Simulations at fixed topology: fixed topology versus ordinary finite volume corrections

A. Dromard[1], I. Bautista[2], W. Bietenholz[2], C. Czaban[1], U. Gerber[2], C. Hofmann[3], H. Mejía[2], L. Prado[2] and M. Wagner[1]

[1]University of Frankfurt am Main
[2]Universidad Nacional Autónoma de México
[3]Facultad de Ciencias, Universidad de Colima

JOHANN WOLFGANG GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Outline

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1. Introduction
   - Introduction to topology
   - Motivation
   - Working at fixed topology
2. Results on different models
   - Method
   - Results
3. Including ordinary finite volume effects
   - Ordinary finite volume effects
   - Combining ordinary and topological finite volume effects

Topology in Field Theory

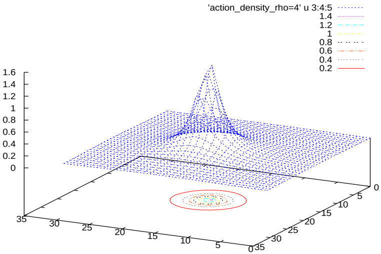
- Path integral formalism: Integral over all possible A_μ (finite action)
- For each A_μ, you can define a topological charge $Q[A]$ which characterizes its topology.

$$Q[A] = \frac{g^2}{32\pi^2} \int F_{\mu\nu}\tilde{F}_{\mu\nu} \in \mathbb{Z}$$

Example of topological object in QCD: Instanton

- Definition: Pseudo Particle which is solution of the Euclidean EOM
- They carry a topological charge +1

$$Q[A] = N_{inst} - N_{anti}$$



Topological sector

- Definition: The all set of configurations with same topological charge is called a topological sector.

- Keeping the action finite, **you can** continuously deform one configuration (one path) to another if they belong to the **same topological sector.**

- **You cannot** deform continuously one configuration to another if they belong to **different topological sectors**.
    - There is a infinite barrier of action between topological sectors

$\theta-$vacuum

To take into account topology in your action:

- Action with topological term:

$$S_E(\theta) = S_E - i\theta \frac{g^2}{32\pi^2} \int F_{\mu\nu}\tilde{F}_{\mu\nu} = S_E - i\theta Q[A]$$

- $|\Omega, \theta\rangle$ (ground state) is depending of θ: the value of θ defines the distribution of Q
- The influence of the Q-distribution on physics is mostly due to the topological susceptibility:

$$\chi_T = \left. \frac{\partial^2 E_0(\theta)}{\partial \theta^2} \right|_{\theta=0} = \frac{\langle Q^2 \rangle}{V}$$

- Experimental value of θ_{QCD}: $\theta_{QCD} \approx 0$

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Topology on lattice

- Configurations are produced by Monte-Carlo algorithm
- Proceed to create new configuration by small **discrete transformations**
  - allow the algorithm to change Q
- Smaller lattice spacing ⇒transformations closer to continuous deformations

Problem

Topology freezes for a too small lattice spacing $a < 0.1 fm$[a]
⇒Find a way to get the QCD observable from one topological sector

[a] Luscher, Martin JHEP 1008 (2010) 071

Other uses of fixed topology:

- High quality fermions → Overlap fermions
- Mixed action, ...

Outline

At fixed topological charge

- Path integral at fixed Q does not correspond to physical theory
  - No Hamiltonian!
  - We can still define a partition function, correlators and masses at fixed topology.
- Partition function at fixed Q

$$Z_Q = \int DAD\psi D\bar{\psi}\delta_{Q,Q[A]} e^{-S_E[A,\psi,\bar{\psi}]}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, Z(\theta) e^{iQ\theta}$$

- Correlator transformation

$$Z_Q C_Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, C(\theta) Z(\theta) e^{iQ\theta}.$$

BCNW-equation

- Using saddle-point approximation and Taylor expansions, you can relate M_Q to the physical mass $M(\theta = 0)$.
  Condition: $\chi_T V > max(|Q|, 1)$.

- The mass at fixed topology of particles is given by the BCNW equation [1]:

$$M_Q = M(0) + \frac{M''(0)}{2\chi_T V}\left(1 - \frac{Q^2}{\chi_T V}\right) + \mathcal{O}\left(\frac{1}{(\chi_T V)^2}\right)$$

- Fixing the topology implies **finite volume effects (TFV effect)**

[1] Brower, R. et al. Phys.Lett. B560 (2003) 64-74, Aoki, Sinya et al. Phys.Rev. D76 (2007) 054508

Outline

Method

$$M_Q = M(0) + \frac{M''(0)}{2\chi_T V}\left(1 - \frac{Q^2}{\chi_T V}\right)$$

Method to extract physical mass from fixed top. simulation:

1. Compute M_Q using only configurations in a single topological sector for different volumes and topological charges.
2. Fit $M(V, Q)$ to the BCNW-equation for $\chi_T V > max(|Q|, 1)$.
   1. get the parameters: $M(\theta = 0), \chi_T$, and $M''(0)$.

- You can also use only one sector if you know χ_T.

To test this fixed topology method:

1. Compute $M(0)$ using traditional method (unfixed and unfrozen topology)
2. Extract $M(0)$ using fixed topology simulations
3. Compare 1. and 2.

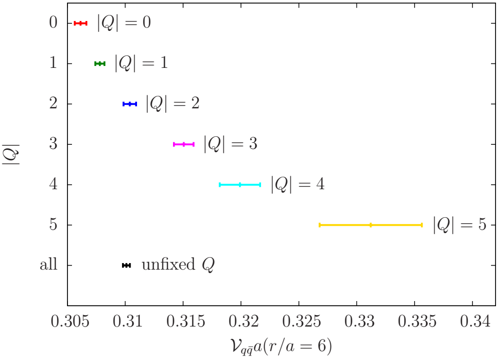
Outline

SU(2) Yang-Mills Theory

$$\mathscr{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$$

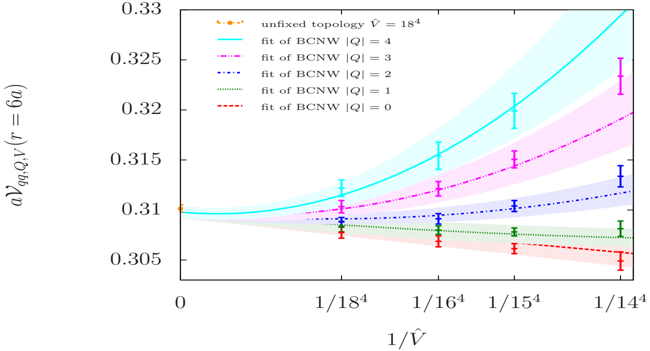
Set-up

- Observable: static potential $V_{qq}(R)$ for $R = 1$ to 6
- $\beta = 2.5$
- Volumes: $14^4, 15^4, 16^4$ & 18^4
- Number of configurations: 4000 per volume



SU(2) Yang-Mills Theory

$$M_Q = M(0) + \frac{M''(0)}{2\chi_T V}\left(1 - \frac{Q^2}{\chi_T V}\right)$$



$V_{qq}(R = 6) = 0.3097(5)$ from fixed Q
$V_{qq}(R = 6) = 0.3101(3)$ unfixed topology simulation (ref)

Schwinger Model

$$\mathscr{L}(\bar{\psi},\psi,A_\mu) = \bar{\psi}\left(\gamma_\mu(\partial_\mu + igA_\mu) + m\right)\psi + \tfrac{1}{2}F_{\mu\nu}F_{\mu\nu}.$$

- Reason of choice :2D Model with fermions and confinement

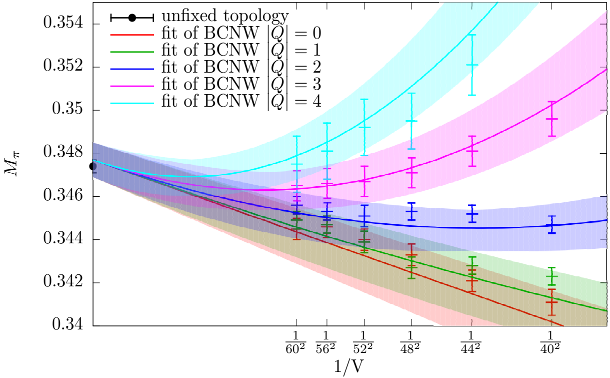
Set-up

- Observable: static potential $V_{qq}(R)$ for $R = 1$ to 4, Pion mass
- Operator for the pion mass:

$$O_\pi = \sum_x \bar{\psi}^{(u)}(x)\gamma_5\psi^{(d)}(x),$$

- $\beta = 4.0$ and $q_m = 0.100$
- Volumes: $40^2, 44^2, 48^2, 52^2, 56^2, 60^2$
- Number of configurations: 500000 per volume

Schwinger Model



- $M_\pi = 0.3477(8)$ from fixed Q
- $M_\pi = 0.3474(3)$ from unfixed topology simulation (ref)

Summary of these results

1. Fixing Q results in topological finite volume effects (TFV).
2. The method is working well to extract the mass under the condition that $\chi_T V > max(|Q|, 1)$
   1. The method is conceptually simple
   2. Precise results for the mass extraction
   3. Applicable with only one top. sector if the topological susceptibility is known
3. Additional difficulties: TFV effects are in competition with ordinary finite volume effects (OFV): small window to apply the method
   1. Small volume: OFV effects dominate
   2. Big volume: too expensive

$\Rightarrow$ Including OFV effects in the equation to increased the window

Outline

Ordinary finite volume effects

Origin of the problem:

- simulation at finite volume
- periodic boundary conditions

$\Rightarrow$The particle can interact with an image of itself !

Ordinary finite volume effects (OFV effect) on a particle of mass M ($SU(N)$ equation):

$$M(V) - M(V = \infty) \propto \frac{1}{L} e^{-\frac{\sqrt{3}}{2} mL}$$

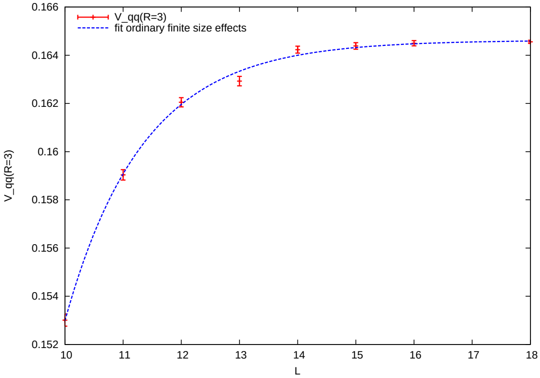
with m: mass of the lightest particle, L: length of the box

Ordinary finite volume effects in QCD:

- Extremely costly to generate configurations $\Rightarrow$small volumes
- m_π is small in QCD.

$\Rightarrow$Difficulties to get rid of ordinary finite volume effects

finite volume effects on SU(2)



- Lightest mass in pure Yang-Mills SU(2): Glueball mass $m = 0.74(4)$ (literature[2]: $m = 0.723(23)$)

[2] hep-th/9812187.pdf

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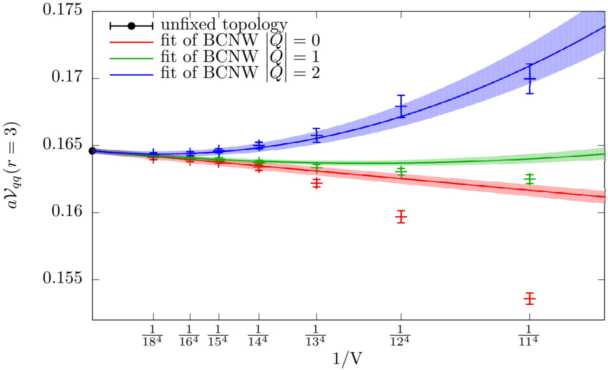
Combining ordinary and topological finite volume effects

- Working at fixed topology with ordinary finite volume effects: need to combine both kind of finite volume effects.
- Leading order (LO): BCNW-equation , OFV

$$M_Q = M(0) + \frac{M''(0)}{2\chi_T V}\left(1 - \frac{Q^2}{\chi_T V}\right) - \frac{A}{m^2 L} e^{-\frac{\sqrt{3}}{2} mL} + \mathcal{O}\left(\frac{1}{(\chi_T V)^2}, \frac{e^{-\frac{\sqrt{3}}{2} m_0 L}}{(\chi_T V)}\right)$$

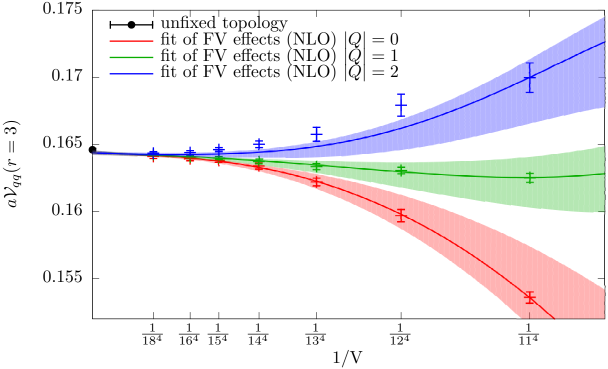
- Next Leading Order (NLO): Ordinary finite volume effects will depend of the topological charge.
  - Possibility to fit but: 4 more parameters for NLO

Ordinary finite volume effects at fixed topology



- Ordinary finite volume effects for $V < 14^4$ (discrepancy with the BCNW fit)
- Different OFV effects for different topological charges
  $\Rightarrow$Need to go to next leading order

Ordinary finite volume effects at fixed topology



- Fit of the next leading order (NLO) equation combining topological and ordinary finite volume effects

	$V_{qq}(R=3)$	m	$\chi_T(\times 10^5)$
fit of FV (NLO)	0.16437(15)	0.67(10)	9.5(2.0)
Unfixed Top. Result (ref)	0.16455(7)	0.723(23)	7.0(0.9)

Summary

1. Worked at fixed topology
   1. Show the efficiency of the method to extract mass from frozen topology simulation
   2. Precise results obtained for the mass
2. Combination of ordinary finite volume effects and topological finite volume effects
   1. Equation combining both finite volume effects
   2. Promising test on SU(2) Yang-Mills theory

- Outlook
  - More tests on the combination of ordinary finite volume effects and topological finite volume effects.
  - Full QCD