Latest developments in anisotropic hydrodynamics

Outline

- Second order viscous hydrodynamics
- Expansion around an anisotropic background
- Leading order in anisotropic hydrodynamics
- Extension of the model

Excited QCD 2015

Leonardo Tinti
Motivations

Hydrodynamic modeling of heavy ion collisions (small viscosity)

Large gradients → viscous corrections

Strong longitudinal expansion,
pressure anisotropy (also AdS/CFT)

Large momentum anisotropy from microscopic models (pQCD, CGC)
Hydrodynamics

\[ \partial_\mu T^{\mu\nu} = 0 \]

\[ T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle \]

Equilibrium?

\[ T^{00}, T^{0i} \quad \Leftrightarrow \quad \varepsilon, u^\mu \]

Gradient expansion

\[ \pi^{\mu\nu} \simeq 2\eta \sigma^{\mu\nu} + \cdots \]
Hydrodynamics

Relativistic Boltzmann equation:

\[ p^\mu \partial_\mu f(x, p) = C[f] \]

First moment:

\[ \int dP \, p^\mu p^\nu \partial_\mu f = \partial_\mu T^{\mu\nu} = \int dP \, p^\nu C[f] \left[ = 0 \right] \]

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From kinetic theory to hydrodynamics

Ansatz for the relativistic Boltzmann distribution

Perfect fluid:

\[ f \approx f_{\text{eq.}} = k \exp \left[ -\frac{p \cdot U(x)}{T(x)} \right] \]

\[
T^{\mu\nu} = k \int dP \ p^{\mu} p^{\nu} \exp \left[ -\frac{p \cdot U}{T} \right] = (\varepsilon + P) U^{\mu} U^{\nu} - g^{\mu\nu} P
\]

\[
T_{\text{L.R.F.}} = \begin{pmatrix}
\varepsilon & 0 & 0 & 0 & 0 \\
0 & P & 0 & 0 & 0 \\
0 & 0 & P & 0 & 0 \\
0 & 0 & 0 & P & 0 \\
0 & 0 & 0 & 0 & P
\end{pmatrix}
\]
Dissipative hydrodynamics

\[ f = f_{eq.} + \delta f \quad \Rightarrow \quad T^{\mu\nu} = T_{eq.}^{\mu\nu} + \delta T^{\mu\nu} \]

\( \delta f \Rightarrow \delta T^{\mu\nu} \) treated as, small, perturbations

Landau frame, massless particles

\[ \delta T^{\mu\nu} = \pi^{\mu\nu} \]

\[ U^{\mu} \pi_{\mu\nu} = 0 \quad g_{\mu\nu} \pi^{\mu\nu} = 0 \]

Four equations, five more degrees of freedom!
Entropy current and entropy source

Obtaining the remaining equations from the second principle of thermodynamics

\[
S^\mu = S^\mu_{eq.} + \delta S^\mu \approx \frac{p}{T} U^\mu + \frac{1}{T} T^{\mu\nu} U_\nu - \frac{\tau_\pi}{4\eta T} \pi^{\alpha\beta} \pi_{\alpha\beta} U^\mu
\]

\[
\partial_\mu S^\mu \geq 0
\]

\[
\partial_\mu S^\mu \approx \frac{1}{T} \pi^{\mu\nu} \left[ \sigma_{\mu\nu} - \frac{\tau_\pi}{2\eta} \Delta^\alpha_\mu \Delta^\beta_\nu D\pi_{\alpha\beta} - \frac{1}{2} T \pi_{\mu\nu} \partial \cdot \left( \frac{\tau_\pi}{2\eta T} U \right) \right]
\]

\[
\pi^{\mu\nu} \pi_{\mu\nu} \geq 0
\]

\[
\tau_\pi \Delta^\alpha_\mu \Delta^\beta_\nu D\pi_{\alpha\beta} + \pi_{\mu\nu} = 2\eta \sigma_{\mu\nu} - \pi_{\mu\nu} T \eta \partial \cdot \left( \frac{\tau_\pi}{2\eta T} U \right)
\]
Example, Bjorken flow

0+1 dimensions: boost invariant in the longitudinal direction, homogeneous in the transverse plane

\[ U = (\cosh \eta_\parallel, 0, 0, \sinh \eta_\parallel) \]
\[ \eta_\parallel = \frac{1}{2} \ln \left( \frac{t + z}{t - z} \right) \]
\[ \tau = \sqrt{t^2 - z^2} \]

Gradients of the four velocity are proportional to \( 1/\tau \)

In the Navier-Stokes limit

\[ \pi_{\mu\nu} \approx 2\eta \sigma_{\mu\nu} \]

therefore

\[ \frac{P_{||}}{P_{\perp}} = \frac{P_{\text{eq.}} + \pi_{ZZ}}{P_{\text{eq.}} + \pi_{XX}} \approx \frac{3T\tau - 16\eta}{3T\tau + 8\eta} \]
Anisotropic hydrodynamics

Reorganization of the hydrodynamic expansion

\[ f = f_{\text{eq.}} + \delta f \]

around an anisotropic background instead of the local equilibrium

\[ f = f_{\text{aniso.}} + \delta \tilde{f} \]

“Romatschke-Strickland” form:

\[ f_{\text{aniso.}} = k \exp \left[ -\sqrt{ \left( p \cdot U(x) \right)^2 + \xi(x) \left( p \cdot Z(x) \right)^2 } \right] \left( \frac{\Lambda(x)}{x} \right) \]
0+1 dimensions

Boost invariant in the longitudinal direction, homogeneous in the transverse plane

\[ T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_\perp & 0 & 0 \\ 0 & 0 & P_\perp & 0 \\ 0 & 0 & 0 & P_\parallel \end{pmatrix} \]

No negative pressure already at the leading order

\[ \left( P_\perp, P_\parallel \right) = \int dP \left( \frac{1}{2} p_\perp^2, p_\parallel^2 \right) f_{RS} > 0 \]

Positive by construction

Exact solution! Test for viscous and anisotropic hydrodynamics
Radial flow

1+1 dimensions, boost invariance in the longitudinal direction, rotation invariance in the transverse plane

Four-velocity

\[
U = \gamma \begin{pmatrix}
1 \\
v_x \\
v_y \\
v_z
\end{pmatrix} = \begin{pmatrix}
\cosh \theta_\perp \cosh \eta_\parallel \\
\sinh \theta_\perp \cos \phi \\
\sinh \theta_\perp \sin \phi \\
\cosh \theta_\perp \sinh \eta_\parallel
\end{pmatrix}
\]

Another unknown function

\[
\tanh \theta_\perp (\tau, r) = \cosh \eta_\parallel \sqrt{v_x^2 + v_y^2}
\]

Orthonormal basis

\[
X = \begin{pmatrix}
\sinh \theta_\perp \cosh \eta_\parallel \\
\cosh \theta_\perp \cos \phi \\
\cosh \theta_\perp \sin \phi \\
\sinh \theta_\perp \sinh \eta_\parallel
\end{pmatrix} \quad Y = \begin{pmatrix}
0 \\
- \sin \phi \\
\cos \phi \\
0
\end{pmatrix} \quad Z = \begin{pmatrix}
\sinh \eta_\parallel \\
0 \\
0 \\
\cosh \eta_\parallel
\end{pmatrix}
\]

*W Florkowski and R Ryblewski, Phys. Rev. C 85, 044902 (2012)*
Improving the starting point of the expansion

For a conformal system

\[ f_{\text{aniso.}} = k \exp \left[ -\frac{\sqrt{(1+\xi_X)(p \cdot X)^2 + (1+\xi_Y)(p \cdot Y)^2 + (1+\xi_Z)(p \cdot Z)^2}}{\lambda(x)} \right] \]

\[ \sum_I \xi_I = 0 \]

Pressure not isotropic in the transverse plane

\[
T_{\text{L.R.F.}} = \begin{pmatrix}
\varepsilon & 0 & 0 & 0 \\
0 & P_X & 0 & 0 \\
0 & 0 & P_Y & 0 \\
0 & 0 & 0 & P_Z
\end{pmatrix}
\]

Still two trivial equations from the local four-momentum conservation and four unknown scalar function: \( \theta_\perp, \lambda \) and two independent anisotropy parameter

\[ \text{L. Tinti and W Florkowski, Phys. Rev. C 89, 034907 (2014)} \]
How to match with second order hydrodynamics

Using a close to equilibrium expansion

\[ f \approx f_{eq} \left( 1 + \frac{\lambda - T}{T^2} (p \cdot U) - \frac{\xi_X (p \cdot X)^2 + \xi_Y (p \cdot Y)^2 + \xi_Z (p \cdot Z)^2}{2T(p \cdot U)} \right) \]

We can find the pressure corrections

\[ T_{\mu\nu} = \int dP p^\mu p^\nu f \]

\[ \pi^{\mu\nu} = \sum_I \pi_I I^\mu I^\nu \]

\[ T_{\mu\nu} \approx T_{eq} - \frac{32\pi}{5} k T^4 \left( \xi_X X^\mu X^\nu + \xi_Y Y^\mu Y^\nu + \xi_Z Z^\mu Z^\nu \right) \]

We can verify the matching if the equations for the anisotropy parameters correspond to the second order hydrodynamics equations for the pressure corrections

\[ \xi_I \approx - \frac{15}{4} \frac{\pi_I}{\varepsilon} \]
The remaining equations

Looking at the second moment of the Boltzmann equation, in relaxation time approximation, to close the system.

\[ p^\lambda \partial_\lambda f = (p \cdot U) \frac{f_{eq.} - f}{\tau_{eq.}} \Rightarrow \int dP \ p^\mu p^\nu p^\lambda \partial_\lambda f = \frac{U_\lambda}{\tau_{eq.}} \int dP \ (p^\mu p^\nu p^\lambda f_{eq.} - p^\mu p^\nu p^\lambda f) \]

Many of the projections of these equations along the vectors of the basis are trivial, but there is a useful combination of the remaining ones.

\[ \partial_\lambda \Theta^{\lambda \mu \nu} = \frac{1}{\tau_{eq.}} \left( U_\lambda \Theta^{\lambda \mu \nu}_{eq.} - U_\lambda \Theta^{\lambda \mu \nu} \right) \]

\[ \Theta^{\lambda \mu \nu} = \int dP \ p^\lambda p^\mu p^\nu f \quad \Theta^{\lambda \mu \nu}_{eq.} = \int dP \ p^\lambda p^\mu p^\nu f_{eq.} \]

\[ \frac{D \xi_I}{1 + \xi_I} + 2 \sigma_I + \frac{\xi_I}{\tau_{eq.}} \left( \frac{T}{\lambda} \right)^5 \sqrt{\prod_j (1 + \xi_j)} - \frac{1}{3} \sum_j \frac{D \xi_j}{1 + \xi_j} = 0 \]

Only two of the three equations are independent, if two of them are fulfilled all of them are
Small anisotropy limit

\[ D \xi_I + 2\sigma_I + \frac{\xi_I}{\tau_{eq.}} \approx 0 \]

Using the first moment equations and after some calculations

\[ \tau_{eq.} D \pi_I + \pi_I = 2 \left( \frac{4}{15} \varepsilon \tau_{eq.} \right) \sigma_I - \pi_I \left[ \frac{4}{15} \varepsilon \tau_{eq.} T \partial \cdot \left( \frac{15}{8\varepsilon T} U \right) \right] \]

Consistent with Israel-Stewart

\[ \tau_\pi D \pi_I + \pi_I = 2\eta \sigma_I - \pi_I \left[ \eta T \partial \chi \left( \frac{\tau_\pi}{2\eta T} U^\lambda \right) \right] \]

provided

\[ \tau_\pi = \tau_{eq.} \quad \eta = \frac{4}{15} \varepsilon \tau_{eq.} \Rightarrow \tau_\pi = \frac{5\eta}{T} \]
Important even without radial expansion


Much better agreement with the exact solution

So far so good, but...

Bulk degree of freedom  (M Nopoush, R Ryblewski, M Strickland, Phys. Rev. C 90, 014908 (2014))

Gubser Flow  (M Nopoush, R Ryblewski, M Strickland, Phys. Rev. D 91, 045007 (2015))
(3+1)-dimensional framework

No symmetry constraints from boost invariance or cylindrical symmetry

Generalized “Romatschke-Strickland” form

\[ f = k \exp \left( -\frac{1}{\lambda} \sqrt{p_\mu \Xi_{\mu\nu} p_\nu} \right) \]

Geometric decomposition

\[ \Xi_{\mu\nu} = U_\mu U_\nu + \xi_{\mu\nu} - \phi \Delta_{\mu\nu} \]

Matching with second order viscous hydrodynamics (Israel-Stewart??)

L. Tinti, arXiv:1411.7268
(3+1)-dimensional framework

Geometric argument to choose the necessary equations

We need higher moments of the Boltzmann equation

\[ \Theta^{\lambda\mu\nu} = \int dP \; p^\lambda p^\mu p^\nu f = \Theta_U U^\lambda U^\mu U^\nu + U^\lambda \Theta^{\mu\nu} + U^\mu \Theta^{\nu\lambda} + U^\nu \Theta^{\lambda\mu} \]

\[ \Theta^{\mu\nu} = U_\alpha \Delta^\mu_\beta \Delta^\nu_\gamma \Theta^{\alpha\beta\gamma} \]

“Shear equation”

\[ D\Theta^{\langle\mu\nu\rangle} + \frac{5}{3} \theta \Theta^{\langle\mu\nu\rangle} + 2\Theta^{\langle\mu\sigma\nu\rangle\lambda} - 2\Theta^{\langle\mu\omega\nu\rangle\lambda} = -\frac{1}{\tau_{eq.}} \Theta^{\langle\mu\nu\rangle} \]
(3+1)-dimensional framework

Close-to-equilibrium limit

\[ \Theta^{\mu\nu} \simeq g(m, T) \pi^{\mu\nu} \quad \Theta_{\text{tr.}} \simeq \Theta_{\text{eq.}}^{\text{tr.}} + h_{\text{tr.}}(m, T) \Pi \]

\[ n \simeq n_{\text{eq.}} - h_0(m, T) \Pi \]

\[ D\pi^{\mu\nu} + \frac{1}{\tau_{\pi}} \pi^{\mu\nu} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi^{\mu\nu}_{\lambda} \omega^{\nu}_{\lambda} - \tau_{\pi\pi} \pi^{\mu\nu}_{\lambda} \sigma^{\nu}_{\lambda} \]

\[ -\delta_{\pi\pi} \pi^{\mu\nu} \theta + \lambda_{\pi\Pi} \Pi \pi^{\mu\nu} \]

\[ D\Pi + \frac{1}{\tau_{\Pi}} \Pi = -\beta_{\Pi} \theta - \delta_{\Pi\Pi} \Pi \theta + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu} \]
Summary & outlook

- Anisotropic hydrodynamics is a reorganization of the hydrodynamic expansion, around an anisotropic distribution.

- Pressure anisotropies already at the leading order. Very good approximation in the exactly solvable 0+1 case.

- Generalized ansatz for the leading order. Well defined in the early stages and consistent with second order viscous hydrodynamics close to equilibrium, providing bulk-shear couplings.

- Next: Test with the exact solutions of the Boltzmann equation (WIP)?