

# Electron beam dynamics in storage rings

## Synchrotron radiation and its effect on electron dynamics

Lecture 1: Synchrotron radiation

Lecture 2: Undulators and Wigglers

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**Summary**

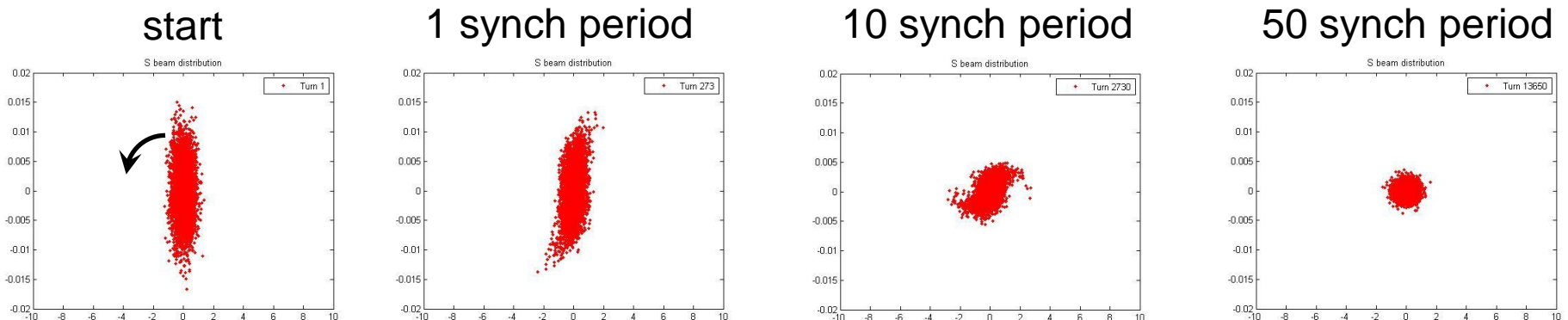
# From the lecture on radiation damping

We have seen that the emission of synchrotron radiation induces a damping of the betatron and synchrotron oscillations; the radiation damping times can be summarized as

$$\frac{1}{\tau_i} = \frac{J_i U_0}{2E_0 T_0} \propto \gamma^3$$

$J_i$  are the damping partition numbers

One would expect that all particle trajectories would collapse to a single point (the origin of the phase space, i.e. 6D the closed orbit). This does not happen because of the quantum nature of synchrotron radiation



**Tracking example: synchrotron period 273 turns, radiation damping of 6000 turns:**

# Quantum nature of synchrotron emission

The radiated energy is emitted in quanta: each quantum carries an energy  $u = \hbar\omega$ ;

The emission process is instantaneous and the time of emission of individual quanta are statistically independent;

The distribution of the energy of the emitted photons can be computed from the spectral distribution of the synchrotron radiation;

The emission of a photon changes suddenly the energy of the emitting electron and perturbs the orbit inducing synchrotron and betatron oscillations.

These oscillations grow until reaching an equilibrium when balanced by radiation damping

Quantum excitation prevents reaching zero emittance in both planes with pure damping.

# From the lecture on synchrotron radiation

## Total radiated power

$$P = \frac{e^2 c}{6\pi\epsilon_0} \frac{\gamma^4}{\rho^2}$$

## Frequency distribution of the power radiated

$$\frac{dI}{d\omega} = \frac{\sqrt{3}e^2\gamma}{4\pi\epsilon_0 c} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x) dx = \frac{2e^2\gamma}{9\epsilon_0 c} S(\xi)$$

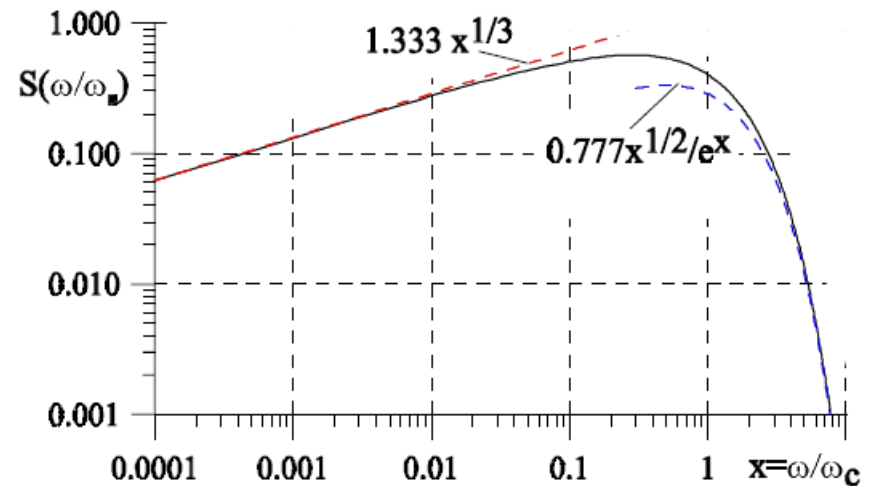
$$\xi = \frac{\omega\rho}{3c\gamma^3} (1 + \gamma^2\theta^2)^{3/2}$$

## Critical frequency

$$\omega_c = \frac{3}{2} \frac{c}{\rho} \gamma^3$$

## Critical angle at the critical frequency

$$\theta_c = \frac{1}{\gamma}$$



# Energy distribution of photons emitted by synchrotron radiation (I)

Energy is emitted in quanta: each quantum carries an energy  $u = \hbar\omega$

From the frequency distribution of the power radiated  $\frac{dP}{d\omega} = \frac{P_\gamma}{\omega_c} S\left(\frac{u}{u_c}\right)$

We can get the energy distribution of the photons emitted per second:

$n(u)$  number of photons emitted per unit time with energy in  $u, u+du$

$u \cdot n(u)$  energy of photons emitted per unit time with energy in  $u, u+du$

$u \cdot n(u)$  must be equal to the power radiated in the frequency range  $du/\hbar$  at  $u/\hbar$

$$u \cdot n(u) du = \frac{dP(u/\hbar)}{du/\hbar} du/\hbar$$

$$n(u) = \frac{1}{\hbar u} \frac{dP}{du/\hbar} \left( \frac{u}{u_c} \right)$$

$$n(u) = \frac{P_\gamma}{u_c^2} \frac{u_c}{u} S\left(\frac{u}{u_c}\right)$$

$$u_c = \hbar\omega_c$$

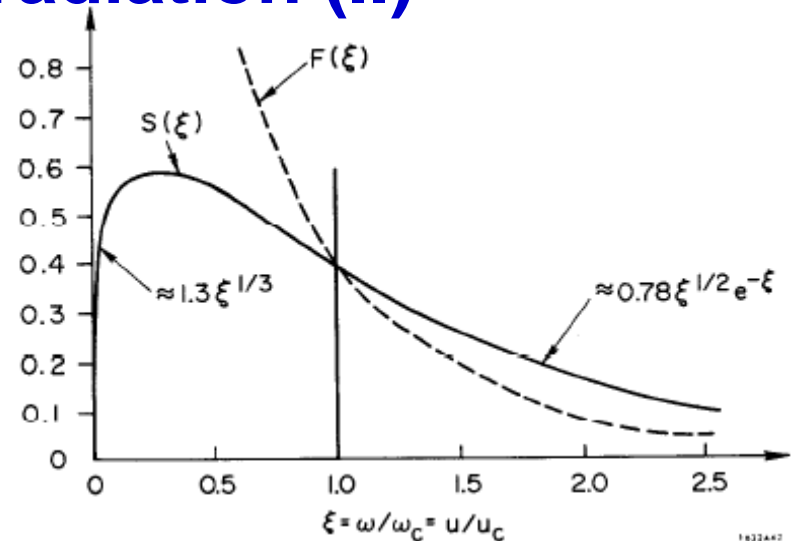
# Energy distribution of photons emitted by synchrotron radiation (II)

Introducing the function  $F(\xi)$

$$F(\xi) = \frac{1}{\xi} S(\xi)$$

we have

$$n(u) = \frac{P_\gamma}{u_c^2} F\left(\frac{u}{u_c}\right)$$



Using the energy distribution of the rate of emitted photons one can compute:

**Total number of photons emitted per second**

$$N_\gamma = \int_0^\infty n(u) du = \int_0^\infty \frac{P_\gamma}{u_c^2} F\left(\frac{u}{u_c}\right) du = \frac{P_\gamma}{u_c} \int_0^\infty F(\xi) d\xi = \frac{15\sqrt{3}}{8} \frac{P_\gamma}{u_c}$$

**Mean energy of photons emitted per second**

$$\langle u \rangle = \frac{1}{N_\gamma} \int_0^\infty u \cdot n(u) du = \frac{8}{15\sqrt{3}} u_c \approx 0.32 u_c$$

**Mean square energy of photons emitted per second**

$$\langle u^2 \rangle = \frac{1}{N_\gamma} \int_0^\infty u^2 n(u) du = \frac{11}{27} u_c^2 \approx 0.41 \cdot u_c^2$$

[See Sands]

# Quantum fluctuations in energy oscillations (IV)

Let us consider again the change in the invariant for linearized synchrotron oscillations

$$A^2 = \varepsilon^2 + \left( \frac{U_s \omega_s}{\alpha} \right)^2 \tau^2$$

After the emission of a photon of energy  $u$  we have

$$\varepsilon \rightarrow \varepsilon - u \quad \tau \rightarrow \tau$$

The time position  $\tau$  w.r.t. the synchronous particle does not change

$$dA^2 = -2\varepsilon u + u^2$$

We do not discard the  $u^2$  term since it is a random variable and its average over the emission of  $n(u)du$  photons per second is not negligible anymore.

**Notice that now also the Courant Snyder invariant becomes a random variable!**



# Quantum fluctuations in energy oscillations (II)

We want to compute the average of the random variable A over the distribution of the energy of the photon emitted

$$\langle dA^2 \rangle_p = -2\varepsilon N_\gamma \langle u \rangle_p - N_\gamma \langle u^2 \rangle_p$$

Quantum excitation

Radiation damping

We have to compute the averages of  $u$  and  $u^2$  over the distribution  $n(u)du$  of number of photons emitted per second.

As observed the term with the square of the photon energy (wrt to the electron energy) is not negligible anymore

# Quantum fluctuations in energy oscillations (VI)

Following [Sands] the excitation term can be written as

$$N_\gamma \langle u^2 \rangle_p = \int_0^\infty u^2 n(u) du = \frac{55}{24\sqrt{3}} r_0 \hbar m c^4 \frac{\gamma^7}{\rho^3}$$

and depends on the location in the ring. We must average over the position in the ring, by taking the integral over the circumference.

$$\langle N_\gamma \langle u^2 \rangle_p \rangle = \frac{55}{24\sqrt{3}} r_0 \hbar m c^4 \gamma^7 \frac{1}{\rho^2 R}$$

The contribution from the term linear in  $u$ , after the average over the energy distribution of the photon emitted, and the average around the ring reads

$$\langle N_\gamma \langle u \rangle_\gamma \rangle_{\Delta t=T_0} = \oint N_\gamma \langle u \rangle_\gamma \frac{ds}{c} = N_\gamma T_0 \langle u \rangle_\gamma = U = \sum_{\text{ring}} u = U_0 + \frac{\partial U}{\partial \varepsilon} \varepsilon$$

Using these expressions...

# Quantum fluctuations in energy oscillations (VII)

**The change in the invariant averaged over the photon emission and averaged around one turn in the ring now reads**

$$\langle dA^2 \rangle_p = -2\varepsilon N_\gamma \langle u \rangle_p + N_\gamma \langle u^2 \rangle_p$$

**The change in the invariant still depends on the energy deviation of the initial particle. We can average in phase space over a distribution of particle with the same invariant A. A will become the averaged invariant.**

**The linear term in u generates a term similar to the expression obtained with the radiative damping. We have the differential equation for the average of the longitudinal invariant**

$$\frac{d \langle A^2 \rangle}{dt} = -\frac{2 \langle A^2 \rangle}{\tau_\varepsilon} + \langle N_\gamma \langle u^2 \rangle_p \rangle$$

# Quantum fluctuations in energy oscillations (VIII)

**The average longitudinal invariant decreases exponentially with a damping time  $\tau_\varepsilon$  and reaches an equilibrium at**

$$\langle A^2 \rangle = \frac{\tau_\varepsilon}{2} \langle N_\gamma \langle u^2 \rangle_p \rangle$$

**This remains true for more general distribution of electron in phase space with invariant A (e.g Gaussian)**

**The variance of the energy oscillations is for a Gaussian beam is related to the Courant-Snyder invariant by**

$$\sigma_\varepsilon^2 = \langle \varepsilon^2 \rangle = \frac{\langle A^2 \rangle}{2}$$

# Quantum fluctuations in energy oscillations (IX)

The equilibrium value for the energy spread reads

$$\sigma_\varepsilon^2 = \frac{55}{32\sqrt{3}} \hbar mc^3 \gamma^4 \frac{I_3}{J_\varepsilon I_2} = \frac{55}{32\sqrt{3}} \hbar mc^3 \gamma^4 \frac{I_3}{2I_2 + I_4}$$

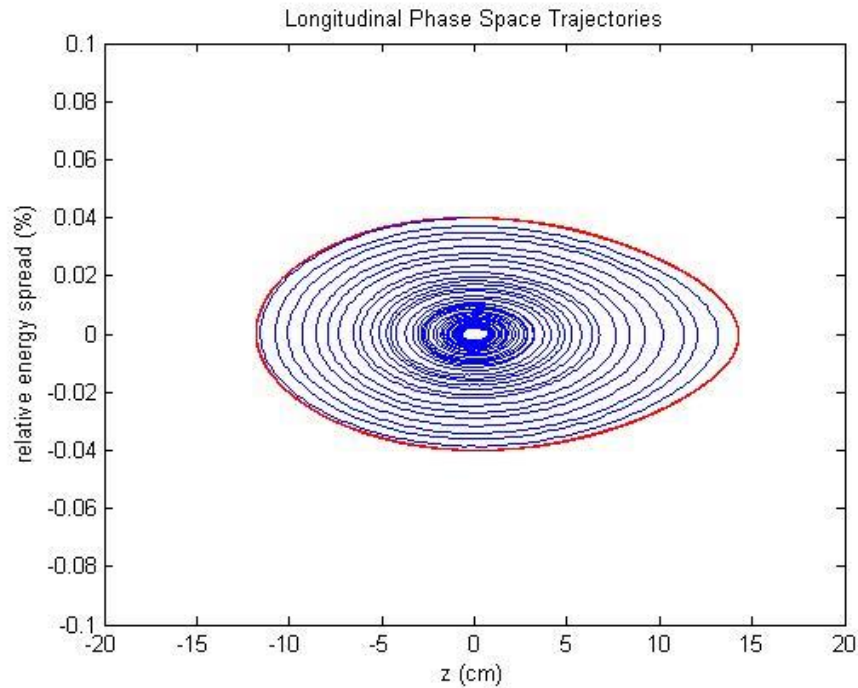
For a synchrotron with separated function magnets

$$\frac{\sigma_\varepsilon^2}{E_0^2} = \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{\rho}$$

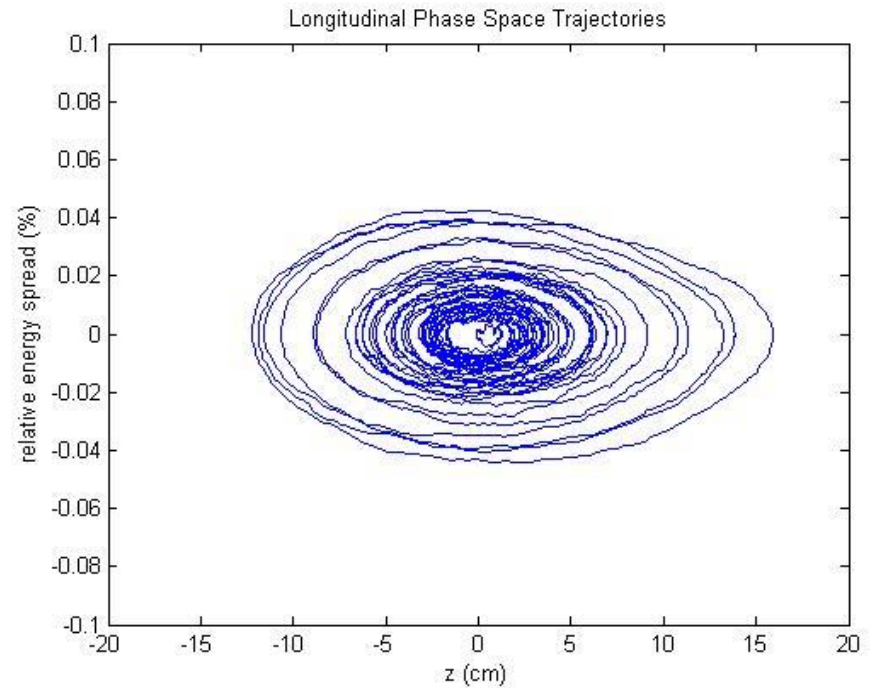
**The relative energy spread depends only on energy and the lattice (namely the curvature radius of the dipoles)**

# A tracking example

**synchrotron period 200 turns; damping time 6000 turns;**



**Diffusion effect off**



**Diffusion effect on**

# Quantum fluctuations in horizontal oscillations (I)

**Invariant for linearized horizontal betatron oscillations**

$$A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

**after the emission of a photon of energy  $u$  we have**

$$\delta x_\beta = -\delta x_\varepsilon = -D_x \frac{u}{U_s} \quad \text{and} \quad \delta x_\beta' = -\delta x_\varepsilon' = -D_x' \frac{u}{U_s}$$

**Neglecting for the moment the linear part in  $u$ , that gives the horizontal damping, the modification of the horizontal invariant reads**

$$dA^2 = (\gamma D_x^2 + 2\alpha D_x D_x' + \beta D_x'^2) \frac{u^2}{U_s^2}$$

**Defining the function**

$$H(s) = \gamma D_x^2 + 2\alpha D_x D_x' + \beta D_x'^2$$

**Dispersion invariant**

**As before we have to compute the effect on the invariant due to the emission of a photon, averaging over the photon distribution, over the betatron phases and over the location in the ring [see Sands]:**

# Quantum fluctuations in horizontal oscillations (II)

**We obtain**

$$\frac{d \langle A^2 \rangle}{dt} = \frac{\langle N_\gamma \langle u^2 \rangle_p H \rangle}{E_0^2}$$

**The linear term in u averaged over the betatron phases gives the horizontal damping**

$$\frac{d \langle A^2 \rangle}{dt} = -\frac{2 \langle A^2 \rangle}{\tau_x}$$

**Combining the two contributions we have the following differential equation for the average of the invariant in the longitudinal plane**

$$\frac{d \langle A^2 \rangle}{dt} = -\frac{2 \langle A^2 \rangle}{\tau_x} + \frac{\langle N_\gamma \langle u^2 \rangle_p H \rangle}{E_0^2}$$



# Quantum fluctuations in horizontal oscillations (III)

At equilibrium

$$\langle A^2 \rangle = \frac{\tau_x}{2} \frac{\langle N_\gamma \langle u^2 \rangle_p H \rangle}{E_0^2}$$

The variance of the horizontal oscillations is

$$\sigma_x^2 = \langle x^2 \rangle = \beta_x \frac{\langle A^2 \rangle}{2} = \beta_x \varepsilon_x$$

Therefore we get the emittance

$$\varepsilon_x = \frac{\langle A^2 \rangle}{2} = \frac{\tau_x}{4E_0^2 L} \oint N_\gamma \langle u^2 \rangle_p H(s) ds$$

The emittance depends on the dispersion function at bendings, where radiation emission occurs

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{J_x} \frac{\langle H / \rho^3 \rangle}{\langle 1 / \rho^2 \rangle}$$

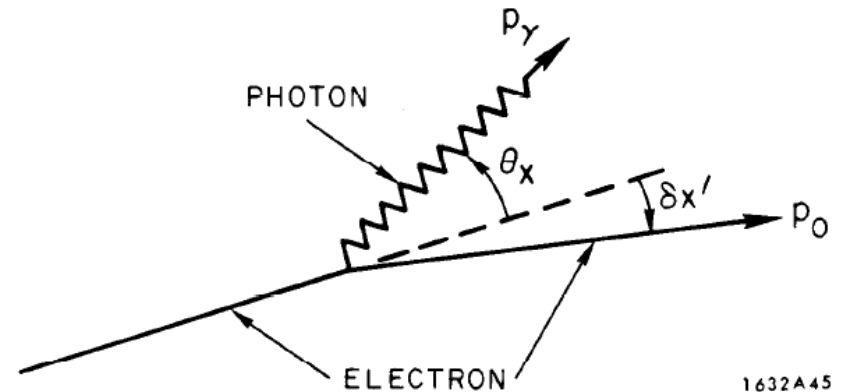
Low emittance lattices strive to minimise  $\langle H / \rho^3 \rangle$  and maximise  $J_x$

# Quantum fluctuations in vertical oscillations (I)

With zero dispersion the previous computation will predict no quantum fluctuations i.e. zero vertical emittance.

However a small effect arises due to the fact that photons are not exactly emitted in the direction of the momentum of the electrons

The electron must recoil to preserve the total momentum



Invariant for linearized vertical betatron oscillations

$$A^2 = \gamma z^2 + 2\alpha z z' + \beta z'^2$$

after the emission of a photon of energy  $u$  the electron angle is changed by

$$\delta z' = \frac{u}{E_0} \theta_z$$

# Quantum fluctuations in vertical oscillations (II)

the modification of the vertical invariant after the emission of a photon reads

$$dA^2 = \frac{u^2}{E_0^2} \theta_z^2 \beta_z(s)$$

Averaging over the photon emission, the betatron phases and the location around the ring:

$$\langle u^2 \theta_z^2 \rangle \approx \langle u^2 \rangle \langle \theta_z^2 \rangle \quad \langle \theta_z^2 \rangle = \frac{1}{2\gamma^2}$$

At equilibrium

$$\langle A^2 \rangle = \frac{\tau_z}{2} \frac{\langle N \langle u^2 \rangle_p \beta_z \rangle}{E_0^2}$$

$$\varepsilon_z = \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} \frac{\langle \beta_z / \rho^3 \rangle}{J_z \langle 1/\rho^2 \rangle}$$

In practice this effect is very small: the vertical emittance is given by vertical dispersion errors and linear coupling between the two planes of motion.

# Related beam quantities: beam size

The horizontal beam size has contributions from the variance of betatron oscillations and from the energy oscillations via the dispersion function:

**Combining the two contributions** we have the bunch size:

$$\sigma_x = \left\{ \varepsilon_x \beta_x(s) + D_x^2(s) \left( \frac{\sigma_\varepsilon}{U_s} \right)^2 \right\}^{1/2}$$

The vertical beam size has contributions from the variance of betatron oscillations but generally not from the energy oscillations ( $D_z = 0$ ).

However the contribution from coupling is usually dominant

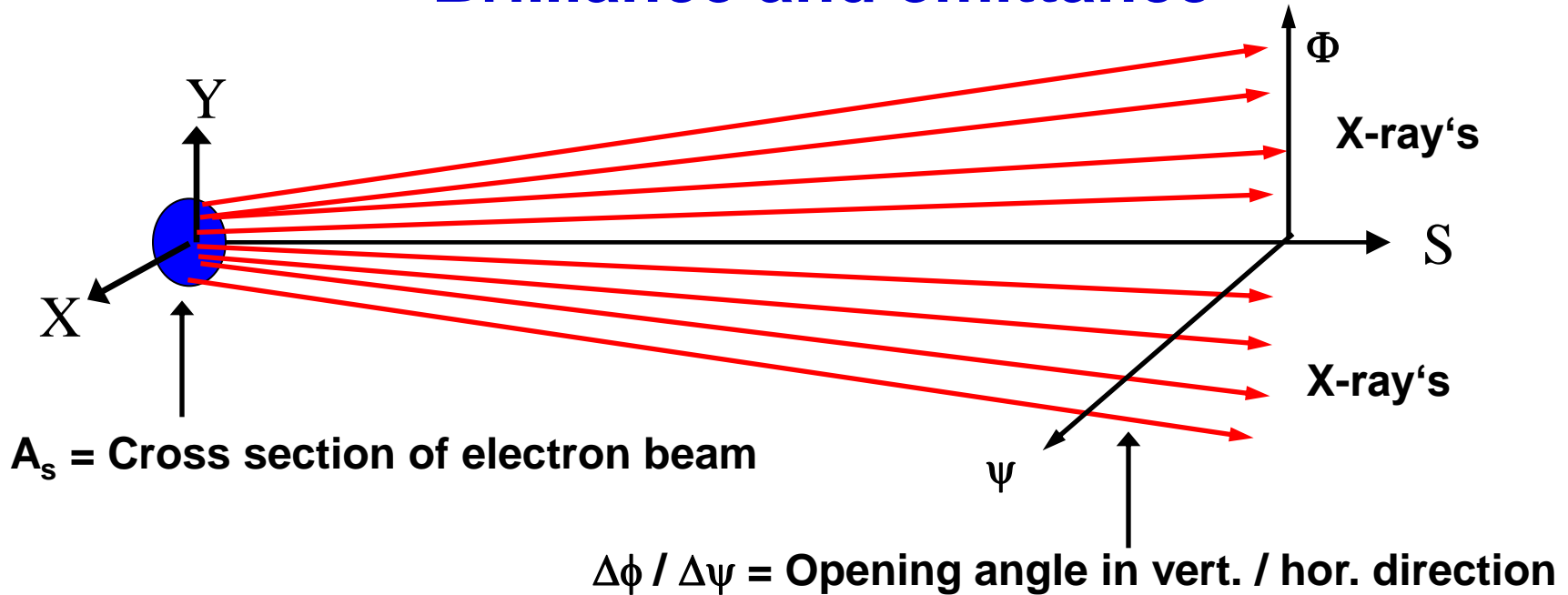
$$\sigma_z = \left( \varepsilon_z \beta_z(s) \right)^{1/2} \quad \varepsilon_z = k \varepsilon_x$$

In 3<sup>rd</sup> generation light sources the horizontal emittance is few nm and the coupling  $k$  is easily controlled to 1% or less, e.g. for Diamond

$$\varepsilon_x = 2.7 \text{ nm}; \quad k = 1\% \rightarrow \quad \varepsilon_y = 27 \text{ pm};$$

$$\sigma_x = 120 \text{ } \mu\text{m} \quad \sigma_y = 6 \text{ } \mu\text{m}$$

# Brilliance and emittance



**Flux = Photons / ( s • BW)**

**Brilliance = Flux / (  $A_s \cdot \Delta\Phi \cdot \Delta\psi$  ) , [ Photons / ( s • mm<sup>2</sup> • mrad<sup>2</sup> • BW )]**

$$\text{brilliance} = \frac{\text{flux}}{4\pi^2 \Sigma_x \Sigma_{x'} \Sigma_y \Sigma_{y'}}$$

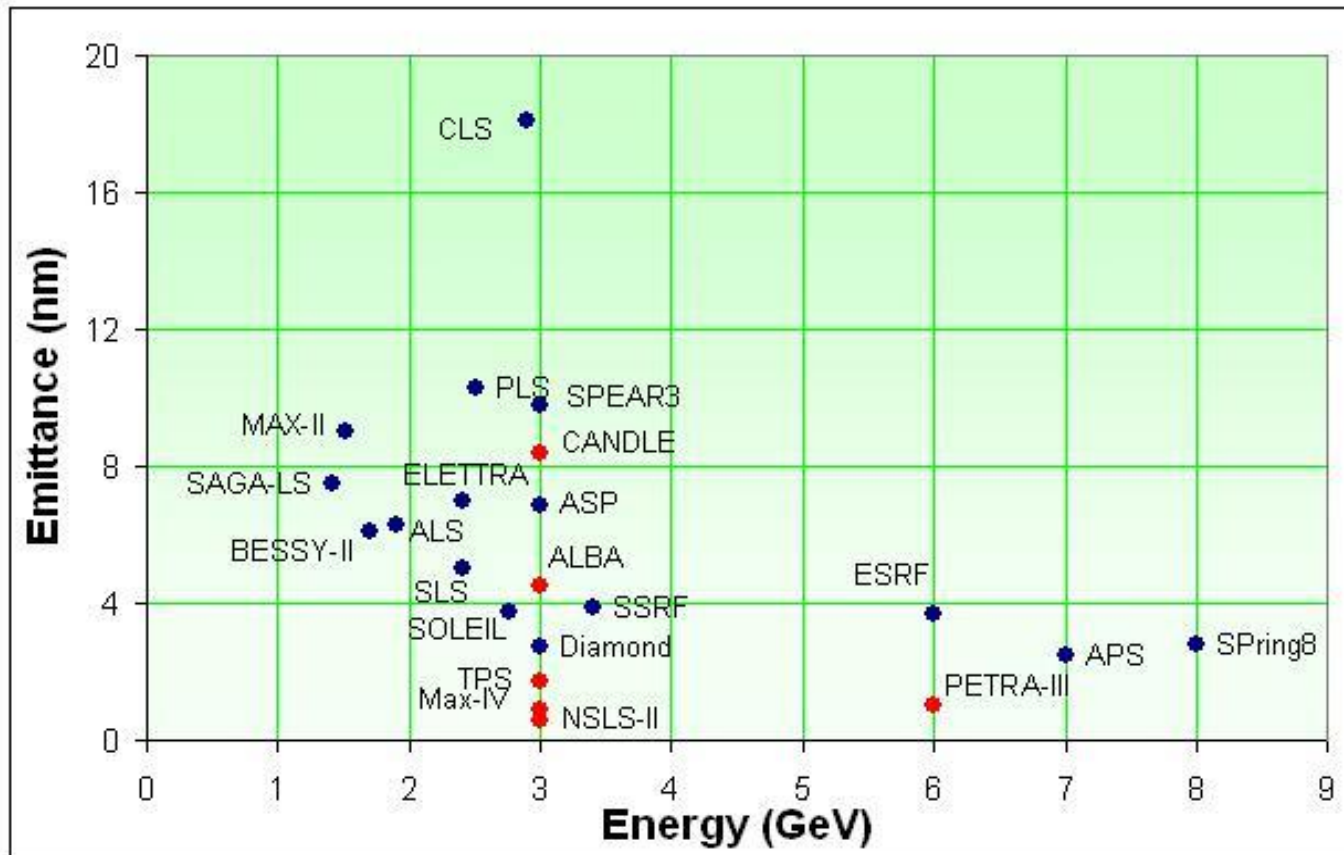
$$\Sigma_x = \sqrt{\sigma_{x,e}^2 + \sigma_{ph,e}^2} \quad \sigma_x = \sqrt{\varepsilon_x \beta_x + (D_x \sigma_\varepsilon)^2}$$

$$\Sigma_{x'} = \sqrt{\sigma_{x',e}^2 + \sigma_{ph,e}'^2} \quad \sigma_{x'} = \sqrt{\varepsilon_x \beta_x + (D'_x \sigma_\varepsilon)^2}$$

**The brilliance represents the number of photons per second emitted in a given bandwidth that can be refocus by a perfect optics on the unit area at the sample.**

# Emittance of third generation light source

The brilliance of the photon beam is determined (mostly) by the electron beam emittance that defines the source size and divergence



# Low Emittance lattices

Lattice design has to provide low emittance **and** adequate space in straight sections to accommodate long Insertion Devices

$$\varepsilon_x = \frac{\gamma^2}{J_x \rho} \langle H \rangle_{\text{dipole}} \quad H(s) = \gamma D^2 + 2\alpha DD' + \beta D'^2$$

Minimise  $\beta$  and  $D$  and be close to a waist in the dipole

Zero dispersion in the straight section was used especially in early machines

avoid increasing the beam size due to energy spread

hide energy fluctuation to the users

allow straight section with zero dispersion to place RF and injection

decouple chromatic and harmonic sextupoles

**DBA and TBA lattices provide low emittance with large ratio between**

$$\frac{\text{Length of straight sections}}{\text{Circumference}}$$

Flexibility for optic control for apertures (injection and lifetime)

# Low emittance lattices

Low emittance and adequate space in straight sections to accommodate long Insertion Devices are obtained in

Double Bend Achromat (DBA)

Triple Bend Achromat (TBA)

DBA used at:

ESRF,  
ELETTRA,  
APS,  
SPring8,  
Bessy-II,  
Diamond,  
SOLEIL,  
SPEAR3

...

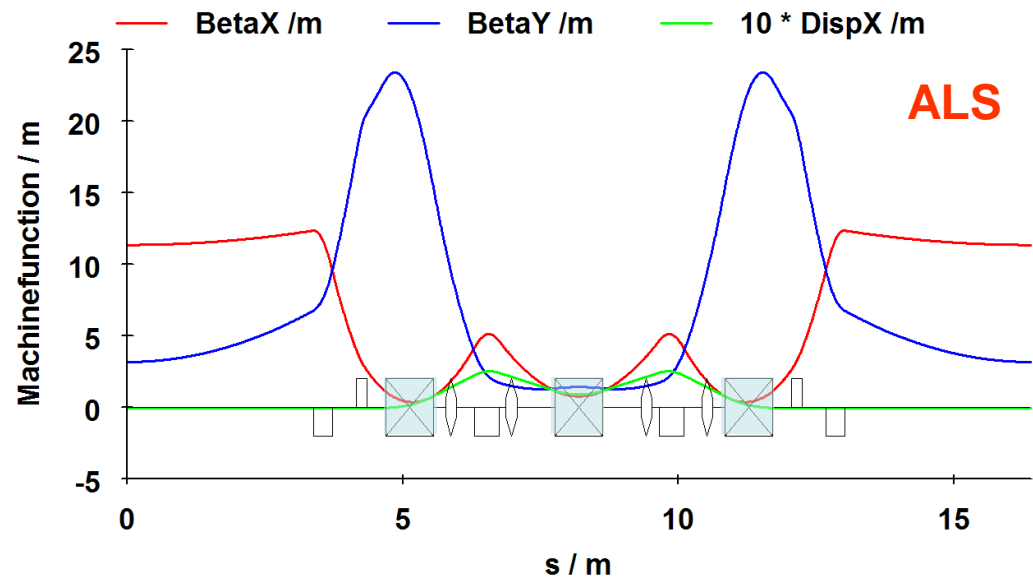
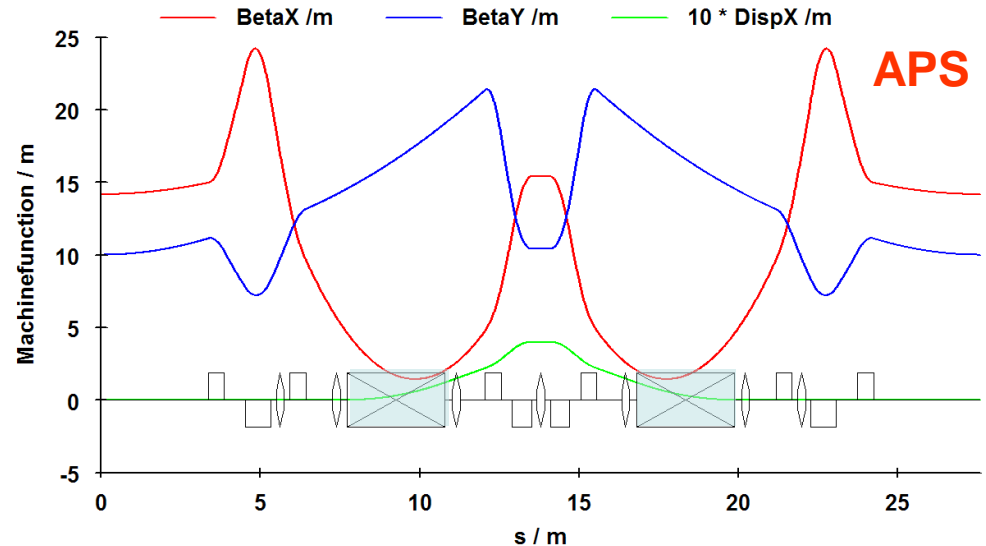
TBA used at

ALS,  
SLS,  
PLS,  
TLS

...

$$\varepsilon_x = F \frac{C_q \gamma^2 \theta_b^3}{J_x} \propto \frac{1}{N_b^3}$$

$$F_{MEDBA} = \frac{1}{4\sqrt{15}} \quad F_{MEDBA-disp} = \frac{1}{12\sqrt{15}}$$





# Low emittance lattices

The original achromat design can be broken, leaking dispersion in the straight section

ESRF	7 nm $\rightarrow$ 3.8 nm
APS	7.5 nm $\rightarrow$ 2.5 nm
SPring8	4.8 nm $\rightarrow$ 3.0 nm
SPEAR3	18.0 nm $\rightarrow$ 9.8 nm
ALS (SB)	10.5 nm $\rightarrow$ 6.7 nm

New designs envisaged to achieve sub-nm emittance involve

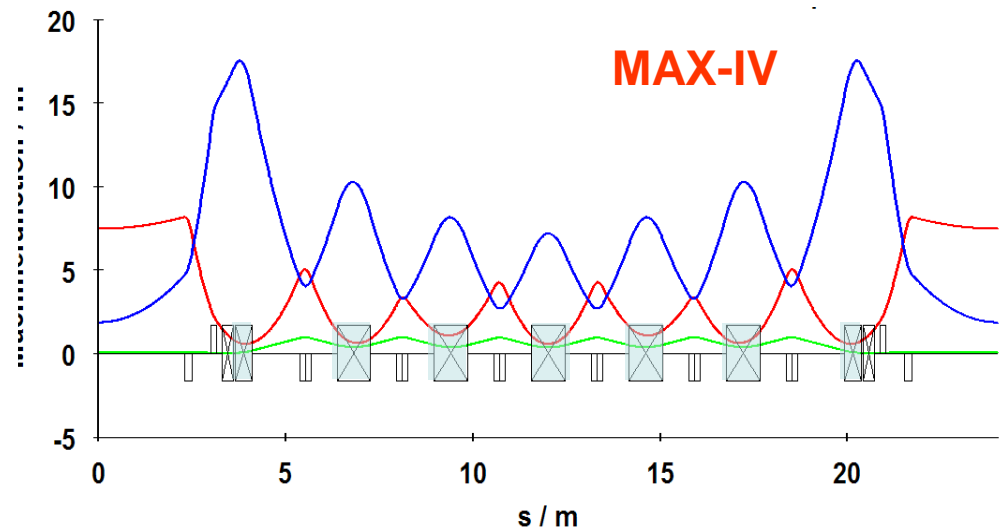
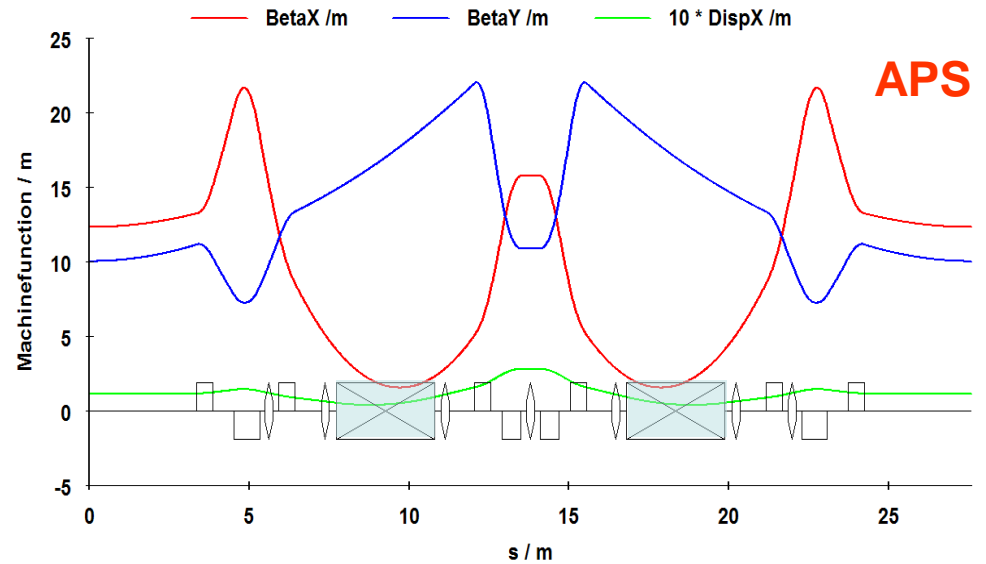
MBA

MAX-IV (7-BA)

Damping Wigglers

NSLS-II

Petra-III



# Related beam quantities: bunch length

## Bunch length from energy spread

$$\sigma_z = \frac{\alpha c}{2\pi f_s} \sigma_\varepsilon \propto \sqrt{\frac{\alpha \gamma^3}{dV_{RF}/dz}}$$

The bunch length also depends on RF parameters: voltage and phase seen by the synchronous particle

$$\alpha = 1.7 \cdot 10^{-4}; V = 3.3 \text{ MV}; \sigma_\varepsilon = 9.6 \cdot 10^{-4} \rightarrow \sigma_z = 2.8 \text{ mm (9.4 ps)}$$

$\sigma_z$  depends on

the magnetic lattice (quadrupole magnets) via  $\alpha$

the RF slope

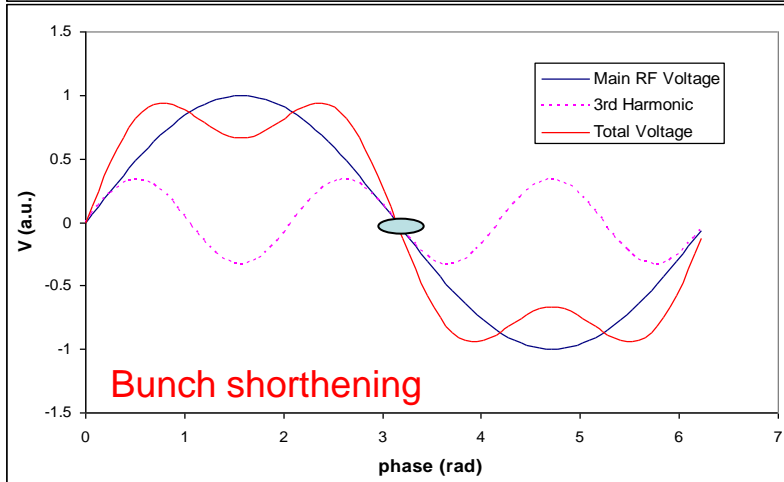
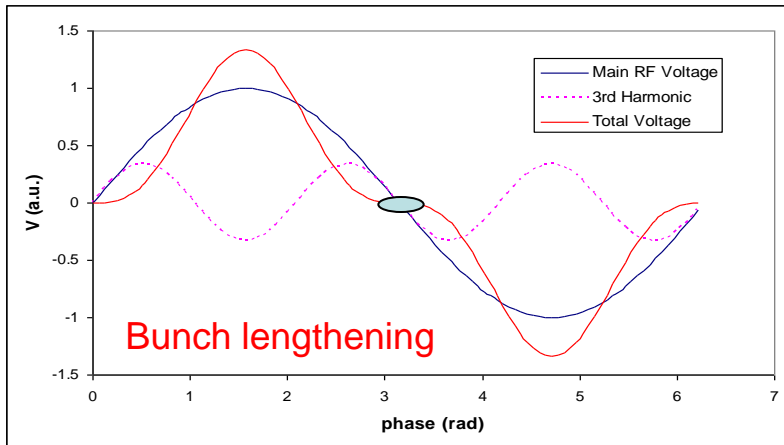
Shorten/Lengthen bunches increasing the RF slope at the bunch (Harmonic cavities)

Shorten bunches decreasing  $\alpha$   
(low-alpha optics)

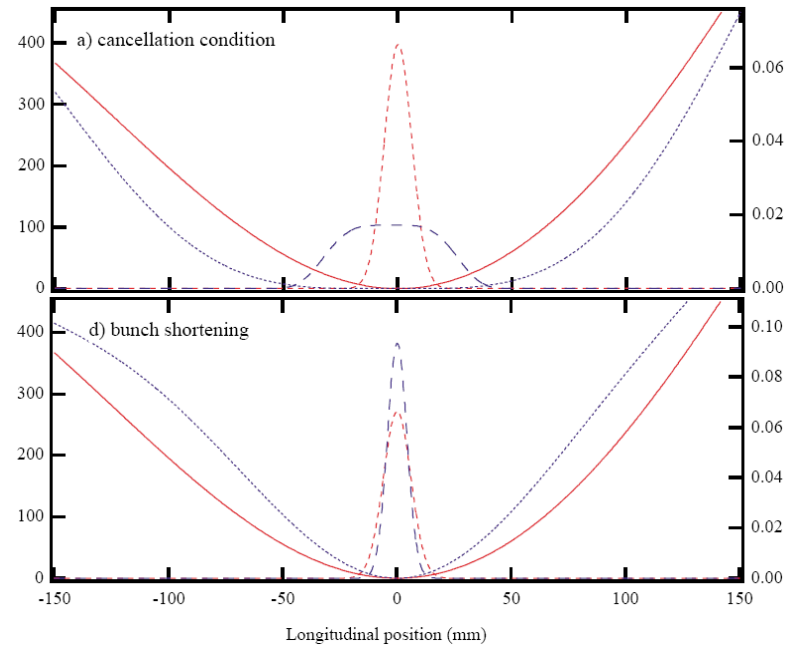
$$\alpha = \frac{1}{L} \oint \frac{D_x}{\rho} ds \approx 10^{-6}$$

# bunch length manipulation: harmonic cavities

RF cavities with frequency equal to an harmonic of the main RF frequency (e.g. 3<sup>rd</sup> harmonic) are used to lengthen or shorten the bunch



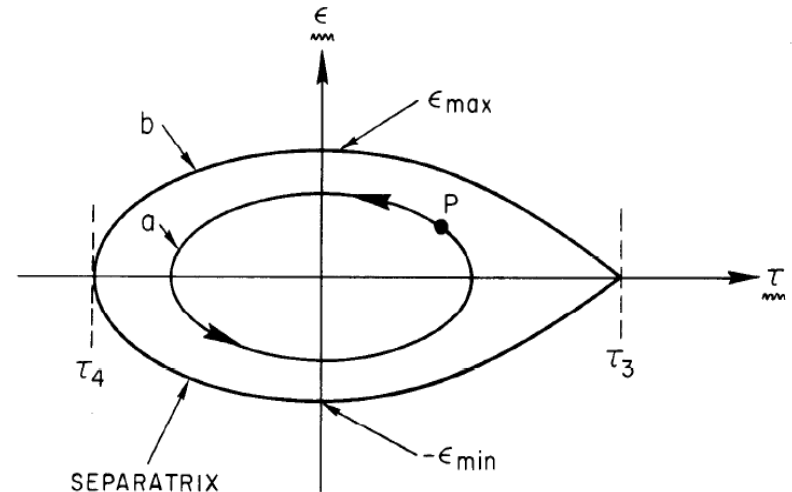
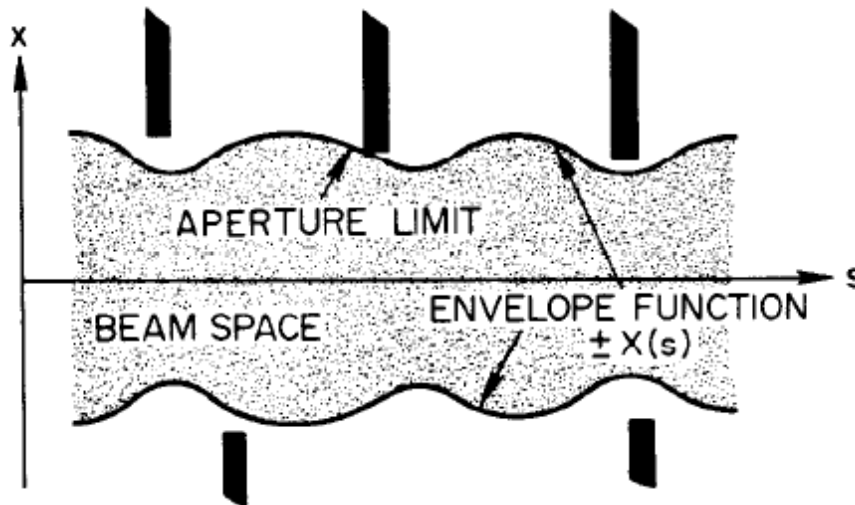
$$\sigma_z = \frac{\alpha c}{2\pi f_s} \sigma_\varepsilon \propto \sqrt{\frac{\alpha \gamma^3}{dV_{RF}/dz}}$$



# Quantum lifetime (I)

Electrons are continuously stirred by the emission of synchrotron radiation photons

It may happen that the induced oscillations hit the vacuum chamber or get outside the RF aperture:



The number of electron per second whose amplitudes exceed a given aperture and is lost at the wall or outside the RF bucket can be estimated from the equilibrium beam distribution [see Sands]

# Quantum lifetime (II)

$$\frac{dN}{dt} = -\frac{N}{\tau_q}$$

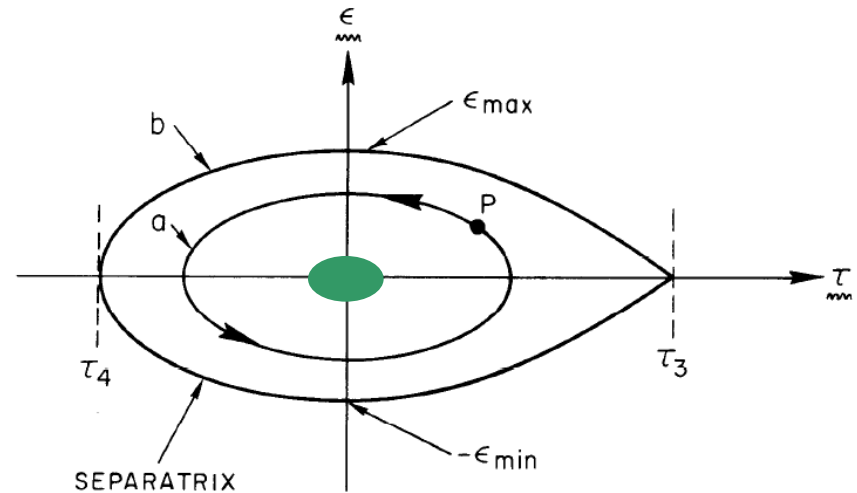
**Exponential decay of the number of particle stored**

$$\xi = \frac{x_{\max}^2}{2\sigma_x^2} \quad \tau_q = \frac{\tau_x}{2} \frac{\exp(\xi)}{\xi} \quad \text{quantum lifetime for losses in the transverse plane}$$

$$\xi = \frac{\varepsilon_{\max}^2}{2\sigma_\varepsilon^2} \quad \tau_q = \frac{\tau_\varepsilon}{2} \frac{\exp(\xi)}{\xi} \quad \text{quantum lifetime for losses in the longitudinal plane}$$

**Given the exponential dependence on the ratio between available aperture and beam size the quantum lifetime is typically very large for modern synchrotron light sources, e.g. Diamond**

$$\xi = \frac{\varepsilon_{\max}^2}{2\sigma_\varepsilon^2} = \frac{(0.04)^2}{2 \cdot (0.001)^2} = 800$$



# Summary

**The emission of synchrotron radiation occurs in quanta of discrete energy**

**The fluctuation in the energy of the emitted photons introduce a noise in the various oscillation modes causing the amplitude to grow**

**Radiation excitation combined with radiation damping determine the equilibrium beam distribution and therefore emittance, beam size, energy spread and bunch length.**

**The excitation process is responsible for a loss mechanism described by the quantum lifetime**

**The emittance is a crucial parameter in the operation of synchrotron light source. The minimum theoretical emittance depends on the square of the energy and the inverse cube of the number of dipoles**