really see the $\Lambda$ effects?\(^1\)

**Prerequisites**

\[ g \ll 1 \]
\[ f \ll g^{-2} \left( Qt \gg 1 \right) \]
\[ f \text{ isotropic: } f(p) \rightarrow f(|p|) = f_p \]

**CSA, }^{1}\mathcal{C} : \mathcal{C}^0 \text{ and then } f \rightarrow f + \frac{1}{2} \text{ } [\text{Mueller, Son (2002)}] \]

\[ F_{e1}[f] = F_Q[f] + \frac{1}{4} \left( f_{p'} + f_{k'} - f_p - f_k \right) \]
\[ \sim f^3 + f^2 \]
\[ \sim f \]

\(^1\text{Gauge case: } [\text{Abraao, Kurkela, Lu, Moore (2014)}] \]
**Kinetic treatment: Numerical results for isotropic $f$**

[TE, Gelis, Tanji, Wu (2014)]

\[
\begin{align*}
\Lambda_{UV} / Q &= 0.5 Q \quad \varepsilon = Q^4 \quad n = 0.75 \varepsilon / m \quad [\text{Classical} + 1/2] \\
|\mu| / Q \quad T / Q \quad n_c / n
\end{align*}
\]

| $|\mu| / Q$ | $T / Q$ | $n_c / n$ |
|---|---|---|
| from [BBSV] | from [BBSV] |

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**Diagram Description**

- The graph shows the dependence of $\Lambda_{UV} / Q$ on $|\mu| / Q$, $T / Q$, and $n_c / n$.
- The curves are labeled for different quantities and data points.
- The graph includes data points and lines indicating the numerical results for isotropic $f$.

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**Additional Notes**

- The graph also includes a legend identifying the datasets and data points.
- The axes are labeled appropriately to represent the quantities being compared.

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**References**

[TE, Gelis, Tanji, Wu (2014)]
Kinetic treatment: Numerical results for isotropic \( f \)

[TE, Gelis, Tanji, Wu (2014)]

\[
m = 0.5 \, Q \quad \epsilon = Q^4 \quad n = 0.75 \, \epsilon / m \quad \text{[Classical + 1/2]}
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Kinetic treatment: Numerical results for isotropic $f$
[TE, Gelis, Tanji, Wu (2014)]
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Kinetic treatment: What to expect for anisotropic $f$?

Prerequisites

$$g \ll 1$$

$$f \ll g^{-2} \ (Qt \gg 1)$$

$f$ anisotropic: $f(p) \rightarrow f(|p_\perp|, p_z) = f_{p_\perp z}$

Boltzmann equation for $2 \leftrightarrow 2$ elastic scattering

$$\partial_t f_{p_\perp z} = \left(\frac{2\pi}{4 E_p}\right)^4 \frac{g^4}{4 E_p} \int_{p', k, k'} \delta^4(P + K - P' - K') F[f]$$
Kinetic treatment: What to expect for anisotropic $f$?

Now suppose $f$ very anisotropic initially

$$f_{p_{\perp z}} \sim \delta(p_z)f_0(p_{\perp})$$

What can happen?
KINETIC TREATMENT: WHAT TO EXPECT FOR ANISOTROPIC \( f \)?

Now suppose \( f \) very anisotropic initially

\[
f_{p_{\perp z}} \sim \delta(p_z)f_0(p_{\perp})
\]

What can happen?

In plane collisions \( \rightarrow \) no isotropization
Kinetic treatment: What to expect for anisotropic $f$?

Now suppose $f$ very anisotropic initially

$$f_{p_{\perp}z} \sim \delta(p_z)f_0(p_{\perp})$$

What can happen?

Out of plane collisions → isotropization
Kinetic treatment: What to expect for anisotropic $f$?

Now suppose $f$ very anisotropic initially

$$f_{p_{\perp z}} \sim \delta(p_z)f_0(p_\perp)$$

What can happen?

Out of plane collisions $\rightarrow$ isotropization

Can these large angle collisions happen?
ANISOTROPIC $f$: PROBLEM WITH THE $C^0$ SCHEME

Remember

\[ F_{c^0}[f] \sim f^3 \]

\[ F_Q[f] = F_{c^0}[f] + f_{p'}f_{k'} - f_p f_k \]

Now take initially $f_{p_{\perp}} = \frac{2\pi}{\delta(p_z)}f_0(p_{\perp})$

Let us inspect the Boltzmann equation in both cases
ANISOTROPIC $f$: PROBLEM WITH THE $C^0$ SCHEME

Remember

$$F_{C^0}[f] \sim f^3$$

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Now take initially $f_{p_{\perp z}} = 2\pi\delta(p_z)f_0(p_{\perp})$

Let us inspect the Boltzmann equation in both cases

$C^0$

$$\partial_t f_{p_{\perp z}} \sim \delta(p_z) \int_{p', k, k'} \delta^3(P_{\perp} + K_{\perp} - P'_{\perp} - K'_{\perp}) f^3$$
ANISOTROPIC $f$: PROBLEM WITH THE $C^0$ SCHEME

Remember

$$F_{c^0}[f] \sim f^3$$

$$F_Q[f] = F_{c^0}[f] + f_p f_k' - f_p f_k$$

Now take initially

$$f_{p_{\perp z}} = \frac{2\pi \delta(p_z)}{p_\perp} f_0(p_{\perp})$$

Let us inspect the Boltzmann equation in both cases

$$Q$$

\[
\partial_t f_{p_{\perp z}} \sim \delta(p_z) \int_{p_{\perp}'} \delta^3(P_\perp + K_\perp - P_{\perp}' - K_{\perp}') f^3
\]

\[
+ \int_{p_{\perp},k_{\perp},k_{\perp}'} \delta^3(P_\perp + K_\perp - P_{\perp}' - K_{\perp}') \delta(k_{\perp} + p_z) f_{p_{\perp}'} f_{k_{\perp}'}
\]

\[
- \delta(p_z) \int_{p_{\perp}',k_{\perp}',k_{\perp}'} \delta^3(P_\perp + K_\perp - P_{\perp}' - K_{\perp}') \delta(k_{\perp}' + p_z') f_{p_{\perp}'} f_{k_{\perp}'}
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Remember

$$F_{c^0}[f] \sim f^3$$

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Now take initially $f_{\perp z} = 2\pi \delta(p_z)f_0(p_{\perp})$

Let us inspect the Boltzmann equation in both cases

$$Q$$

$$\partial_t f_{\perp z} = \delta(p_z) \int_{p'_{\perp},k_{\perp},k_{\perp}'} \delta^3(P_{\perp} + K_{\perp} - P'_{\perp} - K'_{\perp})f_3$$

$$+ \int_{p'_{\perp},k_{\perp},k_{\perp}'} \delta^3(P_{\perp} + K_{\perp} - P'_{\perp} - K'_{\perp})\delta(k_z + p_z)f_{\perp z}'f_{k_{\perp} z}'$$

$$- \delta(p_z) \int_{p',k_{\perp},k'} \delta^3(P_{\perp} + K_{\perp} - P'_{\perp} - K'_{\perp})\delta(k_z' + p_z')f_{\perp z}f_{k'_{\perp} z}$$
ANISOTROPIC $f$: PROBLEM WITH THE $C^0$ SCHEME

Remember

\[ F_{C^0}[f] \sim f^3 \]

\[ F_Q[f] = F_{C^0}[f] + f_p f_{k'} - f_p f_k \]

Now take initially $f_{P_{zz}} = 2\pi \delta(p_z)f_0(p_\perp)$

Let us inspect the Boltzmann equation in both cases

$C^0$ artificially supresses large angle collisions.
ANISOTROPIC $f$: PROBLEM WITH THE $C^0$ SCHEME

Remember

$$F_{C^0}[f] \sim f^3$$

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Now take initially $f_{\perp z} = 2\pi\delta(p_z)f_0(p_{\perp})$

Let us inspect the Boltzmann equation in both cases

$C^0$ artificially supresses large angle collisions.

$C^0$ artificially traps the distribution in an anisotropic state.
ANISOTROPIC $f$: PROBLEM WITH THE $\mathcal{C}^0$ SCHEME

Remember

$$F_{\mathcal{C}^0}[f] \sim f^3$$

$$F_{\mathcal{Q}}[f] = F_{\mathcal{C}^0}[f] + f_p f_k' - f_p f_k$$

Now take initially $f_{p_{A,z}} = 2\pi \delta(p_z)f_0(p_{\perp})$

Let us inspect the Boltzmann equation in both cases

$\mathcal{C}^0$ artificially supresses large angle collisions.

$\mathcal{C}^0$ artificially traps the distribution in an anisotropic state.

None of this happens with $\mathcal{Q}$ or $\mathcal{C}^1$. 
ANISOTROPIC $f$: PROBLEM WITH THE $C^0$ SCHEME

Remember

$$F_{C^0}[f] \sim f^3$$

$$F_Q[f] = F_{C^0}[f] + f_p'f_k' - f_pf_k$$

Now take initially $f_{p_{\perp}} = 2\pi\delta(p_{z})f_0(p_{\perp})$

Let us inspect the Boltzmann equation in both cases

$C^0$ artificially suppresses large angle collisions.

$C^0$ artificially traps the distribution in an anisotropic state.

None of this happens with $Q$ or $C^1$.

Could it be the reason why $C^0$ never isotropizes?
ANISOTROPIC $f$: ILLUSTRATION OF THE PROBLEM WITH THE $C^0$ SCHEME

[BLAIZOT, JIANG, LIAO] WORK IN PROGRESS
ANISOTROPIC $\tilde{f}$: ILLUSTRATION OF THE PROBLEM WITH THE $C^0$ SCHEME

[BLAIZOT, JIANG, LIAO] WORK IN PROGRESS

How expansion affects this?
Work in progress

Classical
Conclusion

- Scalar models: strong hints of hydrodynamical behaviour ✓
- Yang-Mills: Evidences for an early hydrodynamical onset ✓
- Hydrodynamization already happens at weak coupling ✓
- CSA non-renormalizable $\Rightarrow$ Effective theory ✓
ARE THE POSTULATES OF HYDRODYNAMICS SATISFIED DURING THE EARLY STAGES OF A HEAVY-ION COLLISION?

Perspectives

• Boltzmann treatment in the expanding case

• Understanding the BSSV exponents

• Going beyond the CSA $\Rightarrow$ Quantum evolution
Thank you!
Kinetic treatment: Numerical results for isotropic $f$
[TE, Gelis, Tanji, Wu (2014)]

$m = 0.5 \, Q \quad \epsilon = Q^4 \quad n = 0.75 \, \epsilon / m \quad [\text{Classical}]$

$|\mu| / Q$
$T / Q$
$\Lambda_{uv} / Q$

$C_1$