

# New formulation of leading-order anisotropic hydrodynamics

## Outline

- Second order viscous hydrodynamics
- Expansion around an anisotropic background
- Leading order in anisotropic hydrodynamics
- Extension of the model



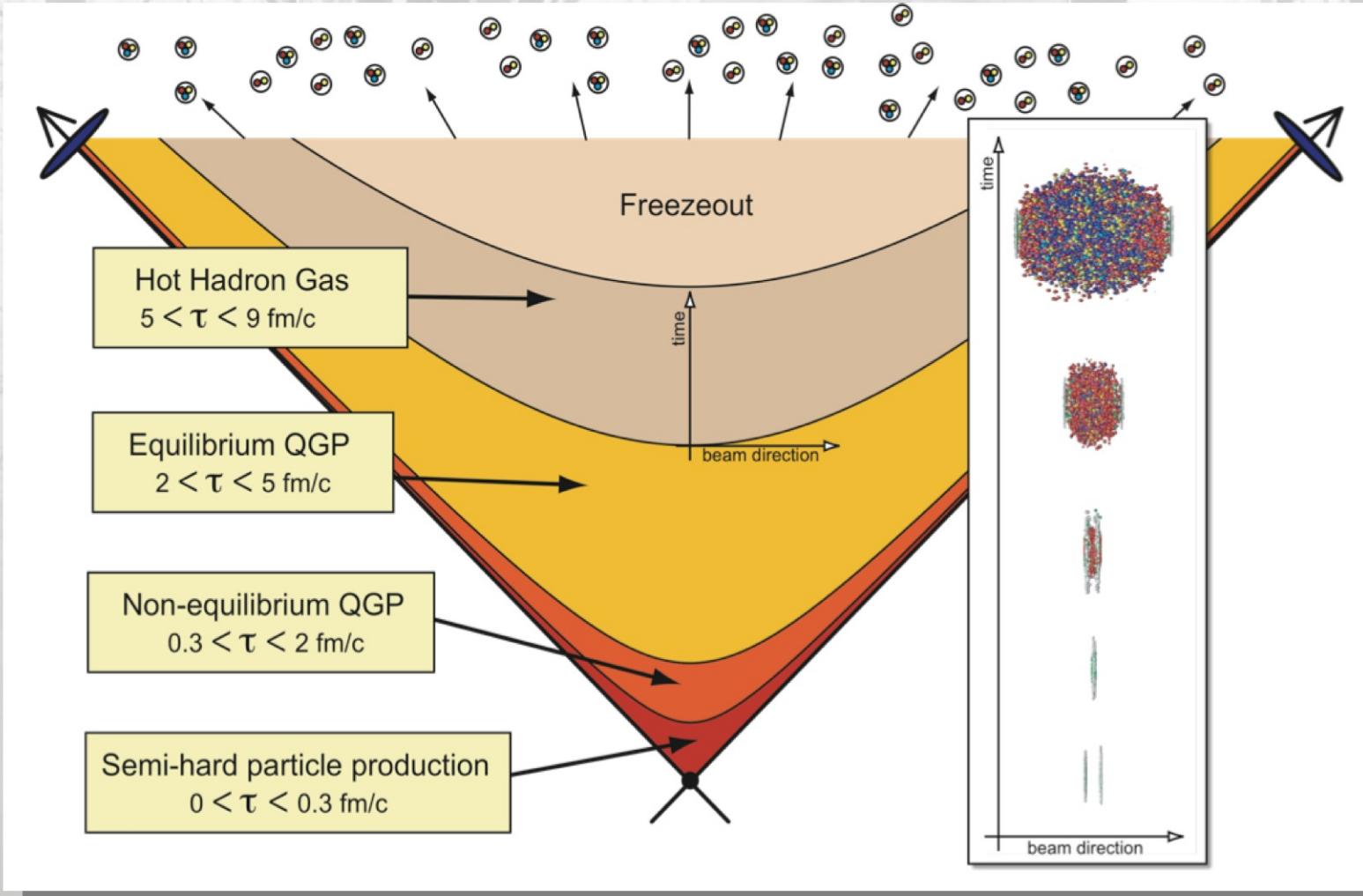
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# Motivations

Hydrodynamic modeling of heavy ion collisions (small viscosity)

Large gradients → viscous corrections



Strong longitudinal expansion,  
pressure anisotropy  
(also AdS/CFT)

Large momentum anisotropy from  
microscopic models  
( $p$ QCCD, CGC)

## *From kinetic theory to hydrodynamics*

Relativistic Boltzmann equation:

$$p^\mu \partial_\mu f(x, p) = \mathcal{C}[f]$$

First moment:

$$\begin{aligned} \int dP p^\mu p^\nu \partial_\mu f &= \partial_\mu T^{\mu\nu} = \\ &= \int dP p^\nu \mathcal{C}[f] \left[ \right] = 0 \end{aligned}$$

•S. R. De Groot, W. A. van Leeuwen, Ch. G. van Weert, *Relativistic kinetic theory*, North Holland (1980).

# Hydrodynamics

*Ansatz for the relativistic Boltzmann distribution*

Perfect fluid:

$$f \simeq f_{\text{eq.}} = k \exp \left[ -\frac{p \cdot U(x)}{T(x)} \right]$$

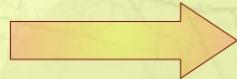


$$T^{\mu\nu} = k \int dP p^\mu p^\nu \exp \left[ -\frac{p \cdot U}{T} \right] = (\varepsilon + P) U^\mu U^\nu - g^{\mu\nu} P$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

# Dissipative hydrodynamics

$$f = f_{\text{eq.}} + \delta f$$



$$T^{\mu\nu} = T_{\text{eq.}}^{\mu\nu} + \delta T^{\mu\nu}$$

$\delta f \Rightarrow \delta T^{\mu\nu}$  treated as, small, perturbations

*Landau frame,  
massless particles*

$$\delta T^{\mu\nu} = \pi^{\mu\nu}$$

$$U^\mu \pi_{\mu\nu} = 0 \quad g_{\mu\nu} \pi^{\mu\nu} = 0$$

**Four equations, five more degrees of freedom!**

# Entropy current and entropy source

Obtaining the remaining equations from the second principle of thermodynamics

$$\mathcal{S}^\mu = \mathcal{S}_{\text{eq.}}^\mu + \delta\mathcal{S}^\mu \simeq \frac{p}{T}U^\mu + \frac{1}{T}T^{\mu\nu}U_\nu - \frac{\tau_\pi}{4\eta T}\pi^{\alpha\beta}\pi_{\alpha\beta}U^\mu$$

$$\partial_\mu \mathcal{S}^\mu \geq 0$$



$$\partial_\mu \mathcal{S}^\mu \simeq \frac{1}{T}\pi^{\mu\nu} \left[ \sigma_{\mu\nu} - \frac{\tau_\pi}{2\eta}\Delta_\mu^\alpha\Delta_\nu^\beta D\pi_{\alpha\beta} - \frac{1}{2}T\pi_{\mu\nu}\partial \cdot \left( \frac{\tau_\pi}{2\eta T}U \right) \right]$$

$$\pi^{\mu\nu}\pi_{\mu\nu} \geq 0$$



$$\tau_\pi\Delta_\mu^\alpha\Delta_\nu^\beta D\pi_{\alpha\beta} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} - \pi_{\mu\nu}T\eta\partial \cdot \left( \frac{\tau_\pi}{2\eta T}U \right)$$

# Example, Bjorken flow

*0+1 dimensions: boost invariant in the longitudinal direction,  
homogeneous in the transverse plane*

$$U = (\cosh \eta_{\parallel}, 0, 0, \sinh \eta_{\parallel}) \quad \eta_{\parallel} = \frac{1}{2} \ln \left( \frac{t+z}{t-z} \right) \quad \tau = \sqrt{t^2 - z^2}$$

Gradients of the four velocity are proportional to  $1/\tau$

*In the Navier-Stokes limit*

$$\pi_{\mu\nu} \simeq 2\eta\sigma_{\mu\nu}$$

*therefore*

$$\frac{P_{\parallel}}{P_{\perp}} = \frac{P_{\text{eq.}} + \pi_{ZZ}}{P_{\text{eq.}} + \pi_{XX}} \simeq \frac{3T\tau - 16\bar{\eta}}{3T\tau + 8\bar{\eta}}$$

# Anisotropic hydrodynamics

*Reorganization of the hydrodynamic expansion*

$$f = f_{\text{eq.}} + \delta f$$

*around an anisotropic background instead of the local equilibrium*

$$f = f_{\text{aniso.}} + \tilde{\delta f}$$

“Romatschke-Strickland” form:

$$f_{\text{aniso.}} = k \exp \left[ -\frac{\sqrt{\left( p \cdot U(x) \right)^2 + \xi(x) \left( p \cdot Z(x) \right)^2}}{\Lambda(x)} \right]$$

# 0+1 dimensions

Boost invariant in the longitudinal direction, homogeneous in the transverse plane

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_\perp & 0 & 0 \\ 0 & 0 & P_\perp & 0 \\ 0 & 0 & 0 & P_\parallel \end{pmatrix}$$

*No negative pressure already at the leading order*

$$\left( P_\perp, P_\parallel \right) = \int dP \left( \frac{1}{2} p_\perp^2, p_\parallel^2 \right) f_{\text{RS}} > 0 \quad \textit{Positive by construction}$$

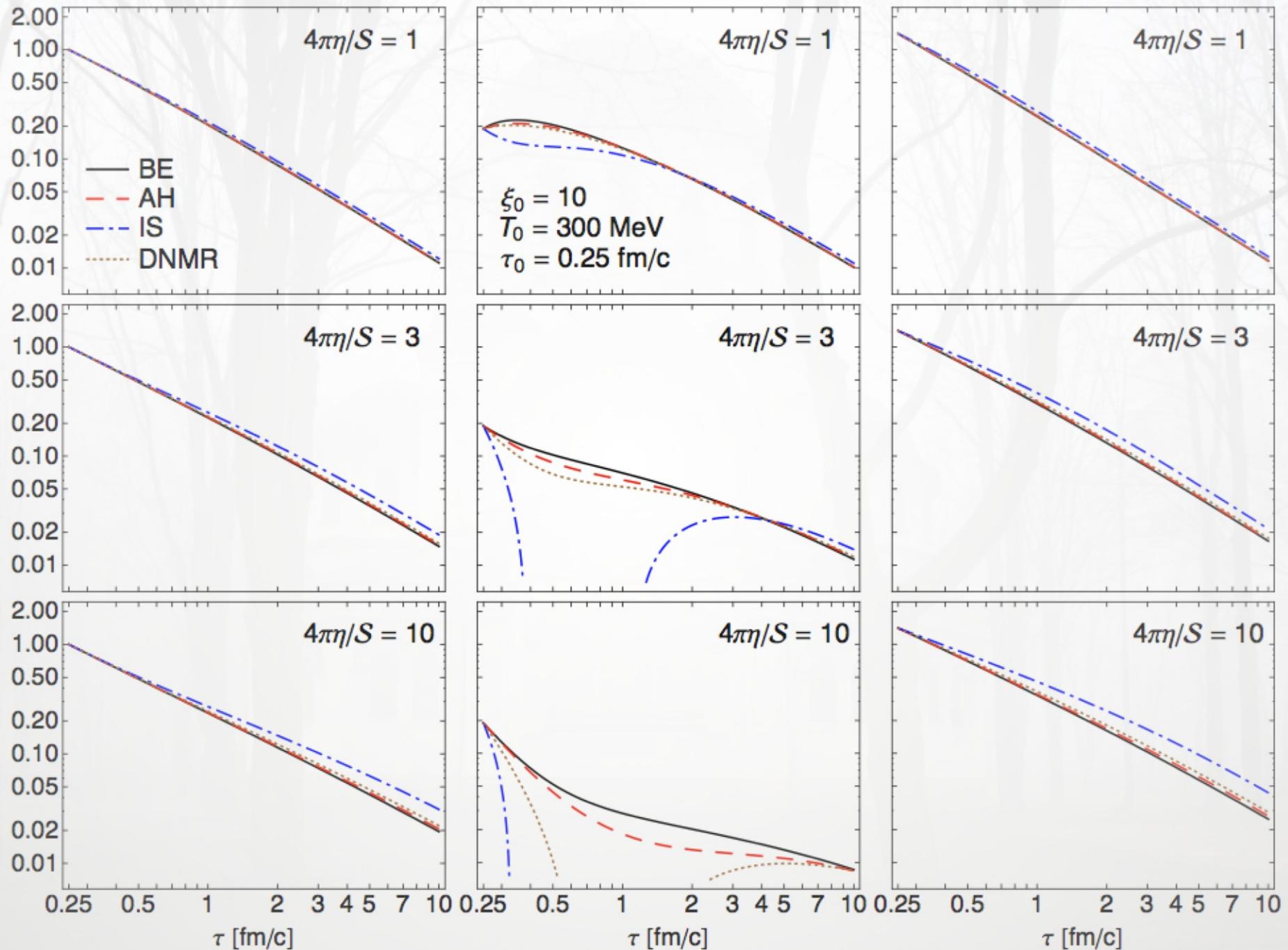
**Exact solution! Test for viscous and anisotropic hydrodynamics**

# Some plots

$\varepsilon(\tau)/\varepsilon(\tau_0)$

$3P_{||}(\tau)/\varepsilon(\tau_0)$

$3P_{\perp}(\tau)/\varepsilon(\tau_0)$



# Radial flow

*1+1 dimensions, boost invariance in the longitudinal direction, rotation invariance in the transverse plane*

Four-velocity

$$U = \gamma \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cosh \theta_\perp \cosh \eta_{\parallel} \\ \sinh \theta_\perp \cos \phi \\ \sinh \theta_\perp \sin \phi \\ \cosh \theta_\perp \sinh \eta_{\parallel} \end{pmatrix}$$

Another unknown function

$$\tanh \theta_\perp(\tau, r) = \cosh \eta_{\parallel} \sqrt{v_x^2 + v_y^2}$$

Orthonormal basis

$$X = \begin{pmatrix} \sinh \theta_\perp \cosh \eta_{\parallel} \\ \cosh \theta_\perp \cos \phi \\ \cosh \theta_\perp \sin \phi \\ \sinh \theta_\perp \sinh \eta_{\parallel} \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \quad Z = \begin{pmatrix} \sinh \eta_{\parallel} \\ 0 \\ 0 \\ \cosh \eta_{\parallel} \end{pmatrix}$$

•W Florkowski and R Ryblewski, Phys. Rev. C 85, 044902 (2012)

# Improving the starting point of the expansion

*For a conformal system*

$$f_{\text{aniso.}} = k \exp \left[ -\frac{\sqrt{(1 + \xi_X) (p \cdot X)^2 + (1 + \xi_Y) (p \cdot Y)^2 + (1 + \xi_Z) (p \cdot Z)^2}}{\lambda(x)} \right]$$

$$\sum_I \xi_I = 0$$

Pressure not isotropic in the transverse plane

$$I \in \{X, Y, Z\}$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_X & 0 & 0 \\ 0 & 0 & P_Y & 0 \\ 0 & 0 & 0 & P_Z \end{pmatrix}$$

Still two trivial equations from the local four-momentum conservation and four unknown scalar function:  $\theta_\perp, \lambda$  and two independent anisotropy parameter

# How to match with second order hydrodynamics

Using a close to equilibrium expansion

$$f \simeq f_{\text{eq}} \left( 1 + \frac{\lambda - T}{T^2} (p \cdot U) - \frac{\xi_X (p \cdot X)^2 + \xi_Y (p \cdot Y)^2 + \xi_Z (p \cdot Z)^2}{2T(p \cdot U)} \right)$$

We can find the pressure corrections

$$\left( T^{\mu\nu} = \int dP p^\mu p^\nu f \right) \quad \pi^{\mu\nu} = \sum_I \pi_I I^\mu I^\nu$$

$$T^{\mu\nu} \simeq T_{\text{eq}}^{\mu\nu} - \frac{32\pi}{5} k T^4 \left( \xi_X X^\mu X^\nu + \xi_Y Y^\mu Y^\nu + \xi_Z Z^\mu Z^\nu \right)$$

We can verify the matching if the equations for the anisotropy parameters correspond to the second order hydrodynamics equations for the pressure corrections

$$\xi_I \simeq -\frac{15}{4} \frac{\pi_I}{\varepsilon}$$

## The remaining equations

*Looking at the second moment of the Boltzmann equation, in relaxation time approximation, to close the system.*

$$p^\lambda \partial_\lambda f = (p \cdot U) \frac{f_{\text{eq.}} - f}{\tau_{\text{eq.}}} \quad \Rightarrow \quad \int dP p^\mu p^\nu p^\lambda \partial_\lambda f = \frac{U_\lambda}{\tau_{\text{eq.}}} \int dP (p^\mu p^\nu p^\lambda f_{\text{eq.}} - p^\mu p^\nu p^\lambda f)$$

$$\partial_\lambda \Theta^{\lambda\mu\nu} = \frac{1}{\tau_{\text{eq}}} (U_\lambda \Theta_{\text{eq}}^{\lambda\mu\nu} - U_\lambda \Theta^{\lambda\mu\nu})$$

$$\Theta^{\lambda\mu\nu} = \int dP p^\lambda p^\mu p^\nu f \quad \Theta_{\text{eq}}^{\lambda\mu\nu} = \int dP p^\lambda p^\mu p^\nu f_{\text{eq.}}$$

Many of the projections of these equations along the vectors of the basis are trivial, but there is a useful combination of the remaining ones

$$\frac{D\xi_I}{1 + \xi_I} + 2\sigma_I + \frac{\xi_I}{\tau_{\text{eq.}}} \left( \frac{T}{\lambda} \right)^5 \sqrt{\prod_J (1 + \xi_J)} - \frac{1}{3} \sum_J \frac{D\xi_J}{1 + \xi_J} = 0$$

**Only two of the three equations are independent, if two of them are fulfilled all of them are**

# Small anisotropy limit

$$D\xi_I + 2\sigma_I + \frac{\xi_I}{\tau_{\text{eq.}}} \simeq 0$$

Using the first moment equations  
and after some calculations

$$\tau_{\text{eq.}} D\pi_I + \pi_I = 2 \left( \frac{4}{15} \varepsilon \tau_{\text{eq.}} \right) \sigma_I - \pi_I \left[ \frac{4}{15} \varepsilon \tau_{\text{eq.}} T \partial \cdot \left( \frac{15}{8\varepsilon T} U \right) \right]$$

Consistent with Israel-Stewart

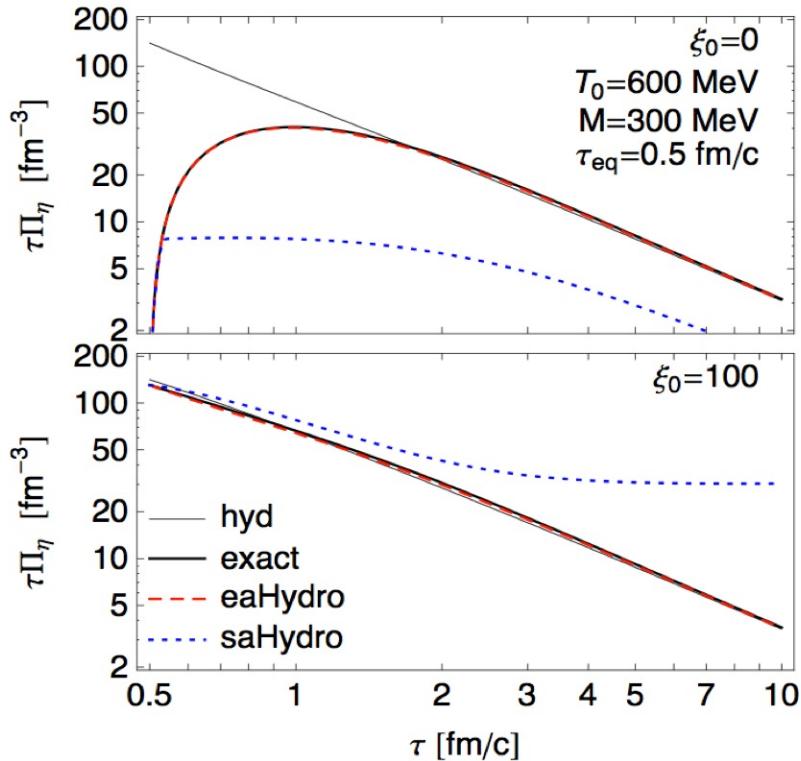
$$\tau_\pi D\pi_I + \pi_I = 2\eta\sigma_I - \pi_I \left[ \eta T \partial_\lambda \left( \frac{\tau_\pi}{2\eta T} U^\lambda \right) \right]$$

**provided**

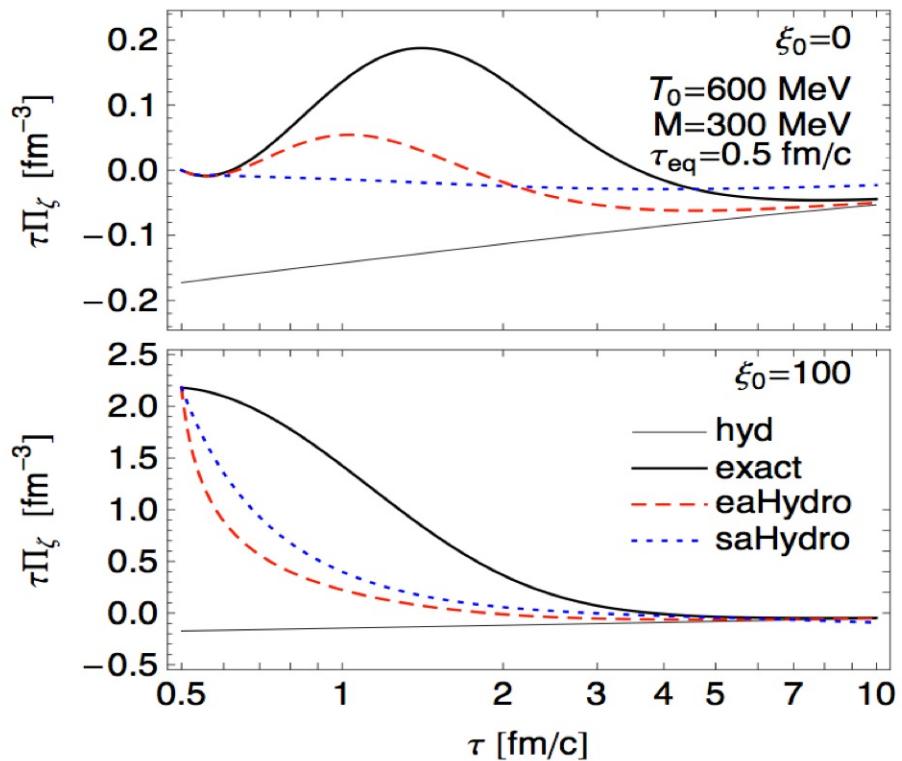
$$\tau_\pi = \tau_{\text{eq.}} \quad \eta = \frac{4}{15} \varepsilon \tau_{\text{eq.}} \Rightarrow \tau_\pi = \frac{5\bar{\eta}}{T}$$

# Important even without radial expansion

• W Florkowski, R Ryblewski, M Strickland, L Tinti, Phys. Rev. C 89, 054909 (2014)



Much better agreement with the exact solution



So far so good, but...

# (3+1)-dimensional framework

*No symmetry constraints from boost invariance or cylindrical symmetry*

*Generalized “Romatschke-Strickland” form*

$$f = k \exp \left( -\frac{1}{\lambda} \sqrt{p_\mu \Xi^{\mu\nu} p_\nu} \right)$$

*Geometric decomposition*

$$\Xi^{\mu\nu} = U^\mu U^\nu + \xi^{\mu\nu} - \phi \Delta^{\mu\nu}$$

**Matching with second order viscous hydrodynamics (Israel-Stewart??)**

- L Tinti, arXiv:1411.7268

# (3+1)-dimensional framework

*Geometric argument to choose the necessary equations*

*We need higher moments of the Boltzmann equation*

$$\Theta^{\lambda\mu\nu} = \int dP p^\lambda p^\mu p^\nu f = \Theta_U U^\lambda U^\mu U^\nu + U^\lambda \Theta^{\mu\nu} + U^\mu \Theta^{\nu\lambda} + U^\nu \Theta^{\lambda\mu}$$

$$\Theta^{\mu\nu} = U_\alpha \Delta_\beta^\mu \Delta_\gamma^\nu \Theta^{\alpha\beta\gamma}$$

**“Shear equation”**

$$D\Theta^{\langle\mu\nu\rangle} + \frac{5}{3}\theta\Theta^{\langle\mu\nu\rangle} + 2\Theta_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} - 2\Theta_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} = -\frac{1}{\tau_{\text{eq}}} \Theta^{\langle\mu\nu\rangle}$$

# (3+1)-dimensional framework

Close-to-equilibrium limit

$$\Theta^{\langle\mu\nu\rangle} \simeq g(m, T) \pi^{\mu\nu} \quad \Theta_{\text{tr.}} \simeq \Theta_{\text{tr.}}^{\text{eq.}} + h_{\text{tr.}}(m, T) \Pi$$

$$n \simeq n_{\text{eq.}} - h_0(m, T) \Pi$$

$$\begin{aligned} D\pi^{\langle\mu\nu\rangle} + \frac{1}{\tau_\pi} \pi^{\mu\nu} &= 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} - \tau_{\pi\pi} \pi_\lambda^{\langle\mu} \sigma^{\nu\rangle\lambda} \\ &\quad - \delta_{\pi\pi} \pi^{\mu\nu} \theta + \lambda_{\pi\Pi} \Pi \pi^{\mu\nu} \end{aligned}$$

$$D\Pi + \frac{1}{\tau_\Pi} \Pi = -\beta_\Pi \theta - \delta_{\Pi\Pi} \Pi \theta + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}$$

# Summary & outlook

- Anisotropic hydrodynamics is a reorganization of the hydrodynamic expansion, around an anisotropic distribution.
- Pressure anisotropies already at the leading order. Very good approximation in the exactly solvable 0+1 case
- Generalized ansatz for the leading order. Well defined in the early stages and consistent with second order viscous hydrodynamics close to equilibrium, providing bulk-shear couplings.
- Next: Test with the exact solutions of the Boltzmann equation?