

Lecture 4 - Transverse Optics II

ACCELERATOR PHYSICS

MT 2014

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- ◆ Vertical Focusing
- ◆ Cosmotron
- ◆ Weak focusing in a synchrotron
- ◆ The “n-value”
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Lecture 4 - Transverse Optics II

Contents

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- ◆ **Solving for a ring**
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Equation of motion in transverse co-ordinates

☿ Hill's equation (linear-periodic coefficients)

$$\frac{d^2 y}{ds^2} + k(s)y = 0$$

– where $k = -\frac{1}{(B\rho)} \frac{dB_z}{dx}$ at quadrupoles

– like restoring constant in harmonic motion

☿ Solution (e.g. Horizontal plane)

$$y = \sqrt{\beta(s)} \sqrt{\varepsilon} \sin [\phi(s) + \phi_0]$$

☿ Condition

$$\phi = \int \frac{ds}{\beta(s)}$$

☿ Property of machine $\sqrt{\beta(s)}$

☿ Property of the particle (beam) ε

☿ Physical meaning (H or V planes)

Envelope $\sqrt{\varepsilon \beta(s)}$

Maximum excursions

$$\hat{y} = \sqrt{\varepsilon \beta(s)}$$

$$\hat{y}' = \sqrt{\varepsilon / \beta(s)}$$

Check Solution of Hill

- ◆ Differentiate substituting $y = \sqrt{\beta(s)} \varepsilon \cos(\phi(s) + \phi_o)$
 $w = \sqrt{\beta}$, $\phi = \phi(s) + \phi_o$

$$y' = \varepsilon^{1/2} \left\{ w'(s) \cos \phi - \frac{d\phi}{ds} w(s) \sin \phi \right\}$$

- ◆ Necessary condition for solution to be true

$$\frac{d\phi}{ds} = \frac{1}{\beta(s)} = \frac{1}{w^2(s)}$$

- ◆ Differentiate again $y' = \varepsilon^{1/2} \left\{ w'(s) \cos \phi - \frac{1}{w(s)} \sin \phi \right\}$

$$y'' = \varepsilon^{1/2} \left\{ w''(s) \cos \phi - \frac{w'(s)}{w^2(s)} \sin \phi + \frac{w'(s)}{w^2(s)} \sin \phi \right\}$$

add both sides $\frac{1}{w^3(s)} \cos \phi$

+ ky + $k w(s) \cos \phi$

cancels to 0

must be zero 0

Continue checking

$$y'' = \varepsilon^{1/2} \left\{ w''(s) \cos \phi - \frac{w'(s)}{w^2(s)} \sin \phi + \frac{w'(s)}{w^2(s)} \sin \phi \right\}$$
$$-\frac{1}{w^3(s)} \cos \phi$$
$$+ky$$
$$+kw(s) \cos \phi$$

cancels to 0

must be zero 0

- ◆ The condition that these three coefficients sum to zero is a differential equation for the envelope

$$w''(s) + kw(s) - \frac{1}{w^3(s)} = 0$$

alternatively

$$\frac{1}{2} \beta \beta'' - \frac{1}{4} \beta'^2 + k\beta^2 = 1$$

Twiss Matrix

- ◆ All such linear motion from points 1 to 2 can be described by a matrix like:

$$\begin{pmatrix} y(s_2) \\ y'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = \mathbf{M}_{12} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix}.$$

- ◆ To find elements first use notation $w = \sqrt{\beta}$
- ◆ We know $y = \varepsilon^{1/2} w \cos(\varphi + \phi_0)$
- ◆ Differentiate and remember $\varphi = \frac{1}{\beta} = \frac{1}{w^2}$
- ◆ Trace two rays one starts $\phi = 0$ “cosine”
- ◆ The other starts with $\phi = \pi/2$ “sine”
- ◆ We just plug in the “c” and “s” expression for displacement and divergence at point 1 and the general solutions at point 2 on LHS
- ◆ Matrix then yields four simultaneous equations with unknowns : a b c d which can be solved

Twiss Matrix (continued)

- ◆ Writing $\phi = \phi_2 - \phi_1$
- ◆ The matrix elements are

$$M_{12} = \begin{pmatrix} \frac{w_2}{w_1} \cos \varphi - w_2 w_1' \sin \varphi & w_1 w_2 \sin \varphi \\ -\frac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2} \sin \varphi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1} \right) \cos \varphi & \frac{w_1}{w_2} \cos \varphi + w_1 w_2' \sin \varphi \end{pmatrix}$$

- ◆ Above is the general case but to simplify we consider points which are separated by only one PERIOD and for which

$$w_1 = w_2 = w, \quad w_1' = w_2' = w', \quad \mu = \phi_2 - \phi_1 = 2\pi Q$$

- ◆ The “period” matrix is then

$$M = \begin{pmatrix} \cos \mu - w w' \sin \mu & w^2 \sin \mu \\ -\frac{1 + w^2 w'^2}{w^2} \sin \mu & \cos \mu + w w' \sin \mu \end{pmatrix}$$

- ◆ If you have difficulty with the concept of a period just think of a single turn.

Twiss concluded

$$M = \begin{pmatrix} \cos \mu - w w' \sin \mu & w^2 \sin \mu \\ -\frac{1 + w^2 w'^2}{w^2} \sin \mu & \cos \mu + w w' \sin \mu \end{pmatrix}$$

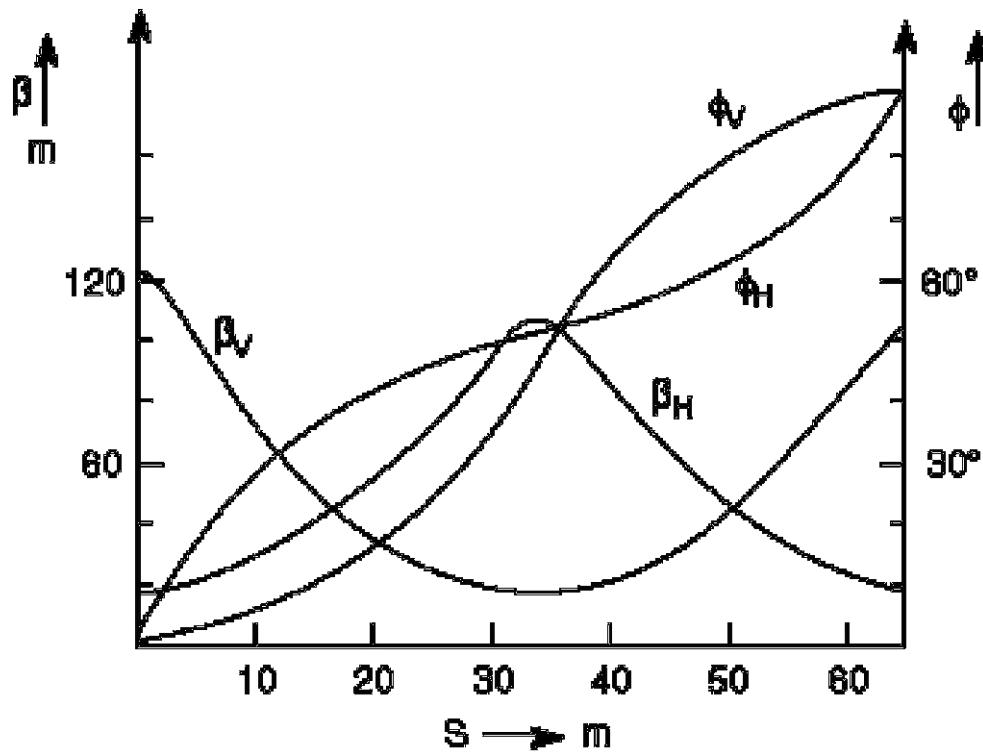
- ◆ Can be simplified if we define the “Twiss” parameters:

$$\boxed{\beta = w^2, \alpha = -\frac{1}{2}\beta', \gamma = \frac{1 + \alpha^2}{\beta}}$$

- ◆ Giving the matrix for a ring (or period)

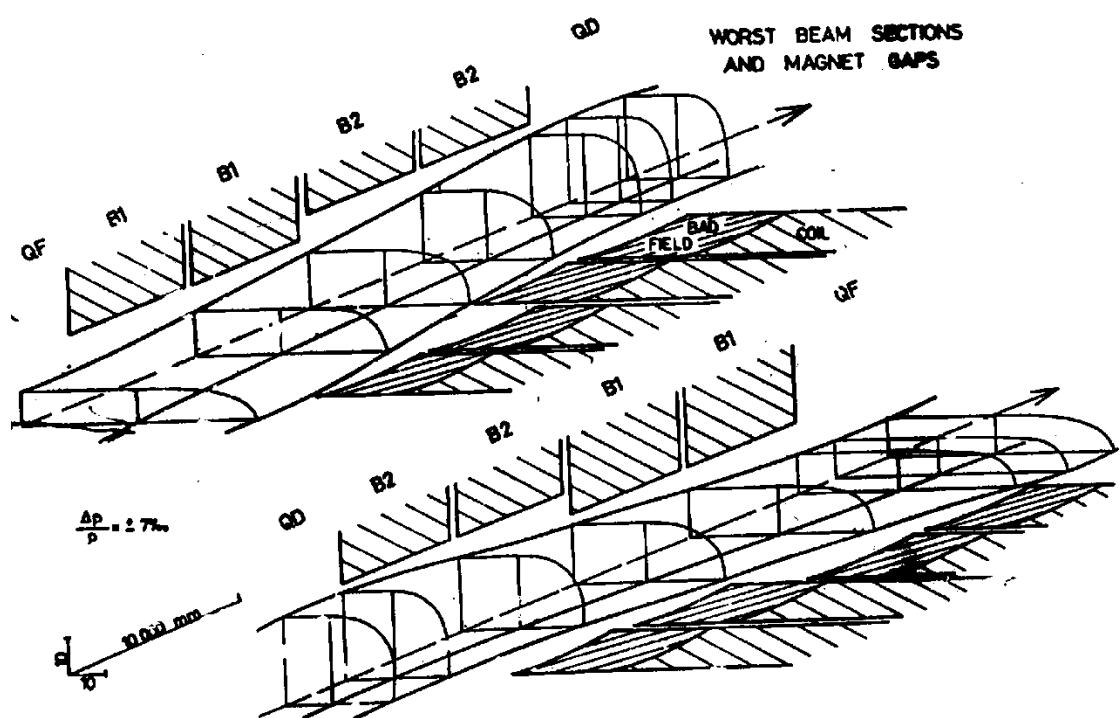
$$\boxed{M = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}}$$

The lattice

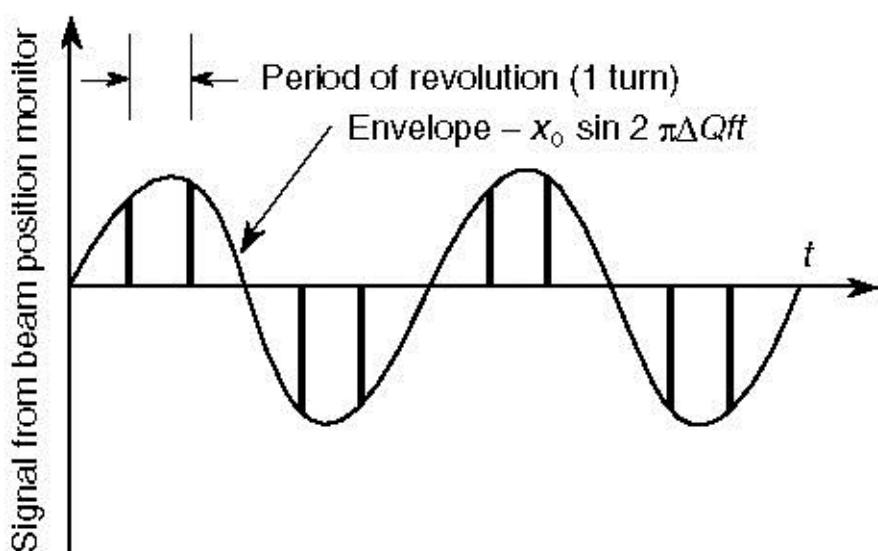
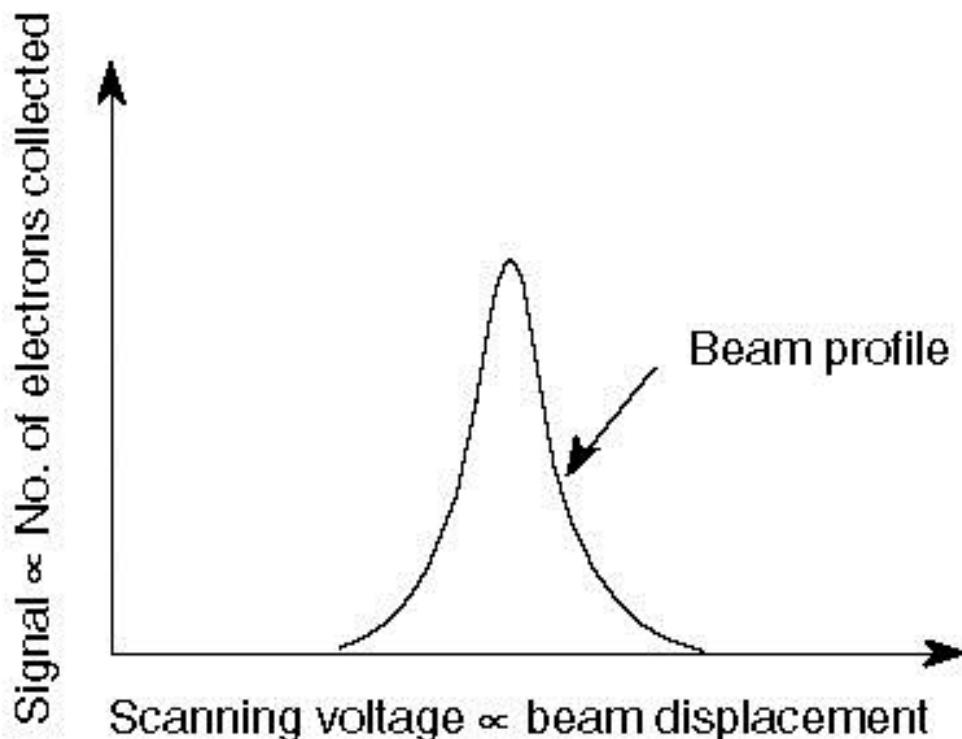


	LENGTH	ANGLE	K(V)	ALPHA(P)	BETA(H)	ALPHA(H)	MUH/2PI	BETA(V)	ALPHA(V)	HUV/2PI	AH/2	AV/2
01	3.085000	0.000000	.015053	1.386440104	.884855	2.452180	.004571	19.011703	.520345	.036571	65.715663	9.917550
02	.360000	0.000000	0.000000	1.374053103	1.279565	2.428089	.005122	19.395014	.544408	.039555	64.547913	10.017039
03	6.260000	.008445	0.000000	1.196124	75.348859	2.009521	.016433	28.628710	.962819	.978198	64.004371	12.212911
04	.400000	0.000000	0.000000	1.186405	73.751941	.988775	.027207	29.659417	.989248	.024377	54.751341	12.376820
05	6.260000	.008445	0.000000	1.060742	51.548094	1.564207	.033474	44.610910	.1.407071	.191988	54.174091	19.192432
06	.390000	0.000000	0.000000	1.054559	50.334182	1.538130	.034692	45.718585	.1.433128	.1.033302	45.428881	15.379447
07	6.260000	.008445	0.000000	1.981762	33.701223	1.119563	.058975	66.274981	.1.850528	.1.01441	44.905056	18.517478
08	.380000	0.000000	0.000000	1.978948	32.860011	1.054184	.060793	67.691002	.1.875896	.1.02344	36.980337	18.713705
09	6.260000	.008445	0.000000	1.959017	21.781569	.675586	.098381	93.787576	.2.292753	.1.04861	36.534921	21.028267
10	2.342700	0.000000	0.000000	1.961480	18.883146	.518942	.116788104	.896272	.2.449038	.3.06621	30.069327	21.295624
11	3.385000	0.000000	.015037	1.034384	18.883088	.518916	1.43368104	.901620	.2.447388	.1.04191	28.349412	23.165625
12	.350000	0.000000	0.000000	1.050730	19.354500	.542318	1.46275103	.986811	.2.424667	.1.03726	28.638022	23.296216
13	6.260000	.008445	0.000000	1.370047	28.764399	.900679	1.89011	73.452122	.2.007802	.1.05027	35.089639	23.106121
14	.380000	0.000000	0.000000	1.391035	29.502432	.988287	.191088	73.935882	.1.982463	.1.05836	35.546047	19.767412
15	6.260000	.008445	0.000000	1.763219	44.472640	.1.404847	.218731	51.724094	.1.565610	.1.071975	43.750378	19.557880
16	.390000	0.000000	0.000000	1.788053	45.578591	.1.430924	.220109	50.513067	.1.539589	.1.073189	44.298587	16.358398
17	6.260000	.008445	0.000000	2.213103	56.113699	.1.849484	.238298	33.849177	.1.122280	.1.073777	53.470174	16.166782
18	.400000	0.000000	0.000000	2.241982	57.303985	.1.876229	.239261	32.962034	.1.096579	.1.092243	54.079134	13.233307
19	6.260000	.008445	0.000000	2.797888	93.714254	.2.294790	.251780	21.859390	.677943	.1.067445	63.830251	13.066741
20	2.352700	0.000000	0.000000	2.904240104	.882261	.2.452099	.255568	19.038985	.520847	.1.06140	67.892709	10.634409
21	3.085000	.008445	.015053	2.946010104	.882268	.2.452096	.260189	19.038106	.320546	.1.01673	68.883088	.924676
22	.360000	0.000000	0.000000	2.925443103	1.25421	.2.428027	.260880	19.421591	.544979	.1.04683	67.668869	10.023890
23	6.260000	.008445	0.000000	2.594240	75.347037	.2.009467	.271992	28.854181	.962177	.1.07246	67.105199	12.218305
24	.400000	0.000000	0.000000	2.574765	73.750162	.1.927222	.272884	29.634602	.988674	.1.09424	57.546933	12.382087
25	6.260000	.008445	0.000000	2.298428	51.546933	.1.564192	.289032	44.628208	.1.406185	.1.069527	56.950187	15.195377
26	.390000	0.000000	0.000000	2.280734	50.337057	.1.538085	.290251	45.735180	.1.432204	.1.068331	47.899867	15.382238
27	6.260000	.008445	0.000000	2.055264	33.700512	.1.119525	.314534	66.276882	.1.849098	.1.076466	47.368928	16.517744
28	.380000	0.000000	0.000000	2.043182	32.899428	.1.094117	.318382	57.691805	.1.374435	.1.077389	39.127022	16.713817
29	6.260000	.008445	0.000000	1.870577	21.781395	.675587	.353941	93.766993	.2.290782	.1.099888	38.663082	22.028838
30	2.342700	0.000000	0.000000	1.851587	19.883101	.518917	.378318104	.865992	.2.448575	.1.036448	31.892336	23.292261
31	3.085000	.008445	.015037	1.873603	18.983178	.518943	.398926104	.862544	.2.447912	.1.098220	30.027986	23.712598

Beam sections



Physical meaning of Q and $\beta\varepsilon\alpha$



Smooth approximation

$$N\mu = 2\pi Q$$

$$\int \frac{ds}{\beta} = \int d\phi$$

$$\frac{2\pi R}{\bar{\beta}} = 2\pi Q$$

$$\therefore \bar{\beta} = \frac{R}{Q}$$

$$\gamma_{tr} \approx Q$$

$$\frac{1}{\gamma_{tr}^2} = \frac{\bar{D}}{R}$$

$$\therefore \bar{D} = \frac{R}{Q^2}$$

Principal trajectories

$$y(s) = C(s)y_0 + S(s)y'_0 + D(s)\frac{\Delta p}{p_0}$$

$$y'(s) = C'(s)y_0 + S'(s)y'_0 + D'(s)\frac{\Delta p}{p_0}$$

$$\begin{pmatrix} C_0 & S_0 \\ C'_0 & S'_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

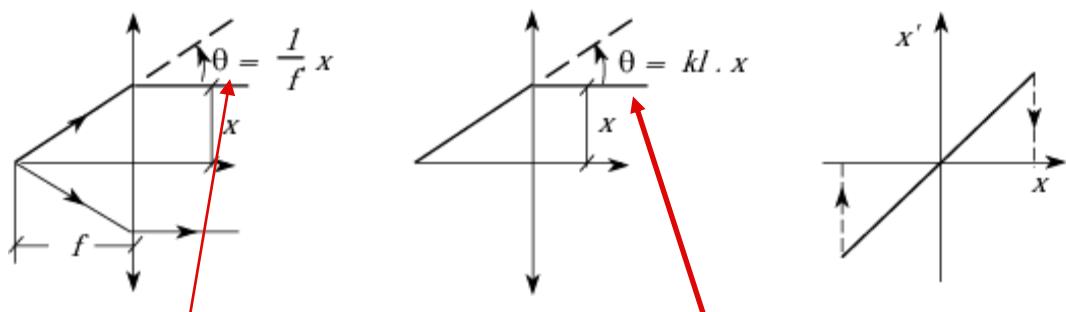
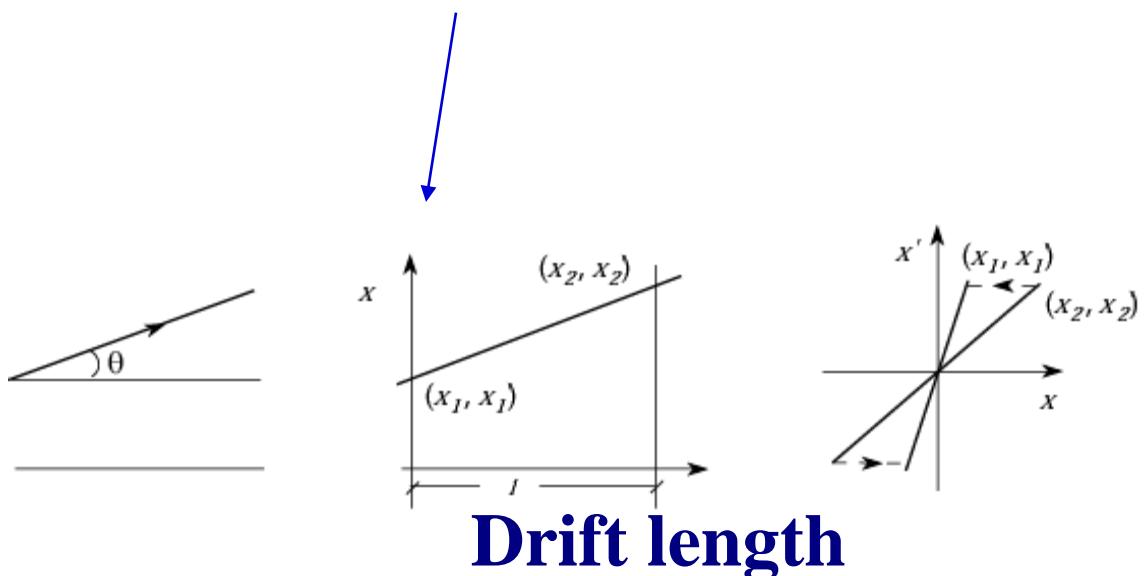
$$\begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} y \\ y' \\ \Delta p/p \end{pmatrix}_s \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \Delta p/p \end{pmatrix}_0$$

Effect of a drift length and a quadrupole

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$



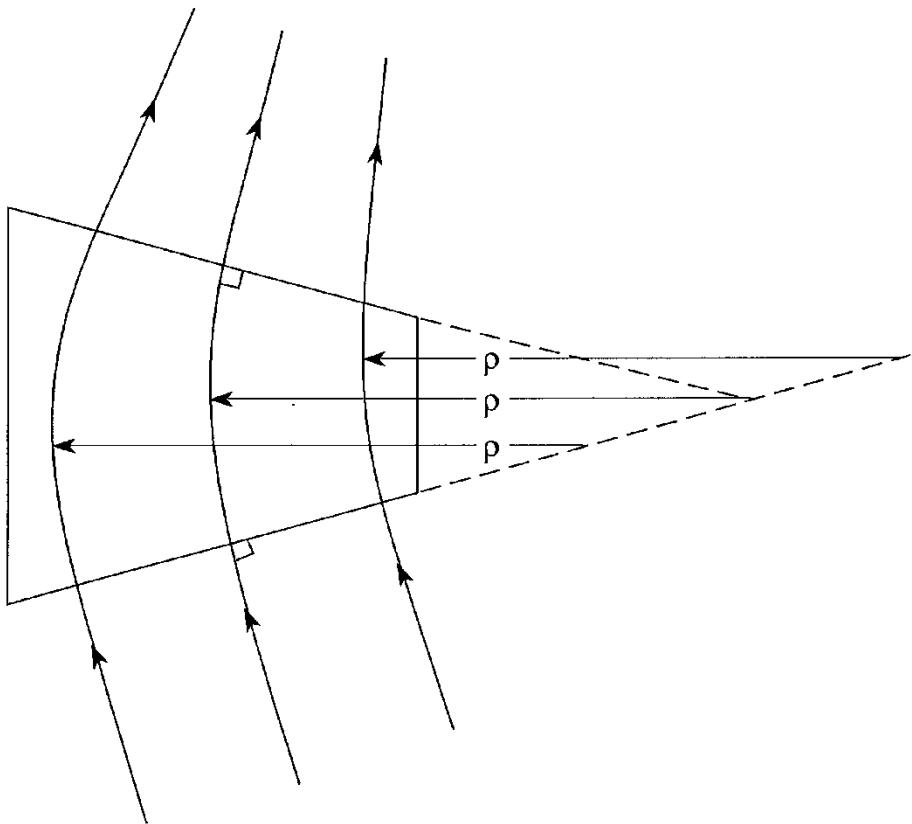
$$\theta = \frac{1}{f} \cdot x$$

Quadrupole

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

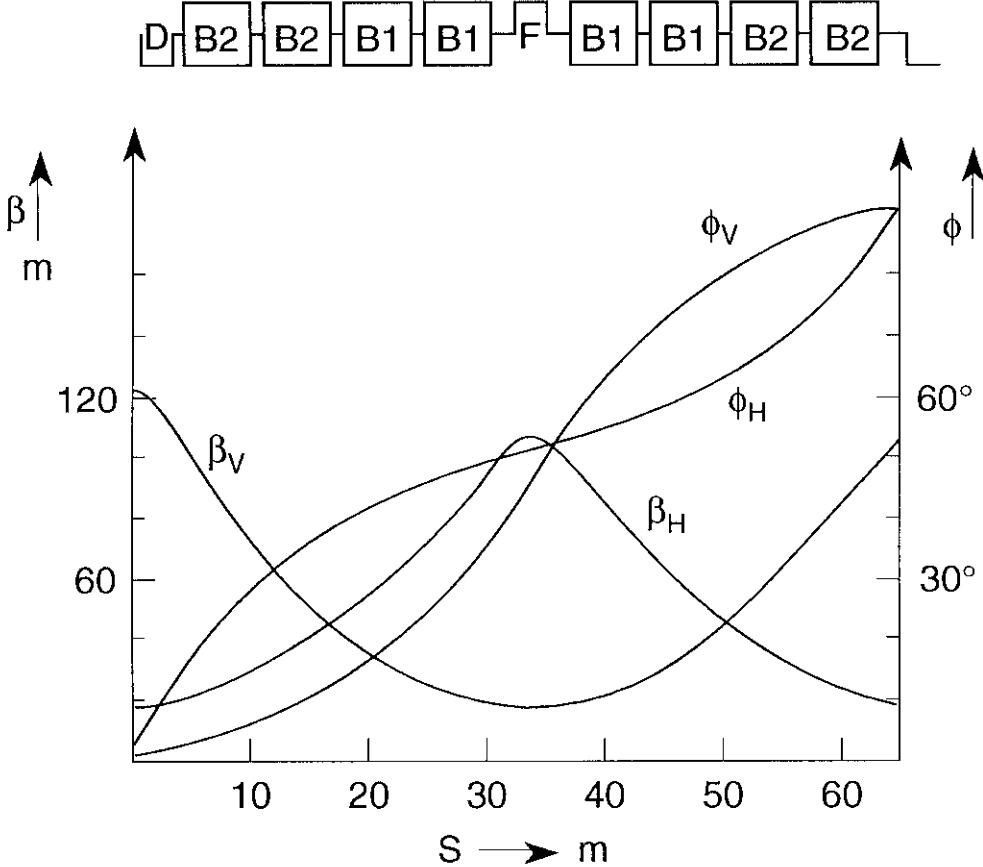
$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Focusing in a sector magnet



$$M_x = \begin{pmatrix} \cos \theta, & \rho \sin \theta \\ -\frac{1}{\rho} \sin \theta, & \cos \theta \end{pmatrix}$$

The lattice (1% of SPS)



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Calculating the Twiss parameters

THEORY

$$M = \begin{pmatrix} \cos \mu + \alpha \sin \mu, & \beta \sin \mu \\ -\gamma \sin \mu, & \cos \mu - \alpha \sin \mu \end{pmatrix}$$

**COMPUTATION
(multiply elements)**

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Real hard numbers

Solve to get Twiss parameters:

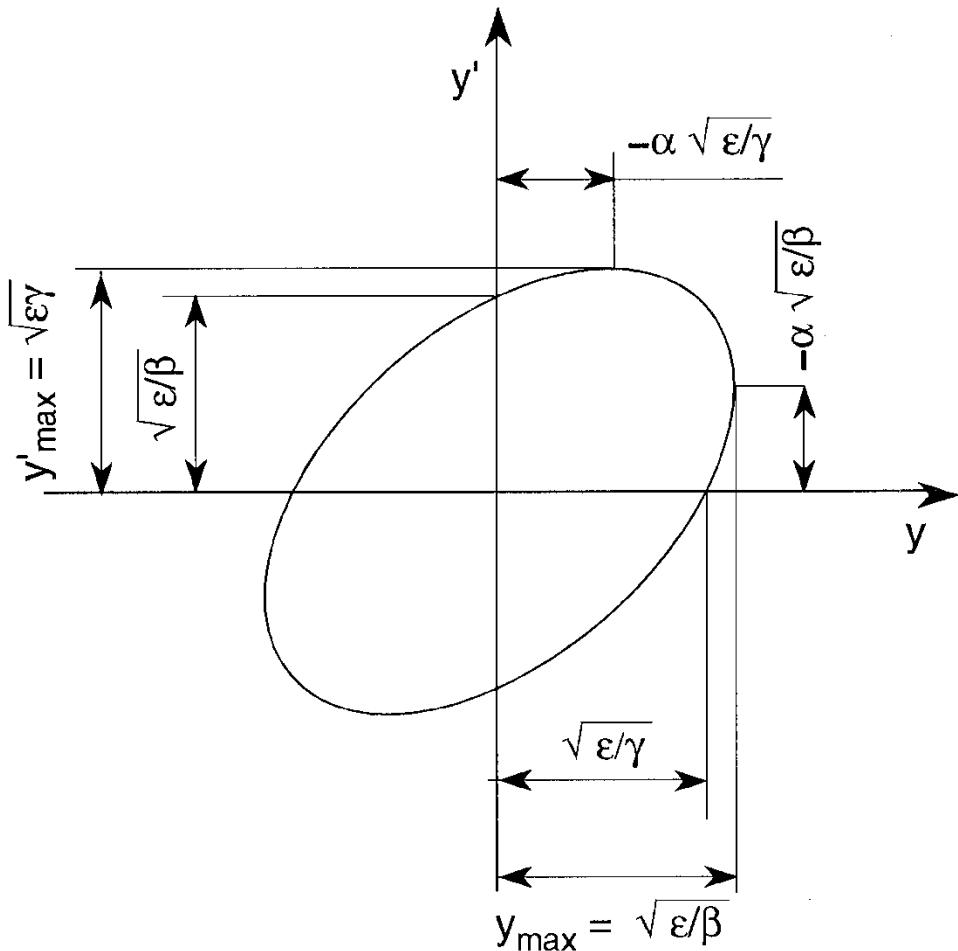
$$\mu = \cos^{-1} \left(\frac{\text{Tr } M}{2} \right) = \cos^{-1} \left(\frac{a + d}{2} \right)$$

$$\beta = b / \sin \mu$$

$$\alpha = \frac{a - d}{2 \sin \mu}$$

$$\gamma = -c / \sin \mu$$

Meaning of Twiss parameters



ꝝ ε is either :

- » Emittance of a beam anywhere in the ring
- » Courant and Snyder invariant from one particle anywhere in the ring

$$\gamma(s)y^2 + 2\alpha(s)yy' + \beta(s)y'^2 = \varepsilon$$

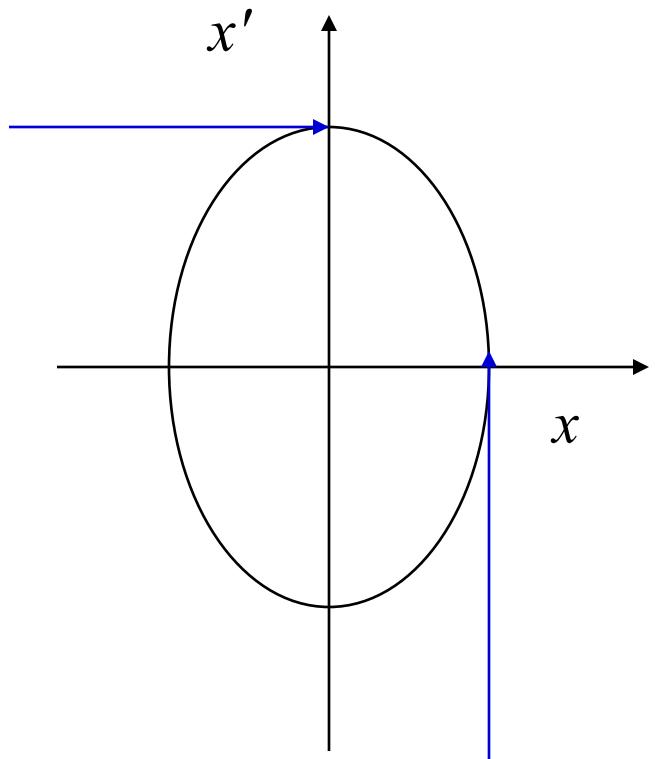
Example of Beam Size Calculation

◆ Emittance at 10 GeV/c

$$\varepsilon = 20\pi \text{ mm.mrad} = 20\pi \times 10^{-6} \text{ m.rad}$$

$$\hat{\beta} = 108 \text{ m}$$

$$\begin{aligned}\sqrt{\varepsilon/\beta} &= \sqrt{20.10^{-6}/108} \\ &= 0.43\sqrt{10^{-6}} \\ &= 0.43 \cdot 10^{-3} \text{ rad} \\ &= 0.43 \text{ mrad.}\end{aligned}$$



$$\begin{aligned}\sqrt{\varepsilon\beta} &= \sqrt{108 \cdot 20 \cdot 10^{-6}} \\ &= 46\sqrt{10^{-6}} \\ &= 46 \cdot 10^{-3} \text{ m} \\ &= 46 \text{ mm.}\end{aligned}$$

Summary

- ◆ **Equation of motion in transverse coordinates**
- ◆ **Check Solution of Hill**
- ◆ **Twiss Matrix**
- ◆ **Solving for a ring**
- ◆ **The lattice**
- ◆ **Beam sections**
- ◆ **Physical meaning of Q and beta**
- ◆ **Smooth approximation**

Further reading

- ◆ The slides that follow may interest students who would like to see a formal derivation of Hill's Equation from Hamiltonian Mechanics

Relativistic H of a charged particle in an electromagnetic field

- ◆ Remember from special relativity:

$$p_x = \frac{mv_x}{\sqrt{1 - \beta^2}} , \quad x$$

$$p_y = \frac{mv_y}{\sqrt{1 - \beta^2}} , \quad y$$

$$p_z = \frac{mv_z}{\sqrt{1 - \beta^2}} , \quad z$$

$$H = p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m_0^2 c^4$$

i.e. the energy of a free particle

- ◆ Add in the electromagnetic field

- » electrostatic energy $e\phi$
- » magnetic vector potential $e\mathbf{A}$ has same dimensions as momentum

$$H(\mathbf{q}, \mathbf{p}, t) = e\phi + c \left[(\mathbf{p} - e\mathbf{A})^2 + m_0^2 c^4 \right]^{1/2}$$

Hamiltonian for a particle in an accelerator

$$H(\mathbf{q}, \mathbf{p}, t) = e\phi + c \left[(\mathbf{p} - e\mathbf{A})^2 + m_0^2 c^4 \right]^{\frac{1}{2}}$$

- ◆ Note this is not independent of q because

$$\mathbf{A} = \mathbf{A}(x, y, s)$$

- ◆ Montague (pp 39 – 48) does a lot of rigorous, clever but confusing things but in the end he just turns H inside out
- ◆ p_s is the new Hamiltonian with s as the independent variable instead of t (see M 48)
- ◆ Wilson obtains
 - » assumes curvature is small
 - » assumes $\phi = 0$
 - » assumes magnet has no ends $A_x = A_y = 0$
 - » assumes small angles $p_x \ll p_s$
 - » ignores y plane
- ◆ Dividing by $P = \sqrt{p_x^2 + p_y^2 + p_z^2}$
- ◆ Finally (W Equ 8)
$$\mathcal{H} = \frac{p^2}{2} - \frac{e}{P} A_s = \frac{(x')^2}{2} - \frac{A_s}{(B\rho)}$$

Multipoles in the Hamiltonian

$$\mathcal{H} = \frac{(x')^2}{2} - \frac{A_s}{(B\rho)}$$

- ◆ We said A_s contains x (and y) dependance

$$A_s = \sum_n A_n x^n$$

- ◆ We find out how by comparing the two expressions:

$$B_y \Big|_{(y=0)} = -\frac{\partial A_s}{\partial x} = -\sum_n n A_n x^{n-1}$$

$$B_y \Big|_{(y=0)} = \frac{1}{(n-1)!} \frac{\partial^{(n-1)} B_y}{\partial x^{(n-1)}} x^{n-1}$$

- ◆ We find a series of multipoles:

$$\mathcal{H} = \frac{(x')^2}{2} + \sum_n \frac{1}{(B\rho)} \frac{1}{n!} \frac{\partial^{(n-1)} B_y}{\partial x^{(n-1)}} x^n$$

- ◆ For a quadrupole $n=2$ and:

$$\mathcal{H} = \frac{(x')^2}{2} + \frac{k(s)x^2}{2}$$

Hill's equation in one (or two) lines

$$\mathcal{H} = \frac{(x')^2}{2} + \frac{k(s)x^2}{2}$$

- ◆ Hamilton's equations give an equation of motion (remember independent coordinate is now s not t)

$$\frac{\partial p}{\partial s} = - \frac{\partial H}{\partial x} \Rightarrow \frac{\partial(x')}{\partial s} = - \frac{\partial H}{\partial x}$$

and:

$$\therefore x'' = - \frac{\partial \mathcal{H}}{\partial x} = -kx$$

$$x'' + kx = 0$$

Other interesting forms

In x plane

$$H \approx -\frac{eA_s}{p} - (1 - \bar{p}_x^2)^{1/2}$$

Small divergences:

$$H \approx -\frac{eA_s}{p} + \frac{\bar{p}_x^2}{2}$$

But: $B_z(z = 0) = \frac{\partial A_s}{\partial x} = n A_n x^{(n-1)}$

Substitute a Taylor series:

$$\frac{eA_s}{p} = \frac{1}{Be} \sum \frac{1}{n!} \frac{\partial(n-1)B_z}{\partial x(n-1)} x^n$$

Multipoles each have a term:

$$H = \frac{p_x^2}{2} + \sum_{n=0}^{\infty} \frac{1}{(Be)} \frac{1}{n!} \frac{\partial(n-1)B_z}{\partial x(n-1)} x^n$$

Quadrupoles:

$$H = \frac{p^2}{2} + \frac{k(s)x^2}{2}$$

