Curvature Singularity in Modified Theories of Gravity

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Modified Gravity

Why do we need to modify Gravity ?

- Late time acceleration of the universe
- Possible explanations :

Einstein Equation

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Modified Gravity :
Modification to the spacetime part of Einstein equation

Dark Energy :
Modification to the matter part of Einstein Equation

Modified Gravity

What is modified Gravity ?

- Generalization of General Relativity
- The early-time as well as the late-time cosmic acceleration may be caused simply by the fact that some sub-dominant terms of more general gravitational action may become essential at large or small curvatures.

For example

$$S_G = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(.... + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + ...\right)$$

f(R) Theories

- f(R) theories are the simplest among the modified gravity theories.

f(R) Theories

- Einstein-Hilbert Action is replaced by

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + S_M$$

- Where, f(R) is an arbitrary function of Ricci Scalar R.
- $f(R) = R + F(R)$ Where, F(R) captures the modification of gravity.
- The trace of the field equation is given by

$$3\Box F_{,R} - 2F - R + RF_{,R} = \kappa^2 T$$

- The non-vanishing term $\phi = \Box F_{,R}$ can give rise to new scalar degree of freedom, dubbed as scalaron, whose dynamics is governed by above equation.

Viability Criteria of f(R) Theories

- Correct Cosmological dynamics
- Free from instabilities and Ghost
- Correct dynamics of cosmological perturbations
- Local Gravity Limit
- Free from Curvature Singularities

Curvature Singularity

- The trace equation is a non-linear equation and hence very difficult to handle.

- The Scalar-Tensor representation can be a rescue by identifying scalar field $\phi = F_{,R}$ and writing the trace equation as

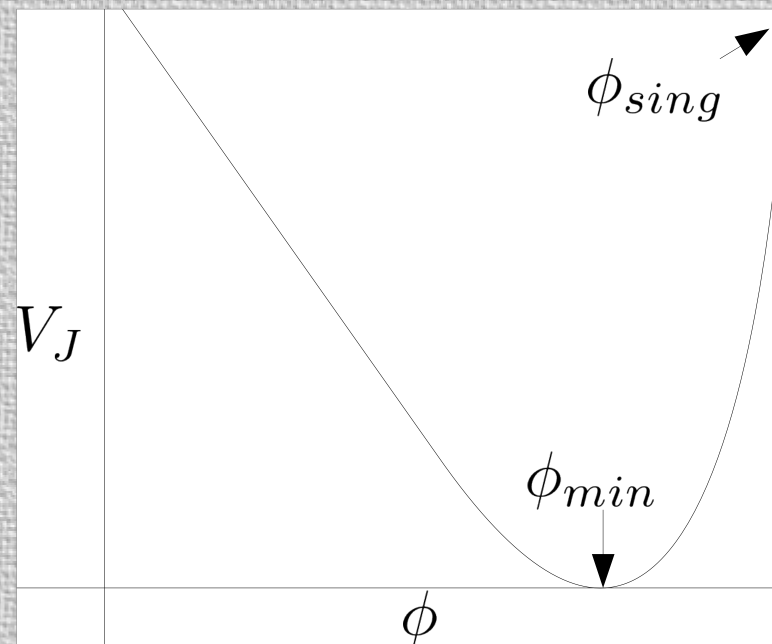
$$\Box\phi = \frac{dV_J}{d\phi} + \frac{\kappa^2 T}{3}$$

where, $$\frac{dV_J}{d\phi} = \frac{1}{3}(R + 2F - RF_{,R})$$

- The dynamics of the field ϕ is governed by its potential V_J originating from the modified gravity, and a force term that is proportional to the stress-energy tensor T.

Curvature Singularity

- The potential $V_J(\phi)$ will have a global minimum at ϕ_{min} where cosmological evolution happens. But, there also exists a point in field space ϕ_{sing} where the curvature scalar R diverges.



- Typically, the points ϕ_{min} and ϕ_{sing} would be separated by finite energy barrier. Therefore, the field can easily reach to the singular point in the process of having small oscillations around its minimum.
- The point ϕ_{min} is also a desitter point since it corresponds to the constant curvature solution for vacuum.

Curvature Singularity

- $$\Box\phi = \frac{dV_J}{d\phi} + \frac{\kappa^2 T}{3} \longrightarrow \Box\phi = \frac{dV_J^{eff}}{d\phi}$$

$$\frac{dV_J^{eff}}{d\phi} = \frac{dV_J}{d\phi} + \frac{\kappa^2 T}{3}$$

- We can also study the dynamics of the scalar field in an astrophysical system whose energy density increases with time.

Weak Gravity Assumption :

Covariant Derivative
↓
Flat Space Partial Derivatives

Homogeneous and Isotropic :

Spatial derivatives can be neglected

- The above equation can be written as $\frac{\partial^2\phi}{\partial t^2} + \frac{\partial V_J^{eff}}{\partial\phi} = 0$.
- Due to non-linear behaviour of the motion, this singular point, can be well reached, in a finite time, during the evolution of the field.

Fifth Force Constraint

d.o.f

f(R) Gravity → Scalaron

→ Spin-2 Graviton

- The effect of the scalar degree of freedom, in the matter sector is evident when the theory is rewritten in the Einstein frame where the gravity part is Einstein-Hilbert type.

- The conformal transformation $\tilde{g}_{\mu\nu} = f_{,R}\, g_{\mu\nu}$ on the metric transforms the action to

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}(\tilde{\nabla}\psi)^2 - V_E(\psi) \right] + \int d^4x \mathfrak{L}_m(\tilde{g}_{\mu\nu} e^{-\frac{2}{\sqrt{6}}\kappa\psi})$$

where, $V_E = \dfrac{Rf_{,R} - f}{2\kappa^2 f_{,R}^2}$.

- Note that the matter part of the Lagrangian in the Einstein frame has direct universal coupling of ψ field to all matter fields.

Fifth Force Constraint

- If the mass of the scalar field is light as the present Hubble constant then there will appear a long range fifth force mediated by the ψ field.
- Local Gravity Test : Need of Screening Mechanism : To screen the fifth force on the surface of the Sun/Earth.
- Chameleon Mechanism : The mass of the scalar field depends on the background density.
- Blending with the environment:

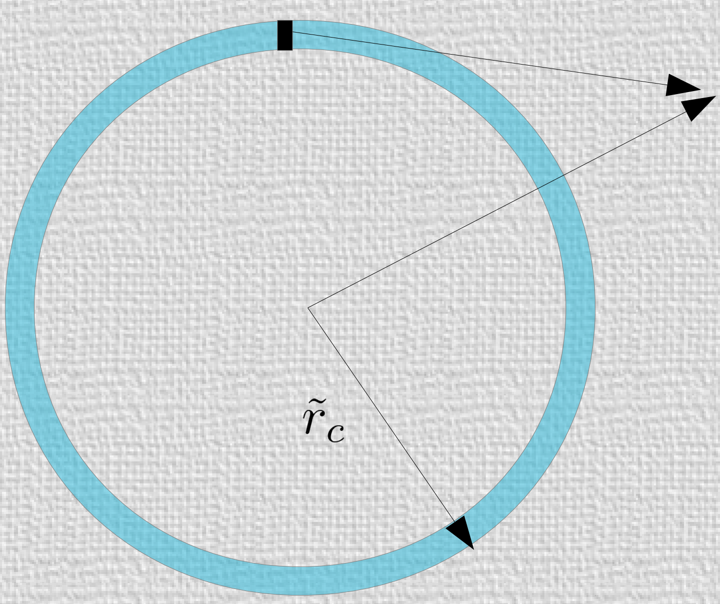
On the surface of Sun/Earth : Heavy scalar field $\Rightarrow$ fifth force is very short range

In the Interstellar space : Light scalar field $\Rightarrow$ fifth force is long range

Fifth Force Constraint

- As long as the scalar field is heavy inside the surface of the earth, it will be frozen at the minimum of its effective potential V_{eff} and cannot contribute to the outside field.

- The only contribution to the outside field can be from a very thin shell around the surface of the earth. This thin shell parameter is given by



$$\frac{\Delta \tilde{r}_c}{\tilde{r}_c} = -\frac{\psi_{out} - \psi_{in}}{(\sqrt{6}/\kappa)\Phi_{test}}$$

$\left.\begin{matrix}\psi_{in} \\ \psi_{out}\end{matrix}\right\}$ f ield values corresponding to the minimum of the effective potential inside and outside the test body.

Φ_{test}:Gravitational potential on the surface at the radius $\tilde{r}_c$ of the test body.

Fifth Force Constraint

- The potential V_{eff} is given by $V_{eff}(\psi) = V_E(\psi) + e^{-\frac{1}{\sqrt{6}}\kappa\psi}$

- Under the Approximation : $F_{,R} << 1 \;\; and \;\; F << R$

  We can write $\psi_{in} << \psi_{out}$

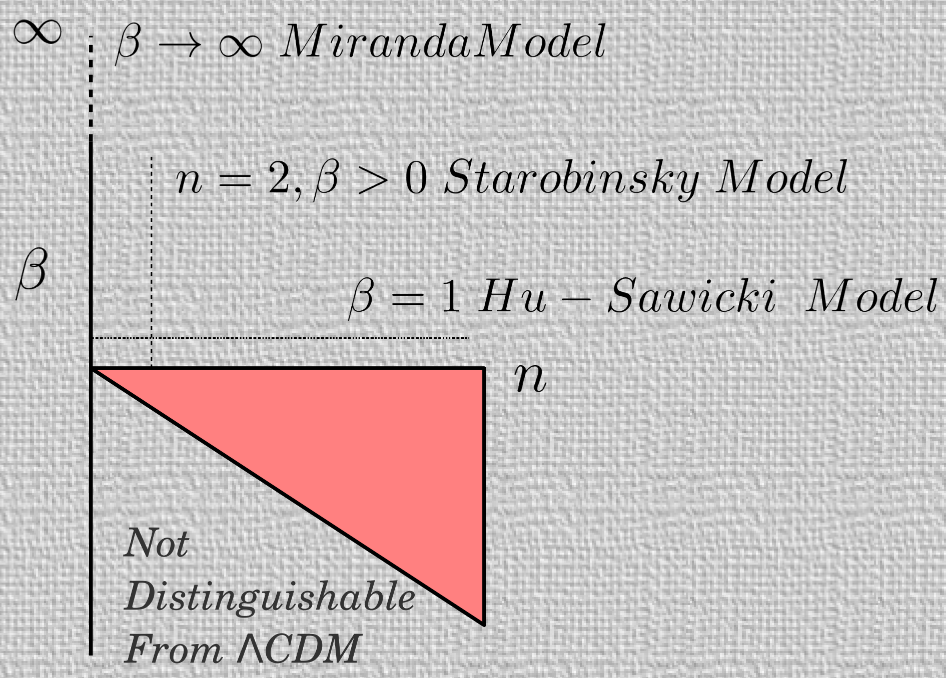
- The thin-shell condition reduces to $|\psi_{out}| \lesssim (\sqrt{6}/\kappa)\Phi_{test}\frac{\Delta\tilde{r}_c}{\tilde{r}_c}$

- The Solar System (SS) test and Equivalence Principle (EP) test put following numerical constraint on (Here, we take $\kappa = 1$)

$$|\psi_{out}| \lesssim \begin{cases} 5.97 \times 10^{-11} & \text{(SS test)}, \\ 3.43 \times 10^{-15} & \text{(EP test)}. \end{cases}$$

General Model

$$f(R) = R + F(R) = R + \alpha R_* \beta \left\{ \left[1 + \left(\frac{R}{R_*} \right)^n \right]^{-\frac{1}{\beta}} - 1 \right\}$$



Behaviour of the model :

$f(R) \to 0 \ as \ R \to 0$

$f(R) \to R - const \ as \ R >> R_*$

Condition for the presence of matter dominated era :

$n > 0 \ and \ (\beta < 0 \ or \ \beta < -n)$

General Model

Local Gravity Test in General Model

For the given form of f(R), the scalar field at the minimum of effective potential outside the test body is given by

$$|\psi_{out}| \sim \frac{\sqrt{6}}{2\kappa} n\alpha \left(\frac{\kappa^2 \rho_{out}}{R_*} \right)^{-\frac{n}{\beta}-1}$$

If R_1 is the curvature at desitter minimum, we can define a dimensionless variable $x_1 = R_1/R_*$ to obtain

$$|\psi_{out}| \sim \frac{\sqrt{6}}{2\kappa} n\alpha \left(\frac{x_1 \kappa^2 \rho_{out}}{R_1} \right)^{-\frac{n}{\beta}-1}$$

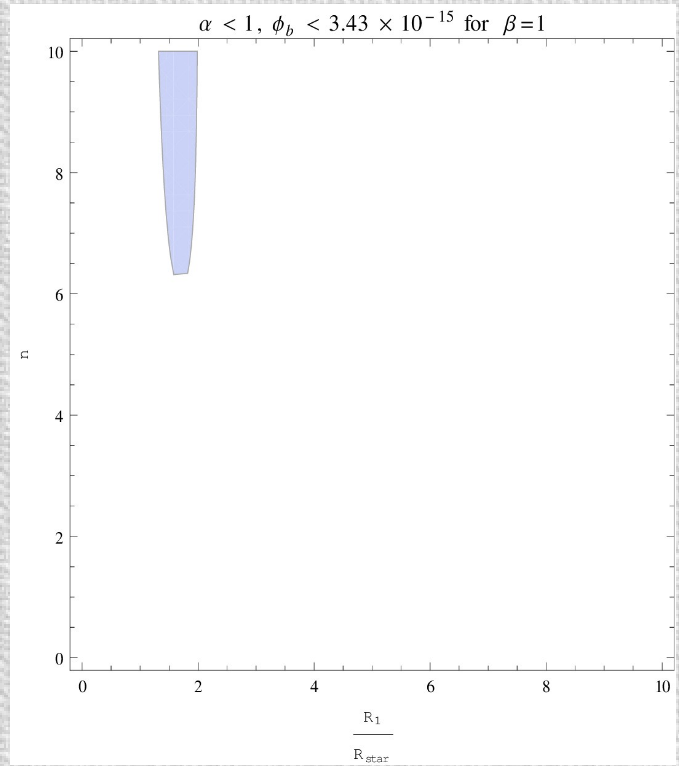
Taking $R_1 = \kappa^2 \rho_c$, we obtain

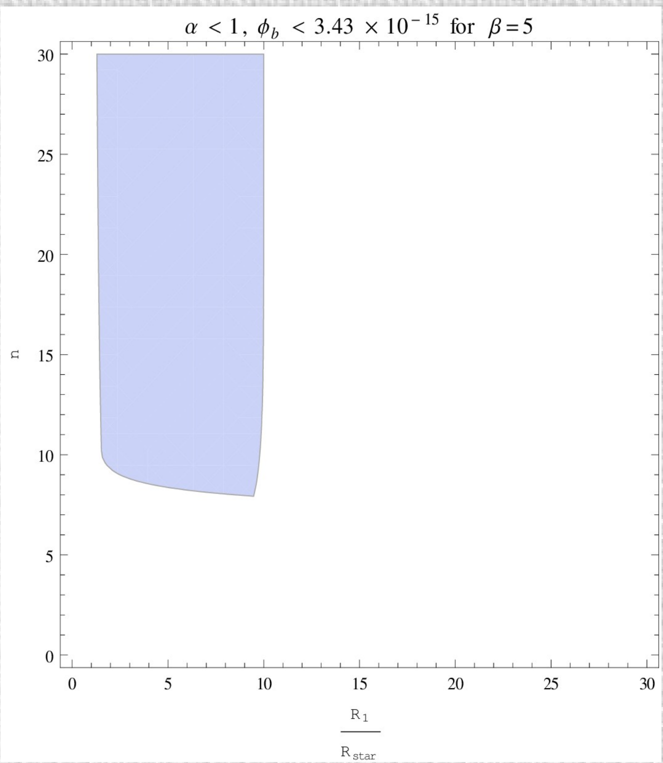
$$|\psi_{out}| \sim \frac{\sqrt{6}}{2\kappa} n\alpha \left(\frac{x_1 \rho_{out}}{\rho_c} \right)^{-\frac{n}{\beta}-1}$$

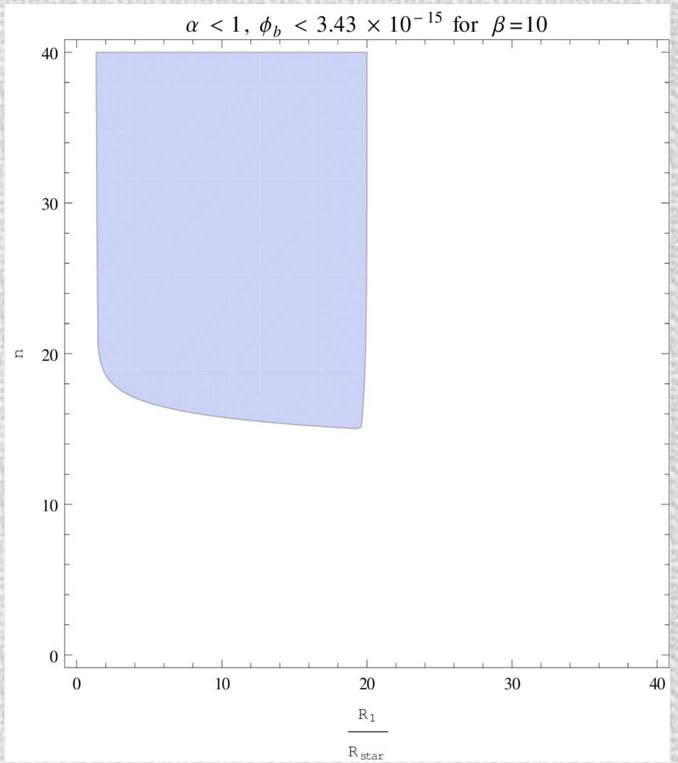
General Model

- Local Gravity Test in General Model

The allowed regions dectated by the SS test and EP test has been shown in parameter space as in figure below.







$$\frac{n}{\beta} \gtrsim 2$$

General Model

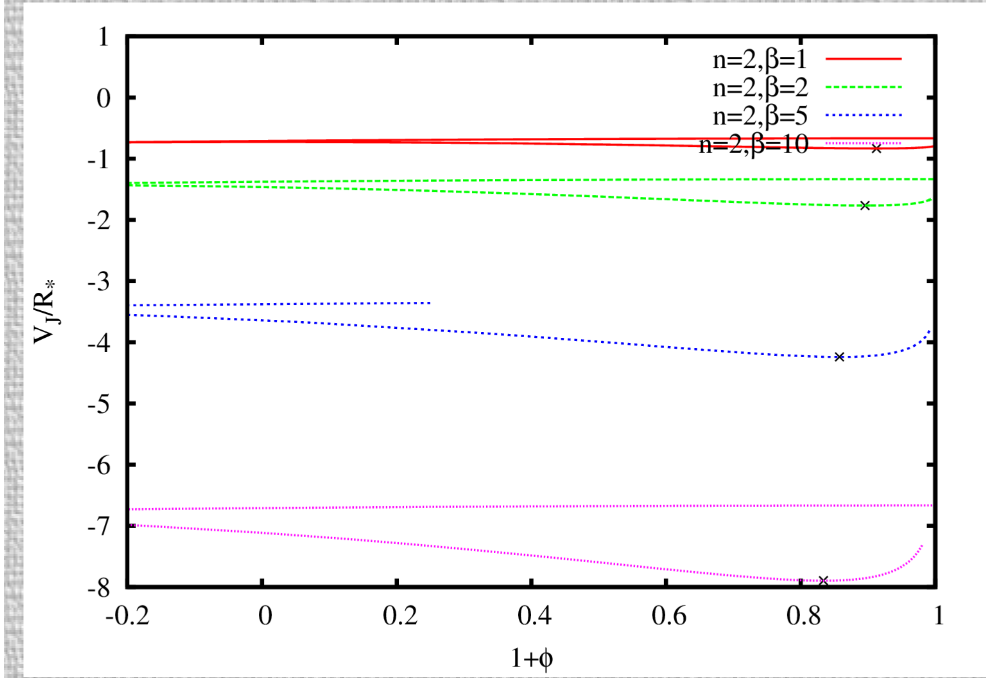
- Curvature Singularity in General Model

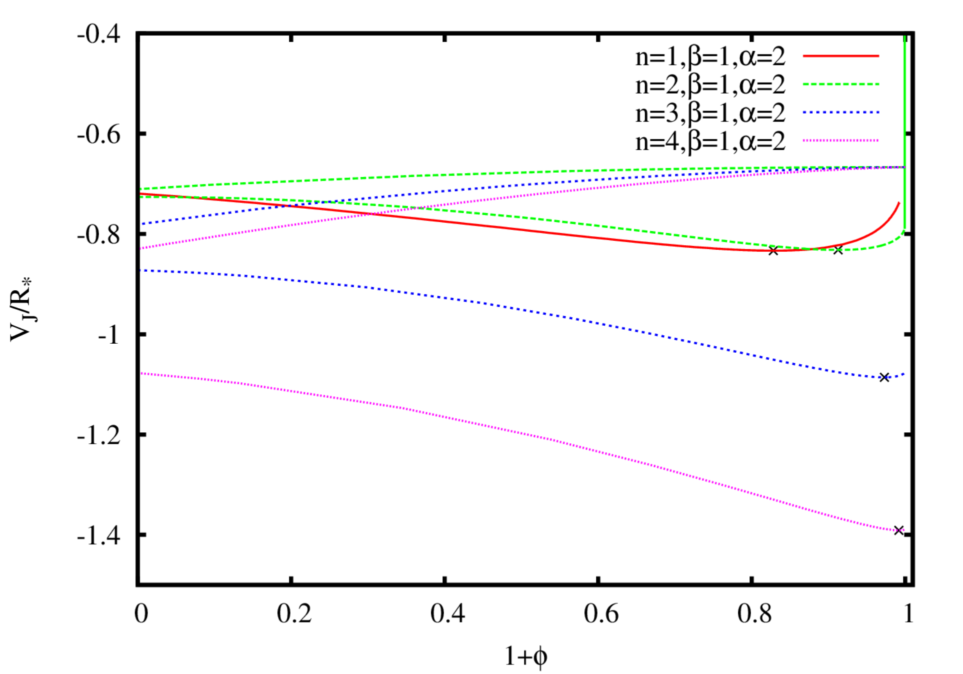
Scalar field :

$$\phi = -n\alpha\left(\frac{R}{R_*}\right)^{n-1}\left[1+\left(\frac{R}{R_*}\right)^n\right]^{-\frac{1}{\beta}-1}$$

$$\phi_{sing} = 0$$

The plots $V_J(\phi)/R_* \to \phi$ for different values of parameters





General Model

Curvature Singularity in General Model

Study of curvature singularity in an astrophysical object :

Consider the dense contracting system of locally homogeneous and isotropic cloud of pressureless dust whose density ρ_m is much greater than critical density ρ_c.

Energy-momentum tensor :

$$T = -T_0\left[1+\frac{t}{t_{ch}}\right]$$

The field Equation : $$3\frac{\partial^2}{\partial t^2}F_{,R} + 2F + R - RF_{,R} - \kappa^2 T_0\left[1+\frac{t}{t_{ch}}\right] = 0$$

Defining dimensionless quantity, $u = R_*/R$ we obtain

$$\ddot{u} + \frac{n}{\beta}\frac{\dot{u}^2}{u} = -\frac{1}{3n\alpha}\frac{\beta}{n+\beta}(u)^{-n/\beta}\left[\kappa^2 T_0\left(1+\frac{t}{t_{ch}}\right) - \frac{R_*}{u} + 2\alpha\beta R_*\right] + \frac{R_*}{3}\frac{\beta}{n+\beta}\left(1+\frac{2\beta}{n}\right)$$

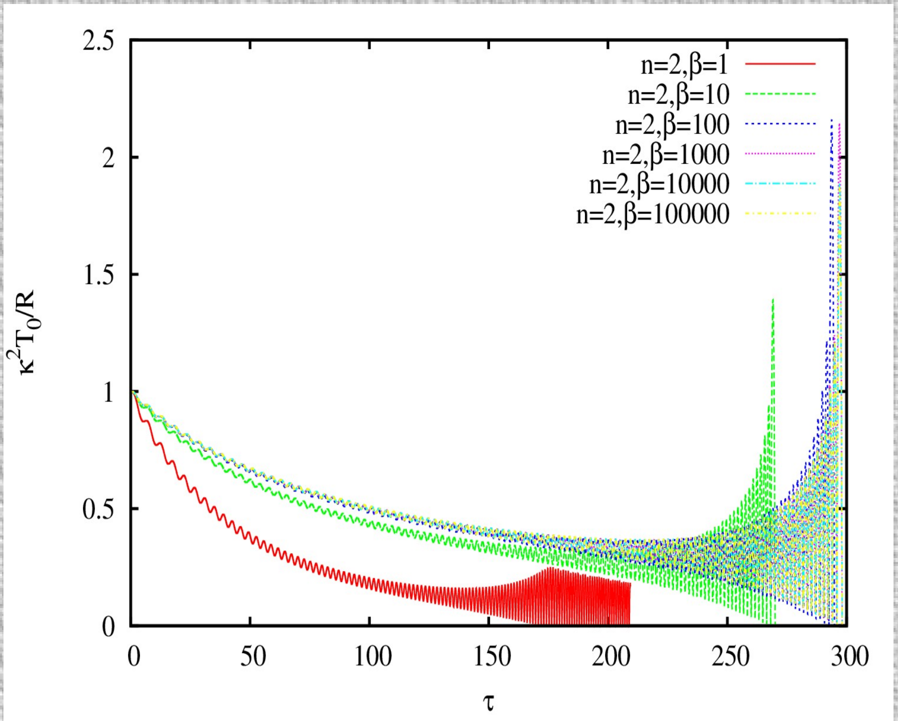
Change of variables $y = \eta u$ and $\tau = \gamma^{-1} t$ gives :

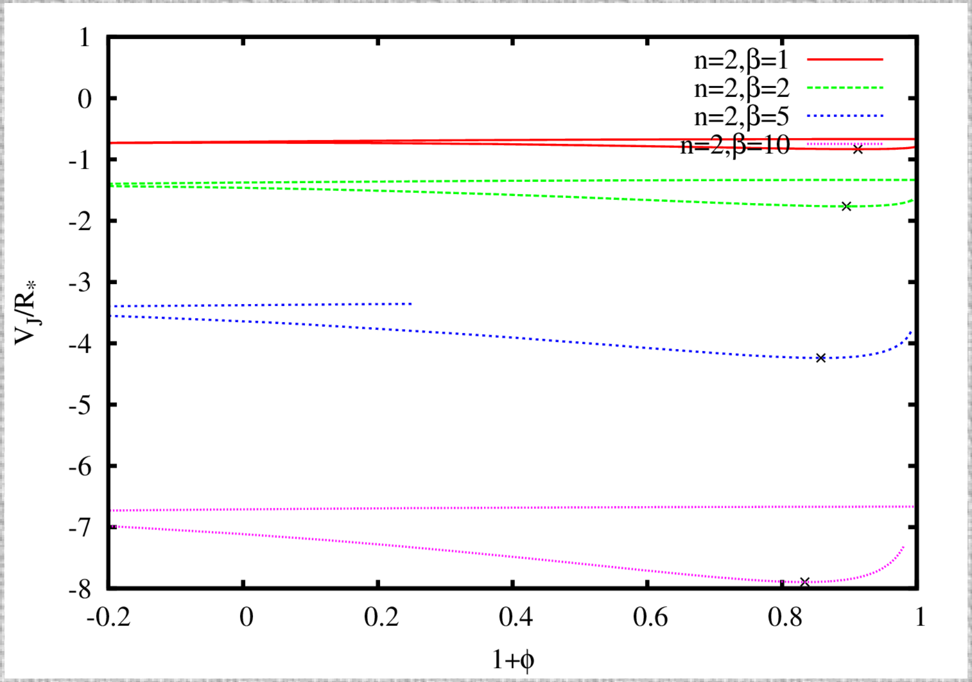
$$y'' + \frac{n}{\beta}\frac{y'^2}{y} = -y^{-n/\beta}\left[\left(1+\frac{\tau}{\tau_*}\right) - \frac{1}{y} + \frac{2\alpha\beta}{\eta}\right] + n\alpha\eta^{-1-n/\beta}\left(1+\frac{2\beta}{n}\right)$$

Where, $\eta = \dfrac{\kappa^2 T_0}{R_*}$ and $$\gamma^2 = \left[\left(\frac{\kappa^2 T_0}{R_*}\right)^{2+n/\beta}\frac{\beta}{n+\beta}\frac{R_*}{3n\alpha}\right]^{-1}$$

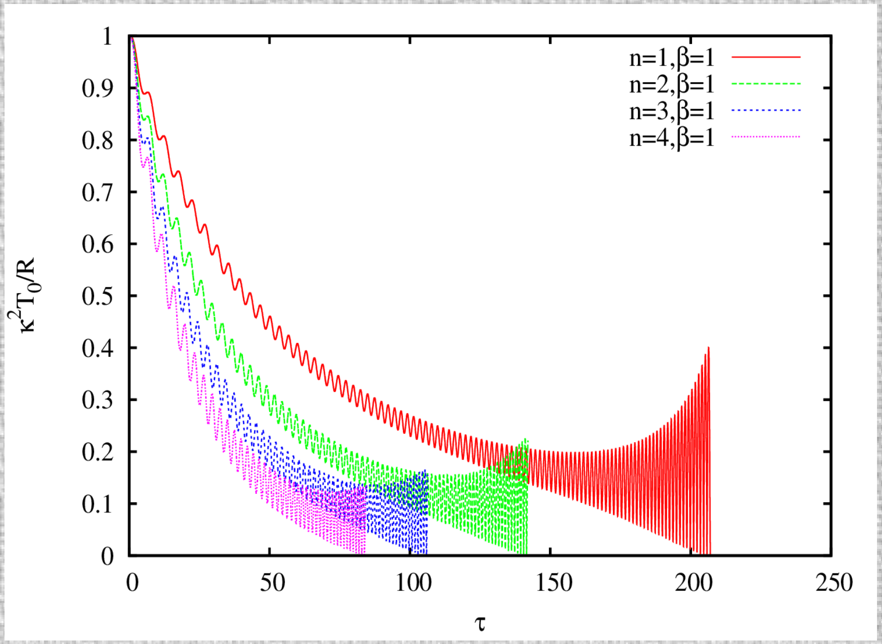
General Model

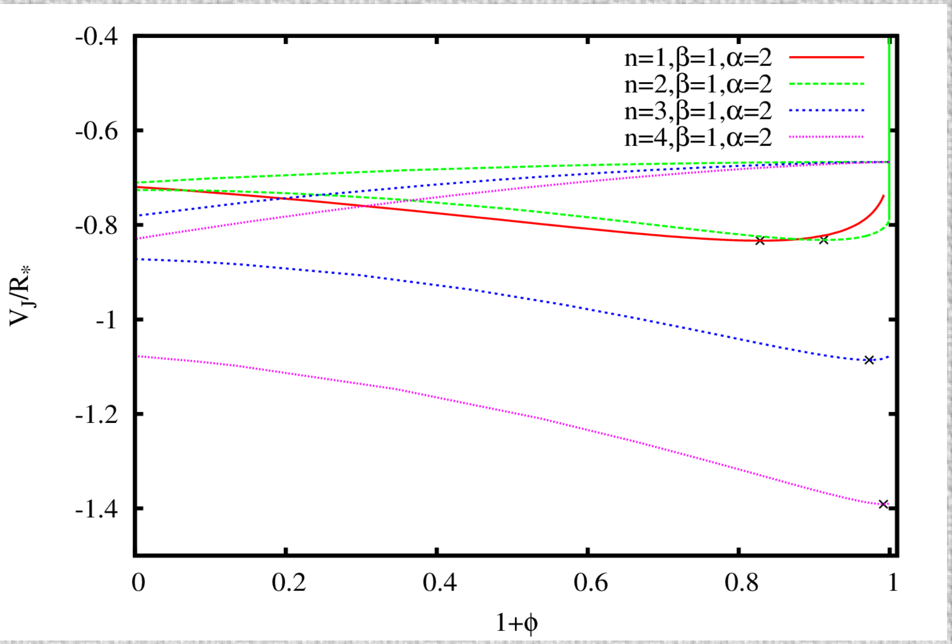
- Curvature Singularity in General Model

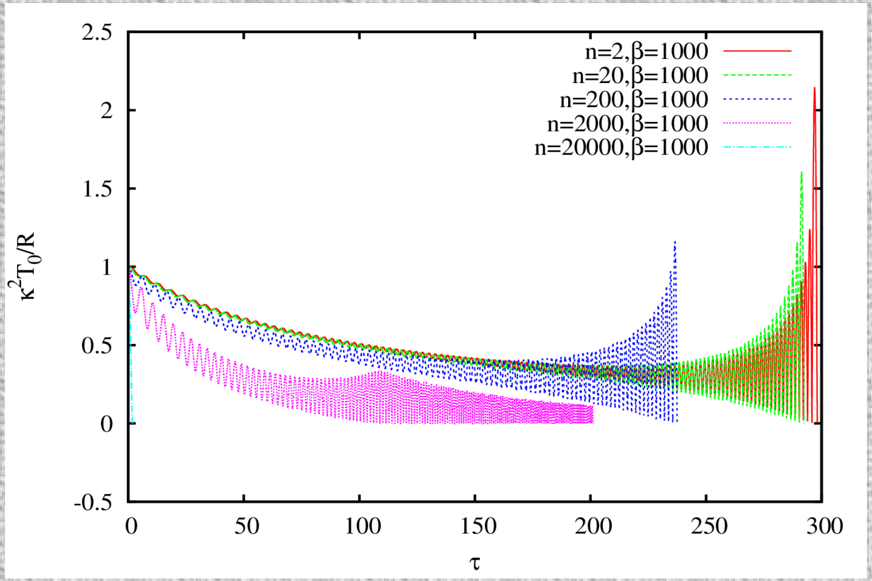




General Model







General Model

n	β	τ_{ch}	t_{ch} (sec)	t_{sing} (sec)
1	1	100	4.0×10^{12}	8.28×10^{15}
2	1	100	9.8×10^{9}	2.05×10^{10}
3	1	100	1.4×10^{6}	1.49×10^{6}
4	1	100	4.4×10^{5}	3.71×10^{5}
2	10	100	2.4×10^{14}	6.48×10^{14}
2	100	100	1.23×10^{15}	3.63×10^{15}
2	1000	100	1.25×10^{15}	3.73×10^{15}
2	10000	100	1.25×10^{15}	3.73×10^{15}
2	100000	100	1.25×10^{15}	3.73×10^{15}
20	1000	100	4.3×10^{15}	1.26×10^{16}
200	1000	100	4.8×10^{15}	1.14×10^{16}
2000	1000	100	$2.4 \times 10^{11.5}$	$4.82 \times 10^{11.5}$
20000	1000	100	6.4×10^{-33}	1.13×10^{-32}

t_{sing} : time singularity is reached within.

t_U (age of the universe) : 4×10^{17} sec

Summary

- The curvature singularity is kinetically possible during the evolution of the scalar field about the minimum of the potential even in the most promising f(R) models.
- In the general model, we found that large values of β can push the singularity away from the de-sitter point and towards higher potential. In order to satisfy fifth-force constraint, n has to be > 2 . But, large n increases the possibility to hit the singularity as in that case de-sitter point shifts closer to singularity. Thus, it is extremely difficult to satisfy fifth-force constraint and avoid the curvature singularity simultaneously. This makes existing f(R) models non-viable.
- In Astrphysical object, the timescale within which the singularity occurs is comparable to age of the universe only for very small n/β ratio.

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...Thank

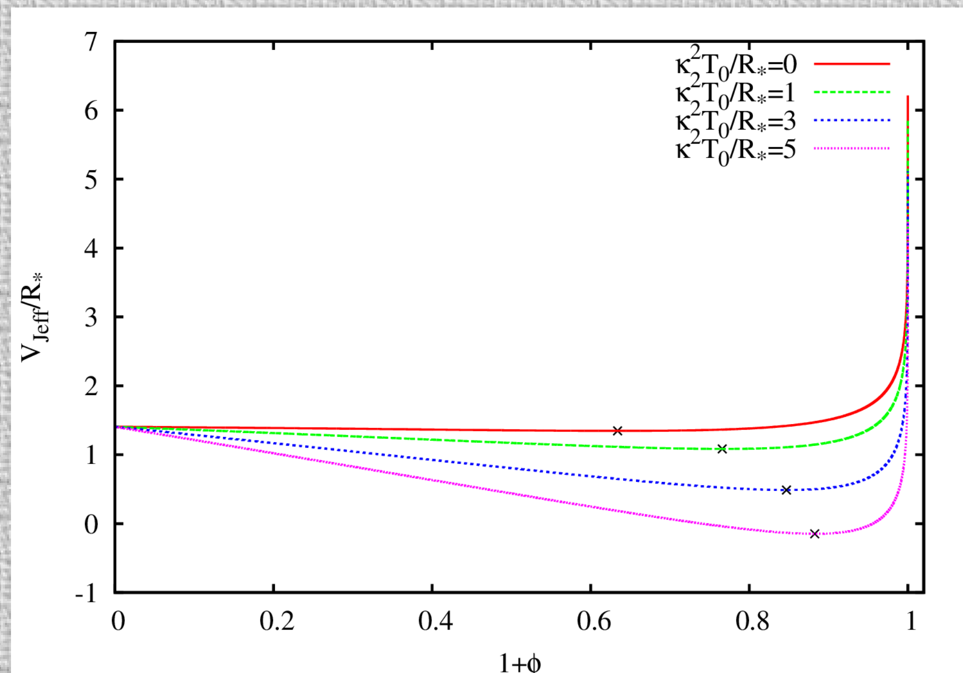
you......

Log Model

- $$f(R) = R - \alpha R_* \ln\left(1 + \frac{R}{R_*}\right)$$
- $$R \to \infty \Rightarrow \phi \to 0$$
- $$\phi = -\frac{\alpha}{1 + \frac{R}{R_*}}$$
- $$\frac{V_J(\phi)}{R_*} = \frac{2}{3}\phi + \frac{2}{3}\alpha\phi - \frac{2}{3}\alpha \ln\phi - \frac{2}{3}\alpha\phi \ln\left(\frac{\phi}{\alpha}\right)$$
- $$\phi_{sing} = 0$$
- As shown in [Miranda], $V_J(\phi) \to \infty$ as $\phi \to \phi_{sing}$.
- Lets take matter into consideration.
- $$\frac{V_J^{eff}(\phi, t)}{R_*} = \frac{2}{3}\phi + \frac{2}{3}\alpha\phi - \frac{2}{3}\alpha \ln\phi - \frac{2}{3}\alpha\phi \ln\left(\frac{\phi}{\alpha}\right) + \frac{\kappa^2}{3}\phi T_0 \left(1 + \frac{t}{t_{ch}}\right)$$

Log Model

- Plotting $V_J^{eff}(\phi)/R_* Vs. 1+\phi$, we obtain



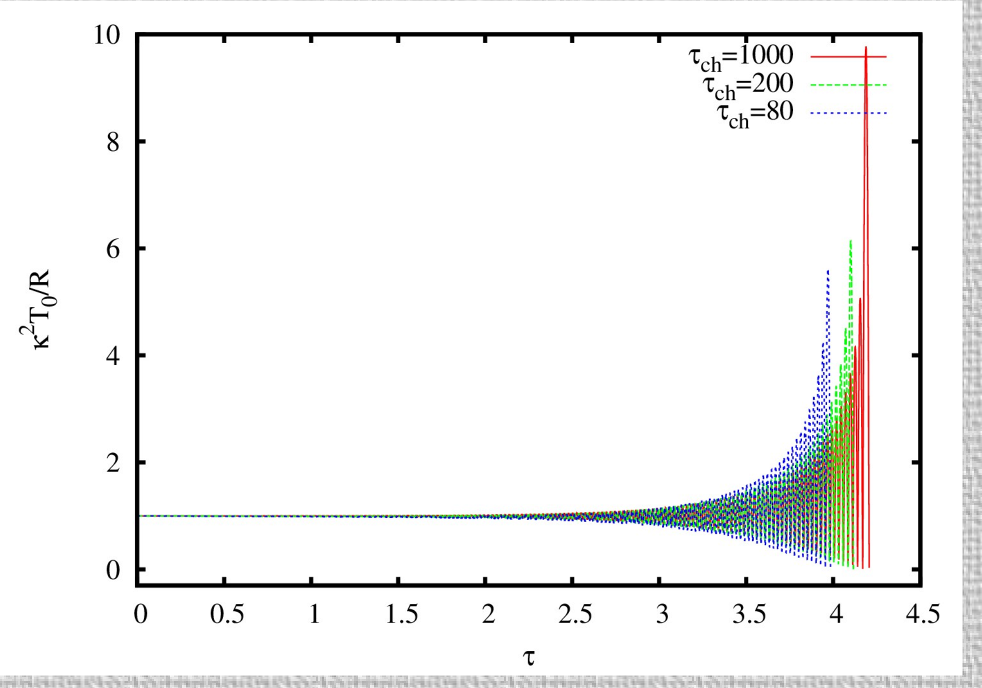
t=0.1 t_{ch}

- Solving the trace equation in terms of dimensionless variables

$y = \frac{\kappa^2 T_0}{R}$ and $\tau = \sqrt{\frac{R_*^2}{3\kappa^2 T_0}} t$ $\qquad \eta = \frac{\kappa^2 T_0}{R_*}$

$$y'' - 2ln(y) - \frac{\eta}{\alpha}\frac{1}{y} - (1 - 2ln(\eta)) + \frac{\eta}{\alpha}\left(1 + \frac{\tau}{\tau_{ch}}\right) = 0$$

Log Model



τ_{ch}=1000
τ_{ch}=200
τ_{ch}=80
$\kappa^2 T_0/R$
τ
0
2
4
6
8
10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5