From singular monopoles to framed BPS states

Dieter Van den Bleeken
Boğaziçi University

Based on work with

- Greg Moore (Rutgers University)
- Andy Royston (Texas A&M)

arXiv:1404.5616 , arXiv:1404.7158 ,
work in progress

Yang-Mills-Higgs Theory

A, X: adjoint valued for some compact simple Lie group G

BPS equation :

$$B = \mathsf{D}X$$

- First order
- Non-linear
- Implies E.O.M.
- General closed form solution not known

Yang-Mills-Higgs Theory

A, X: adjoint valued for some compact simple Lie group G

BPS equation :

$$B = \mathsf{D}X$$

For $SU(2)$, 't Hooft-Polyakov Ansatz

$$X = \frac{1}{2}h(r)H \ , \qquad \begin{pmatrix} H, E_{\pm} \text{ Chevalley basis} \\ r, \theta, \phi \text{ spherical coordinates} \end{pmatrix}$$

$$A = \frac{1}{2}(1 - \cos\theta)\mathsf{d}\phi H + \frac{1}{2}f(r)\left[e^{i\phi}\left(\mathsf{d}\theta + i\sin\theta\mathsf{d}\phi\right)E_+ - e^{-i\phi}\left(\mathsf{d}\theta - i\sin\theta\mathsf{d}\phi\right)E_-\right]$$

Yang-Mills-Higgs Theory

A, X: adjoint valued for some compact simple Lie group G

BPS equation :

$$B = \mathsf{D}X$$

$$\Updownarrow$$

$$f'(r) + f(r)h(r) = 0\,, \qquad r^2 h'(r) + f(r)^2 - 1 = 0$$

$$\Updownarrow$$

$$h(r) = m\coth(mr + c) - \tfrac{1}{r}\,, \qquad f(r) = \tfrac{mr}{\sinh(mr+c)}$$

Yang-Mills-Higgs Theory

A, X: adjoint valued for some compact simple Lie group G

BPS equation :

$$B = \mathrm{D}X$$

$$\Updownarrow$$

$$h(r) = m\coth(mr + c) - \tfrac{1}{r}\ , \qquad f(r) = \tfrac{mr}{\sinh(mr+c)}$$

$c = 0$

- smooth
- magnetic charge: $\gamma_{\mathrm{m}} = H$
- finite energy: $\frac{4\pi m}{g^2}$

$\Rightarrow$ non-abelian monopole

Yang-Mills-Higgs Theory

A, X: adjoint valued for some compact simple Lie group G

BPS equation :

$$B = \mathsf{D}X$$

$$\Updownarrow$$

$$h(r) = m\coth\left(mr + c\right) - \tfrac{1}{r}\ , \qquad f(r) = \tfrac{mr}{\sinh\left(mr+c\right)}$$

$$c > 0$$

- singular at $r = 0$
- magnetic charge: $\gamma_{\mathrm{m}} = H$
- finite energy: $\frac{4\pi m}{g^2}$

$\Rightarrow$ singular monopole

Yang-Mills-Higgs Theory

A, X: adjoint valued for some compact simple Lie group G

BPS equation :

$$B = \mathrm{D}X$$

$$\Updownarrow$$

$$h(r) = m \coth(mr + c) - \frac{1}{r} \; , \qquad f(r) = \frac{mr}{\sinh(mr+c)}$$

$c > 0$

- singular at $r = 0$
- magnetic charge: $\gamma_{\mathrm{m}} = H$
- finite energy: $\frac{4\pi m}{g^2}$

$\Rightarrow$ singular monopole

2 smooth monopoles in a negative defect background

Outline

- Motivation
- Basic definitions and properties
- Moduli space
- Framed BPS states

Motivation

- Defect operators important in field theory
  $\Rightarrow$ semi-classically: singular boundary conditions

  Wilson-'t Hooft operators $\Leftrightarrow$ point singularities
  (line operators)

- Monopoles have very rich mathematical structure
  e.g. Atiyah-Hitchin, Callias, Nahm

  $\Rightarrow$ generalize to singular case?
  e.g. Cherkis-Kapustin, MRV, Kronheimer

- (Framed) BPS states in $\mathcal{N} = 2$
  $\Rightarrow$ various conjectures and results

  checks/predictions in semi-classical regime!

Basic definitions and properties

We'll work on $M = \mathbb{R}_t \times (\mathbb{R}^3 \setminus \{\vec{0}\})$

Boundary conditions (Purely magnetic case)

- As $r \to \infty$

$$
\begin{aligned}
X &= X_\infty - \frac{1}{2r}\gamma_{\mathrm{m}} + O(r^{-(1+\delta)}) \\
F &= \frac{1}{2}\gamma_{\mathrm{m}} \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi + O(r^{-(2+\delta)})
\end{aligned}
$$

$\delta > 0$, $\gamma_{\mathrm{m}}, X_\infty \in \mathfrak{t}$ Cartan sub-algebra, $\exp(2\pi\gamma_{\mathrm{m}}) = 1_G$

$\Rightarrow$ Set by demanding finite energy

Basic definitions and properties

We'll work on $M = \mathbb{R}_t \times (\mathbb{R}^3 \setminus \{\vec{0}\})$

Boundary conditions (Purely magnetic case)

- As $r \to 0$

$$X = -\frac{1}{2r} P + O(r^{-1/2})$$

$$F = \frac{1}{2} P \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi + O(r^{-3/2})$$

't Hooft charge: $P \in \mathfrak{t}$, unique up to Weyl transformations,

$$\exp(2\pi P) = 1_G$$

- more general than considered before $(\exists \alpha, \langle \alpha, P \rangle = 1)$
- well defined variational principle $S_{\text{bndry}} = -\frac{2}{g^2} \int \mathrm{d}t \int_{S^2_\varepsilon} \mathrm{Tr} X F$
- finite energy

Moduli space

$$\overline{\underline{\mathcal{M}}}\,(P;\gamma_{\mathrm{m}};X_\infty) =$$

$$\left\{ (A,X) \;\middle|\; F = \star DX \;,\;\; \begin{array}{l} X = -\frac{1}{2r}P + O(r^{-1/2}) \;,\;\; r \to 0 \;, \\ X = X_\infty - \frac{1}{2r}\gamma_{\mathrm{m}} + O(r^{-(1+\delta)}) \;,\;\; r \to \infty \end{array} \right\} \Big/ \mathcal{G}_{\{P\}}$$

Pretty wild, study through its tangent space $T\overline{\underline{\mathcal{M}}}$

Moduli space

Linearized BPS equation $\quad (\hat{A} = (A, X))$

$$\hat{D}_{[a}\delta\hat{A}_{b]} = \tfrac{1}{2}\epsilon_{ab}{}^{cd}\hat{D}_c\delta\hat{A}_d$$

Metric

$$g(\delta_1, \delta_2) = \tfrac{2}{g_0^2}\int_{\mathbb{R}_0^3} \mathsf{d}^3x\, \mathsf{Tr}\left\{\delta_1\hat{A}_a\delta_2\hat{A}^a\right\}$$

Boundary conditions imply square normalizability!

Hyper-Kähler structure

$$\delta\hat{A}_a \to J_a{}^b\delta\hat{A}_b \qquad J \in \mathbb{H}(\mathbb{R}^4)$$

Moduli space

$$\dim \underline{\overline{\mathcal{M}}} = \dim T \underline{\overline{\mathcal{M}}} = 4 \times ?$$

Moduli space

$$\dim \underline{\overline{\mathcal{M}}} = \dim T\underline{\overline{\mathcal{M}}} = 4\times ?$$

For smooth monopoles (Callias-Weinberg, '79)

- Index of Dirac operator
- Integral of index current
- Reduce to boundary integral
- Trace over free electron states

$$\dim \mathcal{M} = \sum_{\alpha\in\Delta} \frac{\langle \alpha, X_\infty \rangle \langle \alpha, \gamma_{\mathrm{m}} \rangle}{|\langle \alpha, X_\infty \rangle|}$$

Moduli space

$$\dim \underline{\overline{\mathcal{M}}} = \dim T\underline{\overline{\mathcal{M}}} = 4 \times ?$$

For singular monopoles (MRV, '14)

- Index of Dirac operator
- Integral of index current
- Reduce to boundary integral
- Trace over free electron states
  $+$ states of electron in Dirac monopole background

$$\dim \underline{\overline{\mathcal{M}}} = \sum_{\alpha \in \Delta} \left(\frac{\langle \alpha, X_\infty \rangle \langle \alpha, \gamma_{\mathrm{m}} \rangle}{|\langle \alpha, X_\infty \rangle|} + |\langle \alpha, P \rangle| \right)$$

Moduli space

$$\begin{aligned}
\dim\overline{\mathcal{M}} &= \sum_{\alpha\in\Delta}\left(\frac{\langle\alpha,X_\infty\rangle\langle\alpha,\gamma_{\mathrm{m}}\rangle}{|\langle\alpha,X_\infty\rangle|}+|\langle\alpha,P\rangle|\right) \\
&= 2\sum_{\alpha\in\Delta^+}\langle\alpha,\tilde{\gamma}_{\mathrm{m}}\rangle \qquad \tilde{\gamma}_{\mathrm{m}}=\gamma_{\mathrm{m}}-P^- \\
&= 4\sum_{I=1}^{\text{rnk }\mathfrak{g}}\tilde{m}^I \qquad \tilde{\gamma}_{\mathrm{m}}=\tilde{m}^I H_I
\end{aligned}$$

$\tilde{m}^I$ is the number of smooth monopoles

Moduli space

$$\dim \overline{\mathcal{M}} = 4 \sum \tilde{m}^I \qquad \tilde{\gamma}_{\rm m} = \gamma_{\rm m} - P^- = \tilde{m}^I H_I$$

$$\infty\colon\; X = X_\infty - \tfrac{1}{2r}\gamma_{\rm m} + \ldots \qquad 0\colon\; X = -\tfrac{1}{2r} P + \ldots$$

Example

$$\begin{aligned} X &= X_\infty - \frac{P}{2r} \\ A &= \frac{P}{2}(1-\cos\theta)\mathsf{d}\phi \end{aligned}$$

Here $\gamma_{\rm m} = P$

Moduli space

$$\dim\overline{\mathcal{M}} = 4\sum \tilde{m}^I \qquad \tilde{\gamma}_{\rm m} = \gamma_{\rm m} - P^- = \tilde{m}^I H_I$$

$$\infty:\ \ X = X_\infty - \tfrac{1}{2r}\gamma_{\rm m} + \ldots \qquad 0:\ \ X = -\tfrac{1}{2r}P + \ldots$$

Example

$$\begin{aligned} X &= X_\infty - \frac{P}{2r} \\ A &= \frac{P}{2}(1-\cos\theta)\mathsf{d}\phi \end{aligned}$$

Here $\gamma_{\rm m} = P$

Assume $P = P^-$ $\quad \Rightarrow \tilde{\gamma}_{\rm m} = 0$

dim $\overline{\mathcal{M}} = 0$, Pure line defect

Moduli space

$$\dim \overline{\mathcal{M}} = 4 \sum \tilde{m}^I \qquad \tilde{\gamma}_{\rm m} = \gamma_{\rm m} - P^- = \tilde{m}^I H_I$$

$$\infty: \ X = X_\infty - \tfrac{1}{2r}\gamma_{\rm m} + \ldots \qquad 0: \ X = -\tfrac{1}{2r} P + \ldots$$

Example

$$\begin{aligned} X &= X_\infty - \frac{P}{2r} \\ A &= \frac{P}{2}(1 - \cos\theta)\mathrm{d}\phi \end{aligned}$$

Here $\gamma_{\rm m} = P$

When $P \neq P^-$ $\Rightarrow \tilde{\gamma}_{\rm m} \neq 0$ (ok with Weyl redundancy!)

dim $\overline{\mathcal{M}} \neq 0$, smooth monopoles present

Moduli space

$$\dim\overline{\mathcal{M}} = 4\sum \tilde{m}^I \qquad \tilde{\gamma}_{\mathrm{m}} = \gamma_{\mathrm{m}} - P^- = \tilde{m}^I H_I$$

$$\infty\colon\; X = X_\infty - \tfrac{1}{2r}\gamma_{\mathrm{m}} + \dots \qquad 0\colon\; X = -\tfrac{1}{2r}P + \dots$$

Example

$$\begin{aligned} X &= X_\infty - \frac{P}{2r} \\ A &= \frac{P}{2}(1-\cos\theta)\mathrm{d}\phi \end{aligned}$$

Here $\gamma_{\mathrm{m}} = P$

E.g. $P = H \quad \Rightarrow \tilde{\gamma}_{\mathrm{m}} = 2H$

$c = \infty$ case of
't Hooft-Polyakov example

$\dim\,\overline{\mathcal{M}} = 8$, 2 smooth monopoles present

N=2 SYM Theory

Vector multiplet: F, X and Y adjoint valued

BPS equations :

$$B = DX$$
$$D^2 Y + [X,[X,Y]] = 0$$

- $A_0 = Y$ gauge
- Primary and secondary
- Moduli problem identical

Moduli space approximation Manton, Gauntlett

$$\hat{A}(t) = \hat{A}(z(t)) \approx \hat{A} + \delta_m \hat{A}\,\dot{z}^m\,t$$

4d theory reduces to motion on Moduli space

$$L = \tfrac{1}{2} g_{mn} \dot{z}^m \dot{z}^n - \tfrac{1}{2} g_{mn} G^m(z, Y_\infty) G^n(z, Y_\infty)$$

- some z correspond to global gauge phases
  $\Rightarrow$ Electric charge and dyons
- Potential leads to finite radius bound states

Moduli space approximation   Manton, Gauntlett

$$\hat{A}(t) = \hat{A}(z(t)) \approx \hat{A} + \delta_m \hat{A}\, \dot{z}^m\, t$$

4d theory reduces to motion on Moduli space

$$L = \tfrac{1}{2} g_{mn} \dot{z}^m \dot{z}^n - \tfrac{1}{2} g_{mn} G^m(z, Y_\infty) G^n(z, Y_\infty)$$

- Quantize:
  Hilbert space is set of spinors on $\underline{\overline{\mathcal{M}}}$
- Supercharge becomes Dirac operator

$$D_Y = \Gamma^m (i D_m + G_m)$$

Moduli space approximation

Manton, Gauntlett

$$\hat{A}(t) = \hat{A}(z(t)) \approx \hat{A} + \delta_m \hat{A}\, \dot{z}^m\, t$$

4d theory reduces to motion on Moduli space

$$L = \frac{1}{2} g_{mn} \dot{z}^m \dot{z}^n - \frac{1}{2} g_{mn} G^m(z, Y_\infty) G^n(z, Y_\infty)$$

- Quantize:
  Hilbert space is set of spinors on $\overline{\underline{\mathcal{M}}}$
- Supercharge becomes Dirac operator

$$D_Y = \Gamma^m (i D_m + G_m)$$

(framed) BPS states are given by $\ker D_Y$

Framed BPS states

BPS states in the presence of line operators

PSC: $\underline{\overline{\Omega}}(u, L, \gamma; y) = \text{Tr}_{\mathcal{H}^{\text{BPS}}_{u,L,\gamma}} y^{2J_3}(-y)^{I_3}$ Gaiotto, Moore, Neitzke

Semi-classically

$$u = u(X_\infty, Y_\infty, \zeta), \qquad L = (P, Q, \zeta), \qquad \gamma = (\gamma_{\text{m}}, \gamma_{\text{e}})$$

$$\Rightarrow \quad \underline{\overline{\mathcal{M}}}(X_\infty, P, \gamma_{\text{m}}) \quad \text{and} \quad D_{Y_\infty} \qquad (Q = 0)$$

$$\underline{\overline{\Omega}}(X_\infty, Y_\infty, P, \zeta, \gamma_{\text{m}}, \gamma_{\text{e}}; y) = \text{Tr}_{\ker D_{Y_\infty}|_{\gamma_{\text{e}}}} y^{2J_3}(-y)^{I_3}$$

An $\mathfrak{su}(2)$ example

The Blair-Cherkis monopole

- Completely explicit
- Single monopole bound to magnetic singularity

$$P = \tfrac{p}{2} H \qquad \tilde{\gamma}_{\mathrm{m}} = H$$
$$SO(3):\ p \in \mathbb{Z}, \qquad SU(2):\ p \in 2\mathbb{Z}$$

- Moduli space: Taub-NUT$/\mathbb{Z}_p$

$$\dim\ker D_{Y_\infty}\big|_{\gamma_{\mathrm{e}}=\nu\alpha} = \begin{cases} p\nu & \text{when} \quad \nu \le \frac{4\pi}{g_0^2}\frac{(H,Y_\infty)}{(H,X_\infty)} \\ 0 & \text{otherwise} \end{cases}$$

An $\mathfrak{su}(2)$ example

$$\dim\ker D_{Y_\infty}\big|_{\gamma_{\rm e}=\nu\alpha} = \begin{cases} p\nu & \text{when} \quad \nu \leq \frac{4\pi}{g_0^2}\frac{(H,Y_\infty)}{(H,X_\infty)} \\ 0 & \text{otherwise} \end{cases}$$

Prediction form GMN (via laminations from 6d (2,0) theory)

$$F_p(n) = \sum_{\mu,\nu\in\mathbb{Z}} \underline{\overline{\Omega}}_p(\mu,\nu,n)\tilde{x}^\mu\tilde{y}^\nu = \left(\tilde{x}\mathcal{L}_{2n}(\tilde{x}^{-1}\tilde{y}^{1-n},\tilde{x}\tilde{y}^n)\right)^{\frac{p}{2}}$$

$$\mathcal{L}_{2n}(x,y) = x^{-1}\left[T_n\left(\tfrac{xy+y+1}{2\sqrt{xy}}\right) + \tfrac{xy-y-1}{2\sqrt{xy}}U_{n-1}\left(\tfrac{xy+y+1}{2\sqrt{xy}}\right)\right]^2$$

T_n, U_n Tchebyshev polynomials

$$\tilde{\gamma}_{\rm m} = \mu H\,, \qquad \gamma_{\rm e} = \nu H\,, \qquad n = \left\lceil \frac{4\pi}{g_0^2}\frac{(H,Y_\infty)}{(H,X_\infty)} \right\rceil$$

An $\mathfrak{su}(2)$ example

$$\dim\ker D_{Y_\infty}\big|_{\gamma_{\mathrm{e}}=\nu\alpha} = \begin{cases} p\nu & \text{when} \quad \nu \leq \frac{4\pi}{g_0^2}\frac{(H,Y_\infty)}{(H,X_\infty)} \\ 0 & \text{otherwise} \end{cases}$$

Prediction form GMN (via laminations from 6d (2,0) theory)

$$F_p(n) = \sum_{\mu,\nu\in\mathbb{Z}} \underline{\overline{\Omega}}_p(\mu,\nu,n)\tilde{x}^\mu\tilde{y}^\nu = \left(\tilde{x}\mathcal{L}_{2n}(\tilde{x}^{-1}\tilde{y}^{1-n},\tilde{x}\tilde{y}^n)\right)^{\frac{p}{2}}$$

It works!!

$$\dim\ker D_{Y_\infty}\big|_{\gamma_{\mathrm{e}}=\nu\alpha} = \underline{\overline{\Omega}}_p(1,\nu,n)$$

An $\mathfrak{su}(2)$ example

Prediction form GMN (via laminations from 6d (2,0) theory)

$$F_p(n) = \sum_{\mu,\nu\in\mathbb{Z}} \overline{\underline{\Omega}}_p(\mu,\nu,n)\tilde{x}^{\mu}\tilde{y}^{\nu} = \left(\tilde{x}\mathcal{L}_{2n}(\tilde{x}^{-1}\tilde{y}^{1-n},\tilde{x}\tilde{y}^{n})\right)^{\frac{p}{2}}$$

For $\mu > 1$ an infinite number of predictions for Dirac operators on moduli spaces with dim$= 4\mu$

Summary

- Singular monopoles describe BPS states in presence of defect operators
- Rich mathematical structure
- Dimension formula, provides physical interpretation
- Semi-classical description of (framed) BPS states in $N = 2$
- Results/conjectures for BPS spectra lead to predictions/checks on moduli space