The anomalous magnetic moment of the muon from lattice QCD

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The magnetic moment of the muon

- Magnetic moment of a muon: $\vec{\mu} = g \frac{Qe}{2m} \vec{s}$.
- Prediction from free Dirac theory: $g = 2$.
- In interacting quantum field theory: $g$ gets corrections.

Anomalous magnetic moment of muon: $a_\mu = \frac{g-2}{2}$.

$a_\mu^{SM} = a_\mu^{QED} + a_\mu^{EW} + a_\mu^{had}$; Only hadronic contribution can be calculated on lattice.

Sub-ppm measurement is sensitive to corrections beyond the Standard Model.
Current status of muon g-2

- BNL E821 (2001) : $a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$ (0.54 ppm).

Overview of measurement technique

- Muons fed into a storage ring having precise $\vec{B}$
- Muons continually decay into positrons and neutrinos

Weak decay : Positron direction follows muon spin
- Muon’s spin precession obtained from the time distribution of decay positrons

- $g$-2 discrepancy of $3.6\sigma$ between the standard model and the experiment: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 25(9) \times 10^{-10}$.

- Discovery of new physics (if $5\sigma$): Supersymmetric particles, dark photon, multi-Higgs model, extra dimensions ??
Next goals

- Future experiment: Fermilab E989 aims to reduce experimental error in $g-2$ to $\pm 1.6 \times 10^{-10}$ (four-fold improvement).
- Theoretical goal to get SM uncertainty of $3.5 \times 10^{-10}$ (.3-.4%) before results from Fermilab E989 (2017/18).
- $a_{\mu}^{QED}$ calculated phenomenologically with same precision as in experiment; $a_{\mu}^{EW}$ very small.
- The current theoretical uncertainty dominated by that from the lowest order hadronic vacuum polarization (HVP) contribution (connected + disconnected).

HVP contribution from dispersion relation + cross section for $e^+ e^- \rightarrow$ hadrons around $700 \times 10^{-10}$ with a 1% error.

- Our aim: To achieve uncertainty of less than 1% in $a_{\mu}^{HVP,lo}$ using first principle lattice QCD calculations.
Our method for calculating HVP from full lattice QCD

\[ a_{\mu, \text{HVP}}^{(f)} = \frac{\alpha}{\pi} \int dq^2 f(q^2) (4\pi\alpha Q_f^2) \hat{\Pi}_f(q^2). \]

(Lautrup,’72, T. Blum,’02)

- Integrand strongly peaked at 
  \[ q^2 \sim m_{\mu}^2 / 4 \sim 0.003 \text{GeV}^2. \]
- Extrapolating from higher values of \( q^2 \) leads to model uncertainties.


- HVP contribution expressed as a Taylor series of small number of derivatives of the vacuum polarization function \( \hat{\Pi} \) evaluated at \( q^2 = 0 \).
- Defining \( \hat{\Pi}(q^2) = \sum_{j=1}^{\infty} q^{2j}\Pi_j \).
- And then \( \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!} \) where \( G_{2n} = (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2) \bigg|_{q^2=0} \).
- The derivatives at \( q^2 = 0 \) of \( \hat{\Pi} \) readily and accurately given by \( t^n \times \text{vector meson correlators} \) (time moments) for \( n = 4,6,8,10 \):
  \[ G_{2n} = a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_V^2 \langle j^i(\vec{x}, t) j^i(0) \rangle \]
  \[ = (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2) \bigg|_{q^2=0}. \]

- Using Padé approximants instead of Taylor approximation allows us to deal with high momenta.
Calculating meson correlator (connected) on lattice

- Continuum QCD action with gluons and quarks; QCD vacuum with gluonic background and sea quarks, but no valence quarks.
- Expectation values of observables as a path integral over fields: Infinite number of integrals.
- 4-dimensional space-time lattice: Finite number of integrals.
- Discretise the QCD lagrangian: Different formalisms.
- Take an ensemble of lattice gauge configurations (few hundreds) as QCD vacuum.
- A meson generated with a Dirac bilinear and allowed to propagate resulting in a 2-point correlator (Quark-line connected and disconnected).
- Quark-line connected meson correlator: \( G_{\text{conn}}(x_0 - y_0) = \langle (\sum \bar{x} \, Tr[\gamma_k D^{-1}(x, y) \gamma_k D^{-1}(y, x)]) \rangle. \)

\( D^{-1}(x, y) \) is the inverse of lattice Dirac matrix or propagator
- Calculate renormalisation factor \( Z \) to match lattice and continuum currents.
- Averaging over configurations extract meson properties.
- At the end, finite volume extrapolation, continuum extrapolation, extrapolation to chiral limit.
Lattice Configurations and Parameters

- Lattice configurations by MILC using Highly Improved Staggered Quark (HISQ) formalism:
  - Small discretisation error.
  - Light (up/down), strange, charm dynamical sea quarks included.
  - Valence strange and charm quark masses precisely tuned, but valence light quark mass taken to be same as in the sea.
  - Large box size: 5.6 fm on the finest lattices.
  - Multiple volumes on a particular light sea quark mass and a particular lattice spacing.
  - 1000 gluon configurations: High statistics.

![Graph](image.png)

**Graph Details:**
- MILC HISQ, 2+1+1
- Multiple volumes
- $m_{\pi}^2 / \text{GeV}^2$ vs. $a^2 / \text{fm}^2$
Our correlators for vector meson made of strange valence quarks: $\phi$

- Precise determination of mass and decay constant (annihilation amplitude) of $s\bar{s}$ vector meson $\phi$ from large time behaviour of correlator.
- For $a_{\mu}^{HVP}$ we are interested in small time behaviour.
- Renormalization factor $Z_{V,s\bar{s}}$ calculated precisely with 0.1% uncertainty completely non-perturbatively (B.Chakraborty et al., PoS LATTICE2013, 309 (2013)).
- Finite volume effect is negligible.
- Our result of $f_{\phi}$ in the continuum limit agree with the experimental result related to $\Gamma(\phi \rightarrow e^+ e^-)$.
- Disconnected diagrams are not included; but very small contribution expected.
Our results: Connected contributions to $a_{\mu}^s$

Our final result for the connected contribution for $s$ quarks to $g - 2$ is:

$$a_{\mu}^s = 53.41(59) \times 10^{-10}$$

(B. Chakraborty et al., Phys. Rev. D 89, 114501);

with $m_{u/d}^{\text{lat}} = m_{u/d}^{\text{phys}}$, after extrapolation to $a = 0$.

The lower blue points represent $m_{u/d}^{\text{lat}} = m_s/5$ and the upper red points represent $m_{u/d}^{\text{lat}} = m_{u/d}^{\text{phys}}$.

Precision obtained at 1.1% level in $a_{\mu}^s$, the best so far.

Major part of the uncertainty comes from lattice spacing error (1%), can be improved if better precision required.
Comparison of our results for $a_{\mu}^s$ with other results

Comparison of our results for $a_{\mu}^s$ with results from another lattice groups.

<table>
<thead>
<tr>
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<th>HPQCD</th>
<th>ETMC (preliminary)</th>
<th>RBC/UKQCD (preliminary)</th>
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<tbody>
<tr>
<td>$a_{\mu}^s$, $\times 10^{-10}$</td>
<td>53.41(59)</td>
<td>53(3)</td>
<td>52.4(21)</td>
</tr>
</tbody>
</table>

![Graph showing comparison of results](image-url)

- ETMC final
- ETMC results
- HPQCD final
- HPQCD results
Preliminary results of the connected contribution to $a_{\mu}^{u/d}$

- Most significant contribution, challenging due to poor Signal-to-noise ratio at larger timeslices.
- Time-moments calculated from correlators reconstructed from the best fit parameters improves precision.
- Other techniques used namely Gaussian smearing etc.
- $a_{\mu}^{u/d}$ shows significant finite volume effects compared to $a_{\mu}^{s}$, needs finite volume extrapolation.

Indicative (preliminary) result: $a_{\mu}^{u/d} = 631(9) \times 10^{-10}$ ($\sim 1.4\%$ uncertainty)

Need to look more carefully at finite volume dependence.

ETMC preliminary from full QCD (1311.3885) : $a_{\mu}^{u/d} = 567(11) \times e^{-10}$ (2%), no finite volume effect seen.
Preliminary lattice results for full four flavour estimation of $a_{\mu}^{HVP,lo}$

- Lattice can provide full four-flavor estimation.
- Our (HPQCD) preliminary estimation:

$$a_{\mu}^{HVP,lo} = a_{\mu}^{\text{light}} + a_{\mu}^{s} + a_{\mu}^{c} + a_{\mu}^{b}$$

$$= (596(7) + 53.41(59) + 14.42(39) + 0.27(4)) \times 10^{-10}$$

$$\sim 699(9) \times 10^{-10} (\sim 1.3\%)$$

- ETMC preliminary number (1311.3885): $a_{\mu}^{HVP,lo} = 674(28) \times 10^{-10} (\sim 4\%)$
Accuracy still doesn’t match with phenomenological result.
Still result from QCD only without experimental input or model assumption.
Our (HPQCD) result hints at no significant disconnected piece (still needs consistency check!)
Leading order disconnected contribution to $a_{\mu}^{HVP}$

- Expected to be very small, may be a significant source of systematic error, in partially quenched ChPT as large as -10% of the connected one.

- On lattice we calculate:

$$G_{\text{disc}}(x_0 - y_0) = -Z_V \langle \left( \sum_{\vec{x}} \text{Tr}[\gamma_k D^{-1}(x, x)] \right) \left( \sum_{\vec{y}} \text{Tr}[\gamma_k D^{-1}(y, y)] \right) \rangle$$

- All-to-all propagator.
- Poor signal-to noise ratio : challenging.
- We aim to calculate a conservative upper bound for a systematic error from neglecting the disconnected contribution.
- Currently existing upper bound : 4-5% (Mainz, 1411.7592).
The Hadronic light-by-light contribution

- Small compared to HVP, but important for getting less than 1% theoretical errors.

- Blob: All possible hadronic states.
- Model estimates: about \((10-12) \times 10^{-10}\) with a 25-40% uncertainty (0901.0306).
- Lattice calculation involves four-point functions: direct calculations not possible.
- First attempt on lattice includes electromagnetism in simulations (Tom Blum and collaborators, 1407.2923).
- Problem reduces to difference of 3-point functions.
- Calculated for unphysical quark and muon masses: Still looks encouraging.
Conclusion

To summarize:

- \( a_{\mu}^{HVP,lo,conn} = 699(9) \times 10^{-10} (\sim 1.1\%) \) (HPQCD preliminary using HISQ), work in progress on u/d quark piece, finite volume effects; First lattice result at physical quark and pion masses and to take care of finite volume effects.

- A conservative upperbound for not including \( a_{\mu}^{HVP,lo,disc} \) 4-5% (Mainz).

- \( a_{\mu}^{HLbL,conn} \) calculated with unphysical quark and muon mass which looks promising (Tom Blum and collaborators).

Future:

- Ensemble size of 10,000 (on a~ 0.15fm, 0.12fm lattices) being made by MILC to be used by MILC and HPQCD for \( a_{\mu}^{light} \) and \( a_{\mu}^{HVP,lo,disc} \) calculations, 3-fold improvement in the uncertainty expected.

- Our (HPQCD) work in progress; highly optimistic about obtaining uncertainty in \( a_{\mu}^{HVP,lo} \) at less than 1% level by 2017.

- HLbL (connected) on physical quark and muon masses: Finite volume correction and other systematic improvements needed. HLbL disconnected piece tiny, but to be calculated.

- Situation seems to be under control!
For spatial currents at zero spatial momentum:
\[ \Pi_{ii}(q^2) = q^2 \Pi(q^2) = a^4 \sum_t e^{iqt} \sum_{\vec{x}} \langle j_i(\vec{x}, t)j_i(0) \rangle \] with \( q \) the Euclidean energy.

We need the renormalized vacuum polarization function:
\[ \hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0). \]

Time moments of the correlator give the derivatives at \( q^2 = 0 \) of \( \hat{\Pi} \) readily and accurately.

\[
G_{2n} \equiv a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_\nu^2 \langle j_i(\vec{x}, t)j_i(0) \rangle = (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2) \bigg|_{q^2=0}.
\]

Defining \( \hat{\Pi}(q^2) = \sum_{j=1}^{\infty} q^{2j} \Pi_j \).

Then \( \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!} \).
We replace \( \hat{\Pi}(q^2) \) with its \([n, n]\) and \([n, n-1]\) Padé approximants derived from the \( \Pi_j \).

The \( q^2 \) integral \( \int dq^2 f(q^2)(4\pi\alpha Q_f^2)\hat{\Pi}_f(q^2) \) is done numerically.

Using Padé approximants instead of Taylor approximation allows us to deal with high momenta, say \( q \geq 1 \) GeV which is important when better than 1% precision is desired.

The precision of different Padé approximants tested by comparing with the exact results for \( a_\mu \) from the perturbation theory.

The Padés converge exponentially quickly to the correct result, achieving better than 1% precision after only two terms are included.
In calculating $G_{2n}$ the error due to the finite $T$ is exponentially suppressed and is negligible near the edge of the lattice at $T/2$ contributing to higher order moments only.

Thus, the finite $T$ effect becomes important for higher order moments, which we do not need in our present calculation of $a_s^\mu$.

The finite $T$ approximation gives an error of $0.002\%$ in our analysis for configuration set 10.

To test the power of Padé approximants, we compare the precision of different $[n,n]$ and $[n,n-1]$ approximants with exact result for the one-loop quark vacuum polarization function from perturbation theory.

The quark mass is set equal to the kaon mass so that the Taylor expansion has the same radius of convergence as the physical $s$-quark vacuum polarization.

The Padés converge exponentially quickly to the correct result, achieving better than $1\%$ precision after only two terms are included.

The finite precision of exact moments from a simulation limits the precision of the final results for...