The $\pi\pi$ scattering amplitude and the σ meson

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Outline

- Introduction – motivation
- The multichannel $\pi\pi$ amplitudes
- Some results with the amplitudes
  – phase shift, inelasticity
  – final-state interactions in heavy-meson decays
- Modification of the amplitudes via the GKPY relations
  – results of the modification
- The σ pole in the single- and coupled-channel analysis
- Conclusions

Introduction

Analysis of the multichannel $\pi\pi$ scattering data (unitarity)

in S wave :	$\pi\pi$	$\longrightarrow$	$\pi\pi$	in P wave :	$\pi\pi$	$\longrightarrow$	$\pi\pi$
			$K\overline{K}$				$\rho 2\pi$
			$\eta\eta$ $(\eta\eta')$				$\rho\sigma$

allows to study properties of light mesons

- scalar isoscalar states: $f_0(500)(\sigma)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$, ...
- vector isovector states: $\rho(770)$, $\rho(1250)$, $\rho(1450)$, ...

Data: phase shifts, inelasticity parameters (δ_{11}, ϕ_{12}, η_{11}, ...);
energy $<$ 2 GeV.

The scattering matrix is constructed utilizing a uniformizing variable and formulas for analytical continuation of the S-matrix elements to all sheets of the Riemann surface (analyticity).

Motivation to modify the $\pi\pi$ amplitudes via Roy-like equations

- in construction of the three-channel $\pi\pi$ amplitudes the crossing symmetry constraint was not assumed $\rightarrow$ *impose the crossing symmetry* [Surovtsev etal, Phys.Rev.D**81**(2010)016001]
- it is interesting to *check a consistency* of the multichannel approach with that based on the dispersion relations
- the three-channel amplitudes do not describe well data near the threshold due to a necessary approximation $\rightarrow$ *improve the description*
- How does the modification affect parameters of the amplitude, *e.g. positions of poles connected with the σ meson?*

  The original amplitude predicts a heavy and broad σ:

  $$m = 829 \pm 10 \text{ MeV and } \Gamma = 1108 \pm 22 \text{ MeV}$$

  which is inconsistent by many sigmas with results from the dispersion relations and parameters from Particle Data Group 2012:

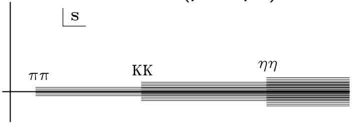
  $$m = 400\text{-}550 \text{ MeV and } \Gamma = 400\text{-}700 \text{ MeV}.$$

The three-channel $\pi\pi$ scattering in S wave

$J^P = 0^+$, $I = 0$ initial state

$$\pi\pi \longrightarrow \begin{array}{ll} \pi\pi & (1) \\ K\overline{K} & (2) \\ \eta\eta & (3) \end{array}$$

$s = (p_\pi + p_\pi)^2$

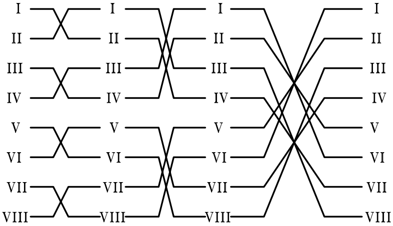


8 sheets of Riemann surface:

sign$(\mathrm{Im}\sqrt{s - s_i})$,

s_i, $i = 1, 2, 3$ thresholds

sheet I: $(+,+,+)$,
sheet II: $(-,+,+)$,
sheet III: $(-,-,+)$



Scalar isoscalar meson resonances: poles on unphysical sheets

Analytic continuation of the S-matrix elements

Process	I	II	III	IV	V	VI	VII	VIII
$1 \to 1$	S_{11}	$\frac{1}{S_{11}}$	$\frac{S_{22}}{D_{33}}$	$\frac{D_{33}}{S_{22}}$	$\frac{\det S}{D_{11}}$	$\frac{D_{11}}{\det S}$	$\frac{S_{33}}{D_{22}}$	$\frac{D_{22}}{S_{33}}$
$1 \to 2$	S_{12}	$\frac{iS_{12}}{S_{11}}$	$\frac{-S_{12}}{D_{33}}$	$\frac{iS_{12}}{S_{22}}$	$\frac{iD_{12}}{D_{11}}$	$\frac{-D_{12}}{\det S}$	$\frac{iD_{12}}{D_{22}}$	$\frac{D_{12}}{S_{33}}$
$2 \to 2$	S_{22}	$\frac{D_{33}}{S_{11}}$	$\frac{S_{11}}{D_{33}}$	$\frac{1}{S_{22}}$	$\frac{S_{33}}{D_{11}}$	$\frac{D_{22}}{\det S}$	$\frac{\det S}{D_{22}}$	$\frac{D_{11}}{S_{33}}$
$1 \to 3$	S_{13}	$\frac{iS_{13}}{S_{11}}$	$\frac{-iD_{13}}{D_{33}}$	$\frac{-D_{13}}{S_{22}}$	$\frac{-iD_{13}}{D_{11}}$	$\frac{D_{13}}{\det S}$	$\frac{-S_{13}}{D_{22}}$	$\frac{iS_{13}}{S_{33}}$
$2 \to 3$	S_{23}	$\frac{D_{23}}{S_{11}}$	$\frac{iD_{23}}{D_{33}}$	$\frac{iS_{23}}{S_{22}}$	$\frac{-S_{23}}{D_{11}}$	$\frac{-D_{23}}{\det S}$	$\frac{iD_{23}}{D_{22}}$	$\frac{iS_{23}}{S_{33}}$
$3 \to 3$	S_{33}	$\frac{D_{22}}{S_{11}}$	$\frac{\det S}{D_{33}}$	$\frac{D_{11}}{S_{22}}$	$\frac{S_{22}}{D_{11}}$	$\frac{D_{33}}{\det S}$	$\frac{S_{11}}{D_{22}}$	$\frac{1}{S_{33}}$

S_{ij} – matrix elements on the physical sheet

$D_{11} = S_{22}S_{33} - S_{23}^2$, $D_{22} = S_{11}S_{33} - S_{13}^2$, $D_{33} = S_{11}S_{22} - S_{12}^2$,

$D_{12} = S_{12}S_{33} - S_{13}S_{23}$, $D_{23} = S_{11}S_{23} - S_{12}S_{13}$

[D. Krupa, V.A. Meshcheryakov, Yu.S. Surovtsev, Nuovo Cimento A 109 (1996) 281]

Uniformization w plane

- a conformal map of the Riemann surface onto the w plane:

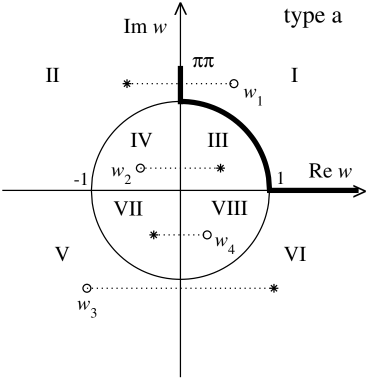
$$w = \frac{\sqrt{s-s_2}+\sqrt{s-s_3}}{\sqrt{s_3-s_2}}$$

  $\rightarrow$ the left-hand cut is not included
  [Surovtsev etal, Phys.Rev.D**81**(2010)016001]

- a necessary approximation (3 channels): the $\pi\pi$-threshold branching point is neglected keeping unitarity on the $\pi\pi$ cut – a model of the R. s.
  $\rightarrow$ the near-threshold data are not described precisely

- clusters of poles and zeros represent resonances,

  7 types of clusters: a, b,..g



cluster of type a ($f_0(500)$):

– a zero only in S_{11}

– only the pole on sheet II is not shifted $S_{11}^{\mathrm{II}} = 1/S_{11}^{\mathrm{I}}$

The scattering matrix

The background and resonant parts of the S-matrix are separated

$$S_{11} = S_{11}^{bgr} S_{11}^{res} = \frac{d_{bgr}(-k_1, k_2, k_3)}{d_{bgr}(k_1, k_2, k_3)} \frac{d_{res}^*(-w^*)}{d_{res}(w)}$$

The resonant part is generated by clusters of poles on uniformization plane

$$d_{res}(w) = w^{-\frac{M}{2}} \prod (w + w_i^*)$$

the product includes all zeros w_i of the chosen resonances.

The background part

$$d_{bgr}(k_1, k_2, k_3) = \exp \left[\mathrm{i} \sum_{j=1}^{3} \frac{k_j}{m_j} (\alpha_j + \mathrm{i}\, \beta_j) \right]$$

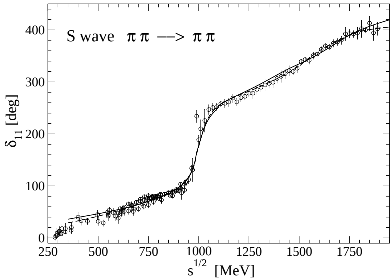
includes less important effects not taken explicitly into account in the formalism.

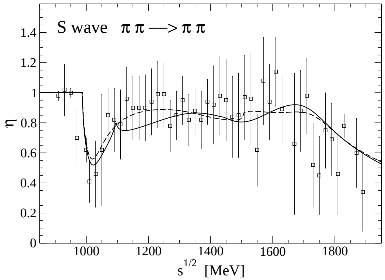
Resonance zeros w_i and background parameters were fitted to the inelasticity parameters and phase shifts in all assumed channels.

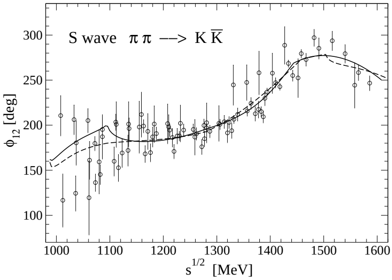
Phase shifts and inelasticities in the isoscalar scalar wave

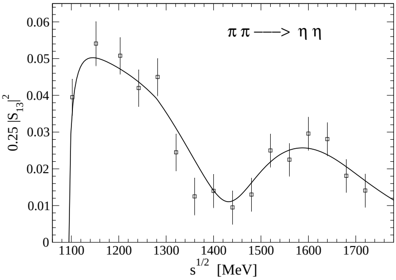
[Surovtsev et al, Phys. Rev. D **81** (2010) 016001]

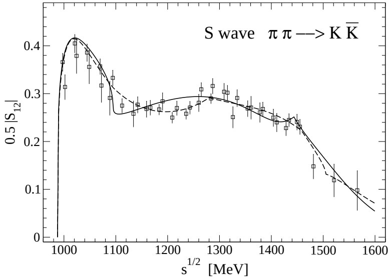
- two variants of the S-wave three-channel data analysis:
  $\pi\pi \longrightarrow \pi\pi$, $K\overline{K}$, $\eta\eta$ (solid line) and $\pi\pi$, $K\overline{K}$, $\eta\eta'$ (dashed line)
- five resonances considered: $f_0(500)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$, $f_0(1710)$
- the best fits: $\chi^2/n.d.f. = 1.24$ (variant I) and 1.11 (variant II)
  a good description of data in the energy range 400–1800 MeV
- the $f_0(500)$ pole on sheet II: $(616.5 - i\,554.)$ MeV $\rightarrow \sigma$ is too wide
  PDG 2014 values: $(400\text{–}550) - i(200\text{–}350)$ MeV

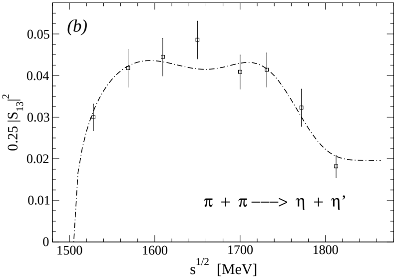





S wave π π ––> K $\overline{K}$
ϕ_{12} [deg]
$s^{1/2}$ [MeV]


π π ––> η η
$0.25\,|S_{13}|^2$
$s^{1/2}$ [MeV]

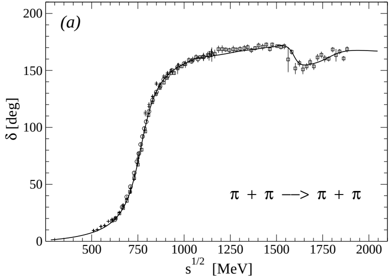

S wave π π ––> K $\overline{K}$
$0.5\,|S_{12}|$
$s^{1/2}$ [MeV]

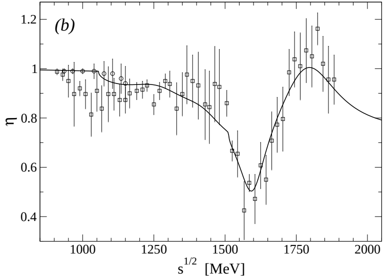

(b)
π + π ––> η + η’
$0.25\,|S_{13}|^2$
$s^{1/2}$ [MeV]

Phase shift and inelasticity in P wave

[Surovtsev et al, Phys. Rev. D **81** (2010) 016001]

- the vector isovector three-channel data analysis:
  $\pi\pi \longrightarrow \pi\pi,\ \rho 2\pi,\ \rho\sigma$
- five resonances considered: $\rho(770)$, $\rho(1250)$, $\rho(1450)$, $\rho(1600)$, $\rho(1800)$
- the best scenario (cluster types: a, e, b, b, c): $\chi^2/n.d.f. = 1.76$
- $\rho(770)$ pole, sheet II: $(766.2 - i\,72.6)$ MeV $\rightarrow$ $m = 769.8$, $\Gamma = 144.9$ MeV
  PDG 2014 values: $m = 775.26 \pm 0.25$, $\Gamma = 147.8 \pm 0.9$ MeV





Final-state interactions in decays of heavy mesons

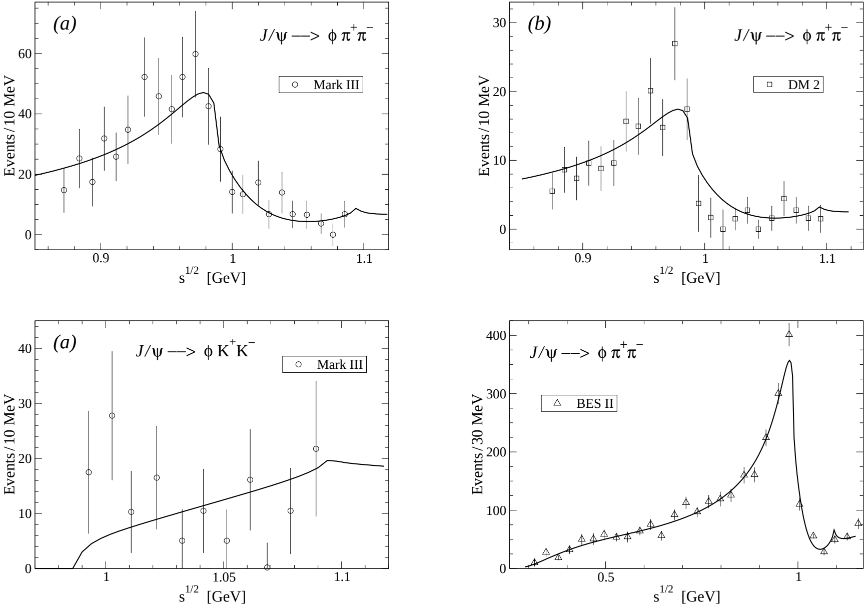
- combined analysis of $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$ and decays: $J/\psi \to \phi\pi\pi$, $\phi K\overline{K}$
  [Yu.S. Surovtsev etal, Phys.Rev.D**89**(2014)036010]
- the simplification: ϕ is a spectator
- the pseudoscalar meson pair is produced in S wave
- the three-channel scattering amplitudes T_{ij} $i,j = 1(\pi\pi),\ 2(K\overline{K})$ are used to describe interactions in the final state:

$$F(J/\psi \to \phi\pi\pi) = \sqrt{2/3}\,[c_1(s)T_{11} + c_2(s)T_{21}],$$

$$F(J/\psi \to \phi K\overline{K}) = \sqrt{1/2}\,[c_1(s)T_{12} + c_2(s)T_{22}],$$

where $c_1(s) = \gamma_{10} + \gamma_{11}s$ and $c_2(s) = \kappa_2/(s - \lambda_2) + \gamma_{20} + \gamma_{21}s$
[D. Morgan and M.R. Pennington, Phys.Rev.D**48**(1993)1185]

- results for the $f_0(500)$ pole on sheet II: $(515 - i466)$ MeV
  confirm our previous results on wide, $\Gamma \approx 900$ MeV, and heavy, $m \approx 700$ MeV, σ meson [Yu.S. Surovtsev etal, Phys.Rev.D**86**(2012)116002]


(a)
J/ψ —> φ π+π−
Mark III
Events/10 MeV
s1/2 [GeV]
(b)
J/ψ —> φ π+π−
DM 2
Events/10 MeV
s1/2 [GeV]
(a)
J/ψ —> φ K+K−
Mark III
Events/10 MeV
s1/2 [GeV]
J/ψ —> φ π+π−
BES II
Events/30 MeV
s1/2 [GeV]

Interference effects in $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$

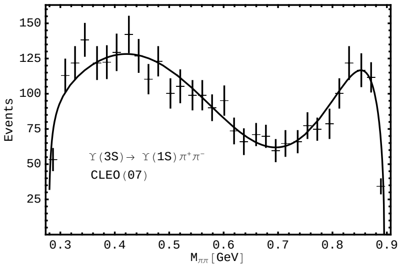
[Yu.S. Surovtsev etal, Phys.Rev.D**91**(2015)037901]

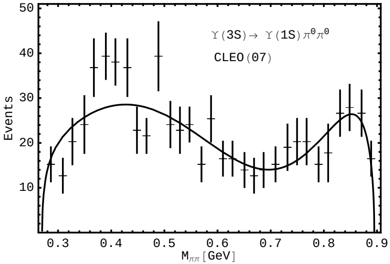
- the decay amplitudes

$$F_n(s) = (\rho_{n0} + \rho_{n1}\, s)\, T_{11} + (\omega_{n0} + \omega_{n1}\, s)\, T_{21},$$

  - T_{11} and T_{21} are the $\pi\pi$ scattering and $K\overline{K} \to \pi\pi$ amplitudes;
  - $n = 1, 2,$ and 3 denotes the considered decays $\Upsilon(2S) \to \Upsilon(1S)\pi\pi$, $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$, and $\Upsilon(3S) \to \Upsilon(2S)\pi\pi$

- a destructive interference between the $\pi\pi$ scattering and $K\overline{K} \to \pi\pi$ describes the two-hump structure in $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$





Modification of the amplitudes using the dispersion relations

[P. Bydžovský, R. Kamiński, V. Nazari, Phys. Rev. D **90** (2014) 116005]

- the multichannel amplitudes are not constrained by crossing symmetry

  – it can be partially included via the left-hand cut in uniformizing variable
  [Yu.S. Surovtsev etal, Phys.Rev.D**85**(2012)036002; **86**(2012)116002]

- the Roy-like dispersion relations can be used to impose crossing symmetry

  – relate the real parts of partial-wave amplitudes (here S and P waves) with a set of imaginary parts of other amplitudes (here S, P, D, and F);

  – provide a correct description of the low-energy $\pi\pi$ scattering;

  – here once subtracted dispersion relations, GKPY, are used.

- the modified S (i=0) and P (i=1) wave amplitudes are from [Surovtsev et al, Phys. Rev. D **81** (2010) 016001]

- the input S (i=2), D (i=0,2), and F (i=1) wave amplitudes are from [R. García-Martín et al, Phys. Rev. D **83** (2011) 074004]

The GKPY relations for S (l=0, i=0) and P (l=1, i=1) waves

- dispersion relations with one subtraction and crossing symmetry
  $\longrightarrow$ relations for the partial-wave projected amplitudes $f_l^i(s)$

$$\mathrm{Re} f_l^i(s) = ST_l^i + \sum_{i'=0}^{2} \sum_{l'=0}^{3} \mathrm{vp} \int_{4m_\pi^2}^{s'_{max}} ds' \, K_{ll'}^{ii'}(s,s') \, \mathrm{Im} f_{l'}^{i'}(s') \; + \; d_l^i(s)$$

ST_l^i, $K_{ll'}^{ii'}(s,s')$ and $d_l^i(s)$ are the subtracting, kernel and driving terms

- $\mathrm{Re}\, f_l^i(s)$ and $\mathrm{Im}\, f_{l'}^{i'}(s')$ are the output and input amplitudes
- Condition for crossing symmetry:

$$\mathrm{Re}\, f_l^i(s)^{out} - \mathrm{Re}\, f_l^i(s)^{in} = 0$$

[R. García-Martín et al, Phys. Rev. D **83** (2011) 074004]

[R. Kamiński, Phys. Rev. D **83** (2011) 076008]

Improvement of description near the $\pi\pi$ threshold

- due to neglecting the $\pi\pi$ branching point description of the phase shifts is not good near the $\pi\pi$ threshold
- re-definition of S- and P-wave amplitudes below a matching energy $\sqrt{s_{0l}}$ using the low-energy expansion ($\sqrt{s_1} < \sqrt{s} < \sqrt{s_{0l}} < \sqrt{s_2}$)

$$\mathrm{Re} f_l^i(s) = \frac{\sqrt{s}}{4k_1} \sin 2\delta_l^i = m\, k_1^{2l} \left[a_l^i + b_l^i\, k_1^2 + c_l^i\, k_1^4 + d_l^i\, k_1^6 + \mathcal{O}(k_1^8) \right]$$

- the scattering length and slope parameters are fixed

$a_0^0 = 0.22\, m_\pi^{-1}$, $b_0^0 = 0.278\, m_\pi^{-3}$, $a_1^1 = 0.0381\, m_\pi^{-3}$, $b_1^1 = 0.00523\, m_\pi^{-5}$

[R. García-Martín et al, Phys. Rev. D **83** (2011) 074004]

- c_l^i and d_l^i are calculated from the matching conditions for the phase shift and its first derivative at $\sqrt{s_{0l}}$
- the low-energy corrected amplitudes $\equiv$ extended amplitudes

Fitting the extended amplitudes to the GKPY relations and data

- the matching energies $\sqrt{s_{00}}$, $\sqrt{s_{01}}$,
  the parameters of $f_0(500)$, $\rho(770)$, $f_0(980)$, $f_0(1500)$,
  and the parameters of $\pi\pi$ background were fitted:

$$\chi^2 = \sum_{\alpha=1}^{2} \chi^2_{Data}(\alpha) + \sum_{\alpha=1}^{3} \chi^2_{DR}(\alpha)$$

$\alpha = 1, 2, 3$ denotes the S(i=0), P(i=1), and S(i=2) partial waves

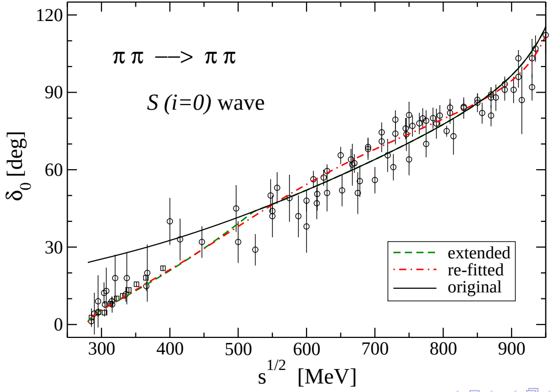
$$\chi^2_{Data}(\alpha) = \sum_{n=1}^{N^\alpha_\delta} \frac{(\delta^{exp}_n - \delta^{th}_n)^2}{(\Delta\delta^{exp}_n)^2} + \sum_{n=1}^{N^\alpha_\eta} \frac{(\eta^{exp}_n - \eta^{th}_n)^2}{(\Delta\eta^{exp}_n)^2}$$

$$\chi^2_{DR}(\alpha) = \sum_{n}^{N_{DR}} \left(\frac{\mathrm{Re} f^{out}_\alpha(s_n) - \mathrm{Re} f^{in}_\alpha(s_n)}{\Delta_{DR}} \right)^2$$

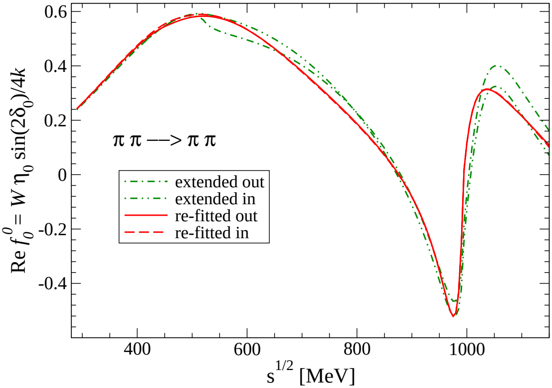
$\mathrm{Re} f^{out}_\alpha$ are calculated using the GKPY relations
$\mathrm{Re} f^{in}_\alpha$ are the input amplitudes

Results:

- $\chi^2/n.d.f.$ for the extended amplitudes: 2.12
  the re-fitted amplitudes: 1.14
- the matching energies: $\sqrt{s_{00}} = 382$ MeV and $\sqrt{s_{01}} = 635$ MeV
- the absolute values of the $\pi\pi$ background parameters are small
- the phase shifts for S(i=0) wave in the low energy region:



- consistency with dispersion relations
  - the difference between $(\mathrm{Re} f_l^i)^{out}$ and $(\mathrm{Re} f_l^i)^{in}$ shows a consistency of the amplitude with the GKPY relations and crossing symmetry
  - before fitting:
    "extended in" – the extended (low-energy corrected) amplitude
    "extended out" – calculated from GKPY with the extended amplitude
  - after fitting:
    "re-fitted in" – the re-fitted amplitude
    "re-fitted out" – calculated from GKPY with the re-fitted amplitude

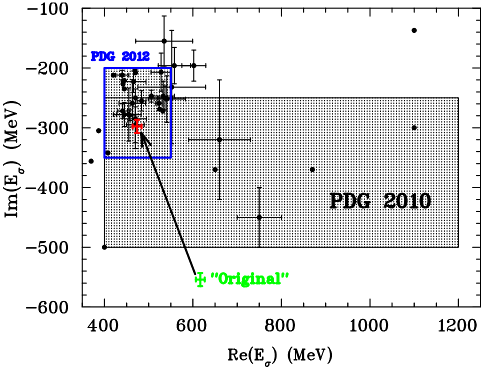


- poles on sheet II:

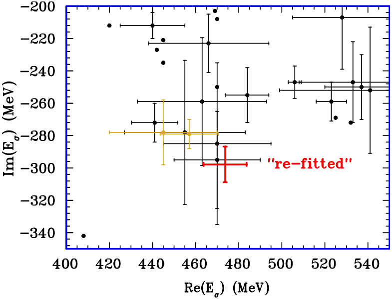
	original	re-fitted
$f_0(500)$	$616.5 - i554.0$	$455.9 - i295.4$
$f_0(980)$	$1009.2 - i31.3$	$998.8 - i23.4$
$f_0(1500)$	$1498.3 - i198.8$	$1441.7 - i164.1$
$\rho(770)$	$766.4 - i72.4$	$765.1 - i73.2$

mass and width (m, Γ):

re-fitted	PDG 2014
(999.1, 46.8)	(990±20, 40–100)
(1451., 328.2)	(1505±6, 109±7)
(768.6, 146.4)	(775.26±0.25,147.8±0.9)



ranges as given by PDG 2012



The σ meson in the single- and two-channel analysis

[Yu.S. Surovtsev etal, Phys.Rev.D**86**(2012)116002, J.Phys.G**41**(2014)025006]

- two-channel formalism: $\pi\pi \longrightarrow \pi\pi,\ K\overline{K}$
  – correct description in the $\pi\pi$ threshold region (the scattering length)
  – the left-hand cut included in the uniformizing variable
- single- ($\pi\pi$ data only) and two-channel analysis to study the $f_0(500)$ pole
- in single-channel analysis the $f_0(500)$ pole is $(448 - i267)$ MeV
  – PDG2012 [Caprini,Colangelo,Leutwyler]: $(441^{+16}_{-8} - i\,272^{+9}_{-13})$ MeV

– the scattering length:

$0.222 \pm 0.008\ m_\pi^{-1}$

$0.220 \pm 0.005\ m_\pi^{-1}$

[Colangelo etal]

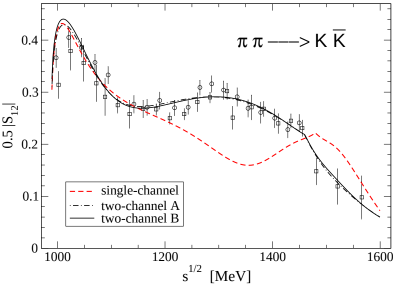
– the slope parameter:

$b_0^0 = 0.295 \pm 0.021\ m_\pi^{-3}$

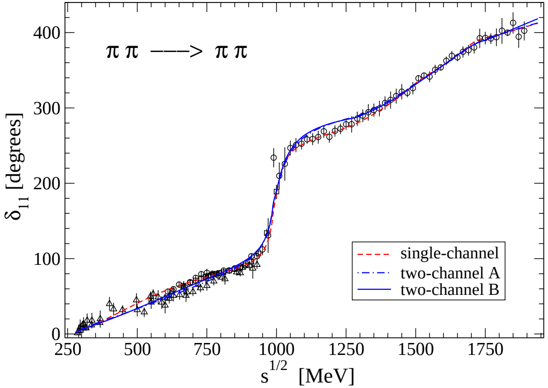
$b_0^0 = 0.278 \pm 0.005\ m_\pi^{-3}$

[García-Martín etal]

– BUT a bad description of $\pi\pi \longrightarrow K\overline{K}$:



- in two-channel analysis – two solutions:
  - $f_0(500)$ pole: $(517 - i394)$ MeV (A), $(551 - i502)$ MeV (B)
    (three-channel analysis $(650 - i343)$ MeV (A), $(568 - i519)$ MeV (B))
  - the scattering length $[m_\pi^{-1}]$: 0.230 ± 0.004 (A), 0.282 ± 0.003 (B)
  - the slope parameter $[m_\pi^{-3}]$: 0.210 ± 0.010 (A) and 0.201 ± 0.007 (B)



Conclusions

- multichannel analyses of the $\pi\pi$ scattering data were done
  – the resonance part of the S-matrix was constructed
  to be analytic and unitary and using a uniformizing variable
  – the background part comes out in the analyses as a small correction
- crossing symmetry was imposed on the multichannel amplitudes
  using the once-subtracted GKPY dispersion relations
  – the $f_0(500)$-pole position on sheet II is strongly affected
- a broad and heavy $f_0(500)$ seems to be a peculiar feature of our multichannel analysis $\rightarrow$ imposing crossing symmetry or in the single-channel analysis the $f_0(500)$-pole position is consistent with values preferred in PDG tables
- the final-state interactions in specific heavy-meson decays,
  $J/\psi, \psi(2S), \Upsilon(nS)$ decaying into V$\pi\pi$, V$K\overline{K}$, or V$\eta\eta$,
  can be described by the multichannel amplitudes
  – the two-humped structure in the di-pion mass distribution in
  $\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi$ can be explained as the coupled-channel effect