Lecture 5

ACCELERATOR PHYSICS

MT 2014

E. J. N. Wilson

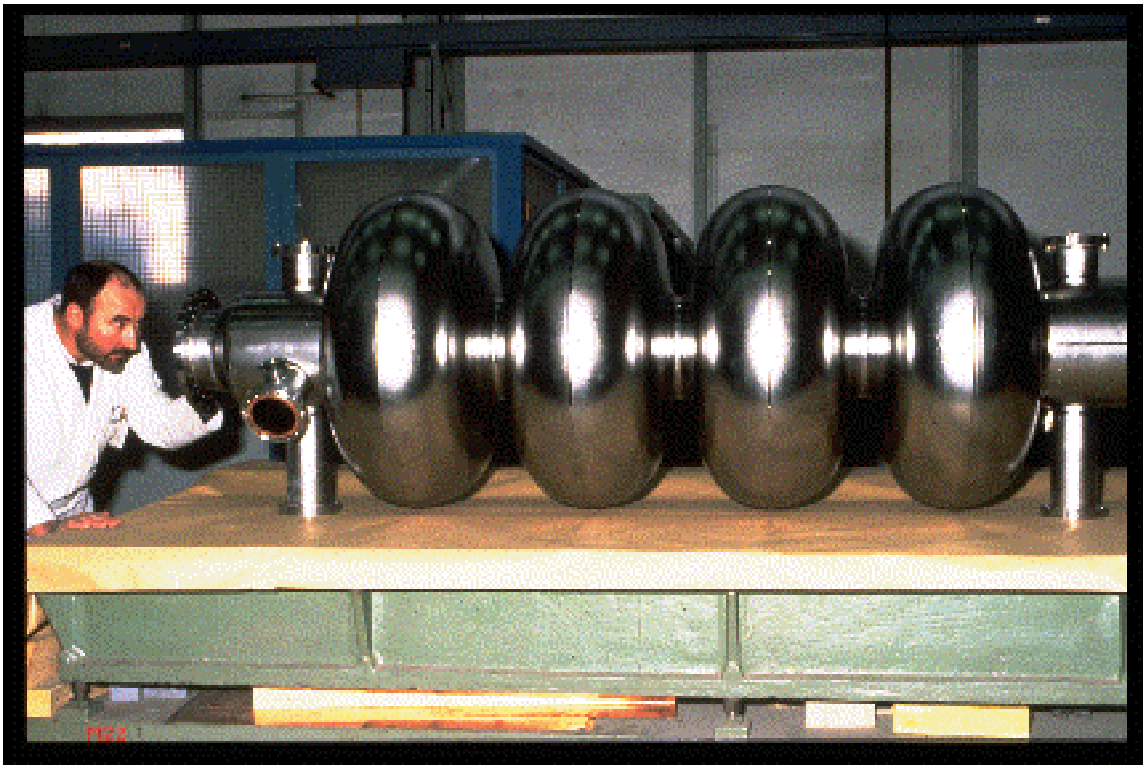
Recap of previous lecture
- Transverse dynamics II

- Equation of motion in transverse co-ordinates
- Check Solution of Hill
- Twiss Matrix
- Solving for a ring
- The lattice
- Beam sections
- Physical meaning of Q and beta
- Smooth approximation

Lecture 5 - Longitudinal dynamics -contents

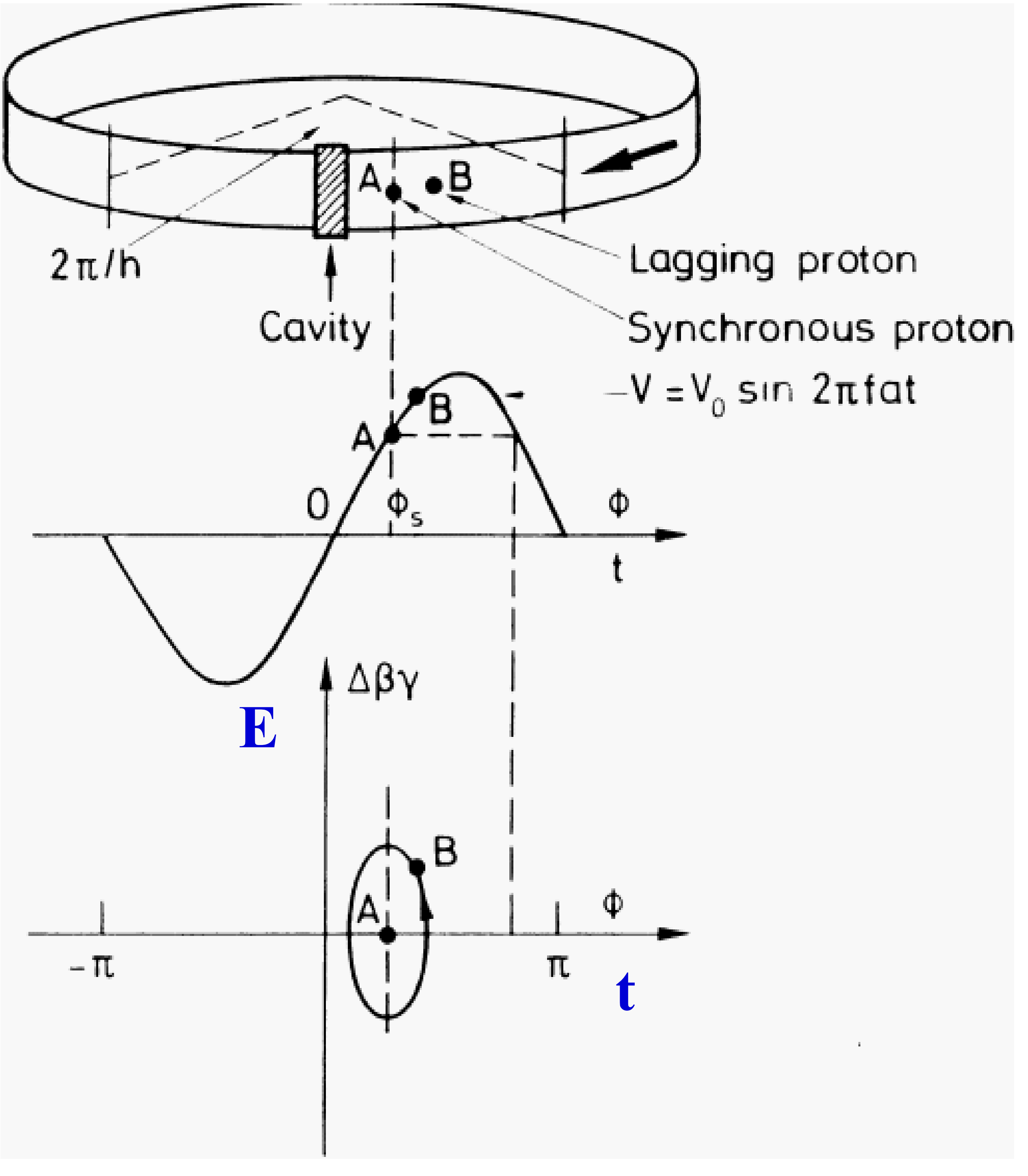
- **RF Cavity Cells**
- **Phase stability**
- **Bucket and pendulum**
- **Closed orbit of an ideal machine**
- **Analogy with gravity**
- **Dispersion**
- **Dispersion in the SPS**
- **Dispersed beam cross sections**
- **Dispersion in a bend (approx)**
- **Dispersion – from the "sine and cosine" trajectories**
- **From "three by three" matrices**

RF Cavity Cells

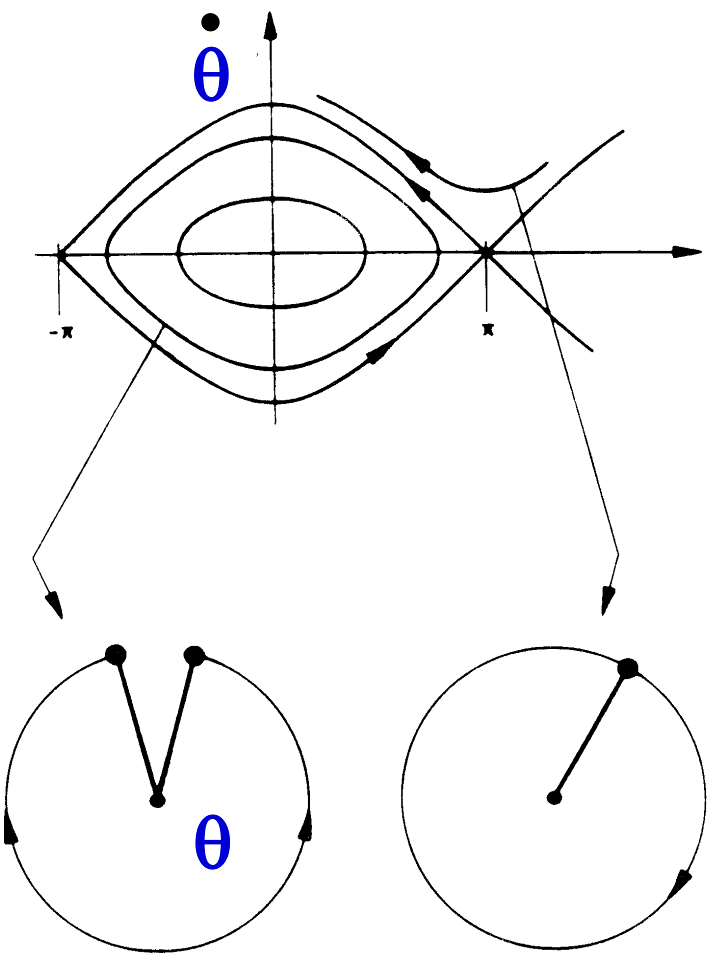


CAV.GIF

Phase stability

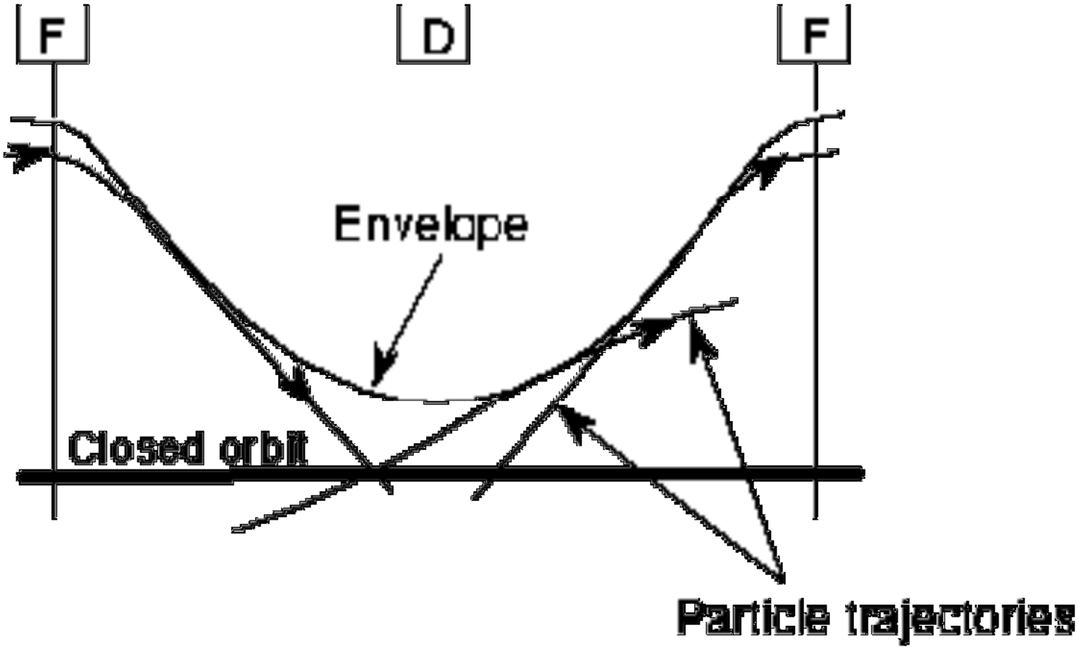


Bucket and pendulum

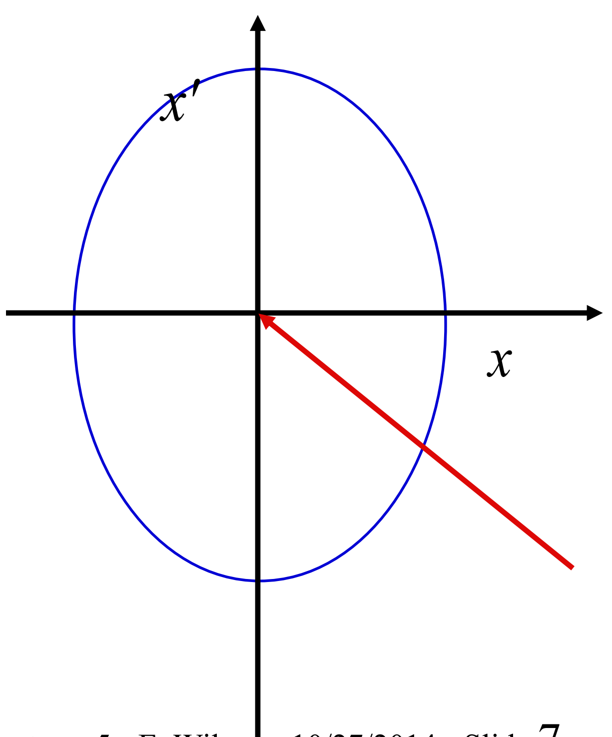


- The "bucket" of synchrotron motion is just that of the rigid pendulum
- Linear motion at small amplitude
- Metastable fixed point at the top
- Continuous rotation outside

Closed orbit of an ideal machine

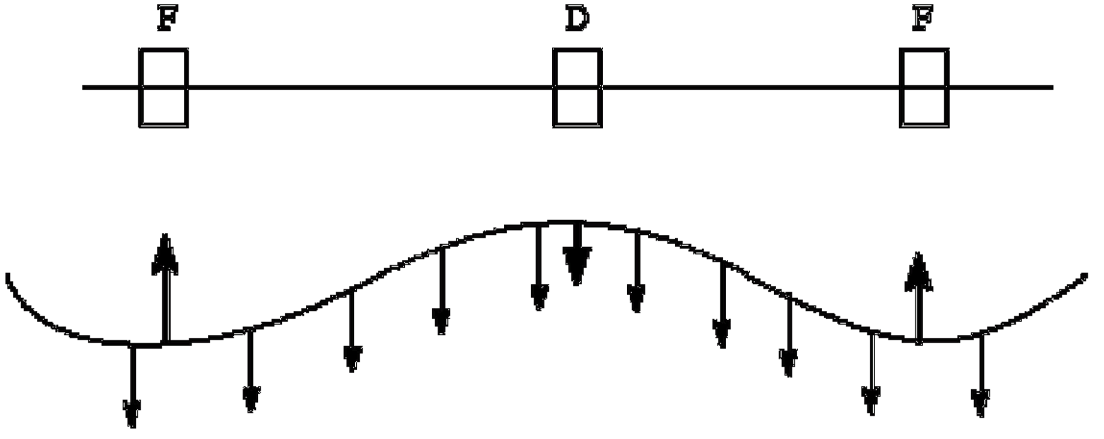


- In general particles executing betatron oscillations have a finite amplitude
- One particle will have zero amplitude and follows an orbit which closes on itself
- In an ideal machine this passes down the axis



Closed orbit Zero betatron amplitude

Analogy with gravity



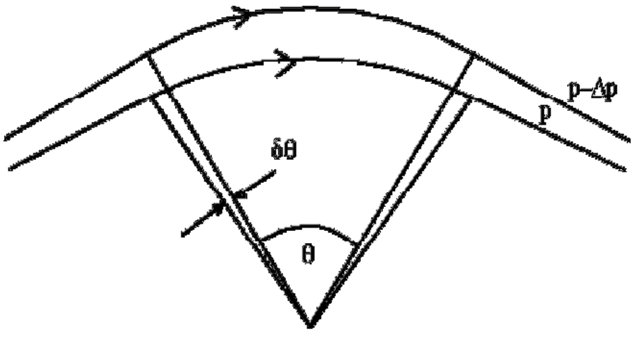
- What keeps particles in the machine
- There is a solution to Hills Equation
- It is closed and symmetric
- It is closer to the axis at vertically Defocusing Quadrupoles

$$\Delta z' = \frac{\Delta \ell \, \beta}{(\beta\rho)} = k \ell z$$

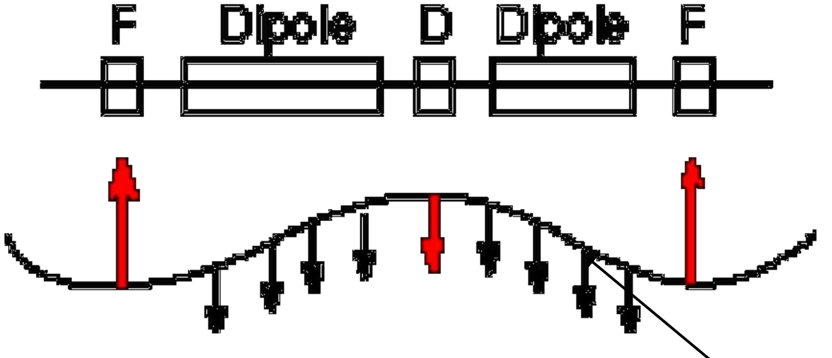
- Deflection is larger in F than D and cancels the force of gravity elsewhere.
- We could call the shape the "suspension" function.

Fig. cas 1.4C

Dispersion



- Low momentum particle is bent more
- It should spiral inwards but:
- There is a displaced (inwards) closed orbit
- Closer to axis in the D's
- Extra (outward) force balances extra bends

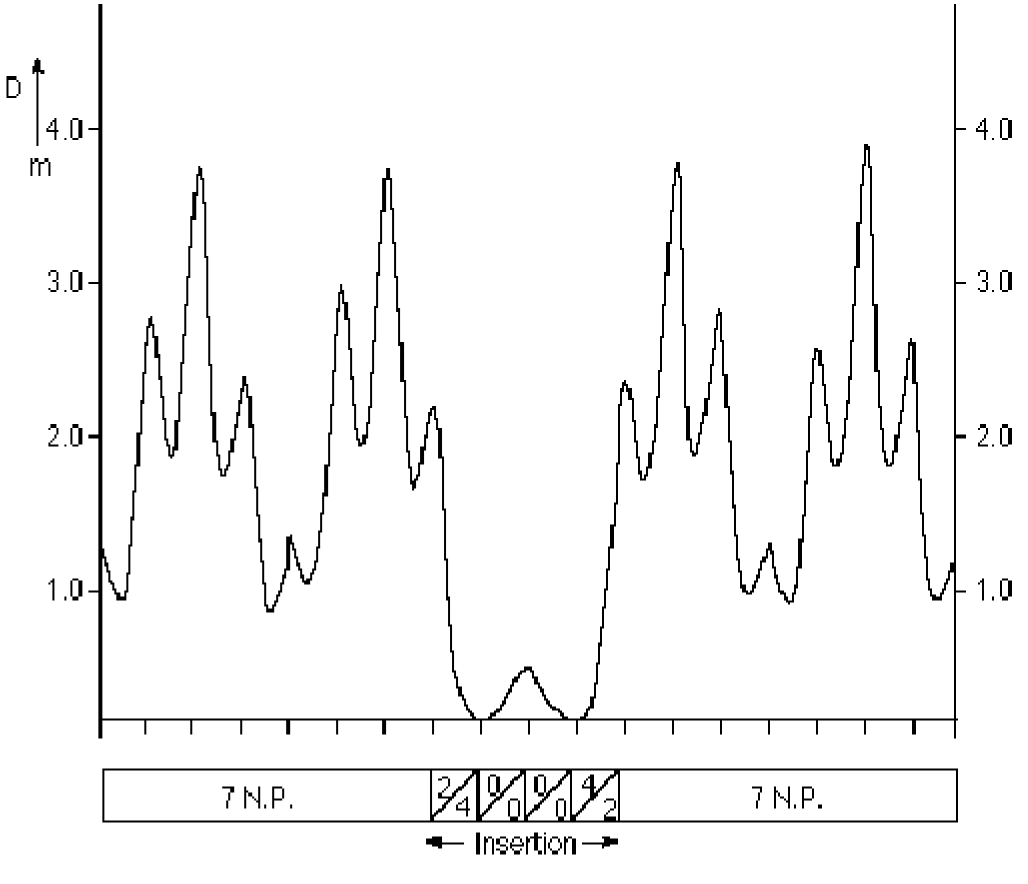


- *D(s)* is the "dispersion function"

$$x = D(s)\frac{\Delta p}{p}$$

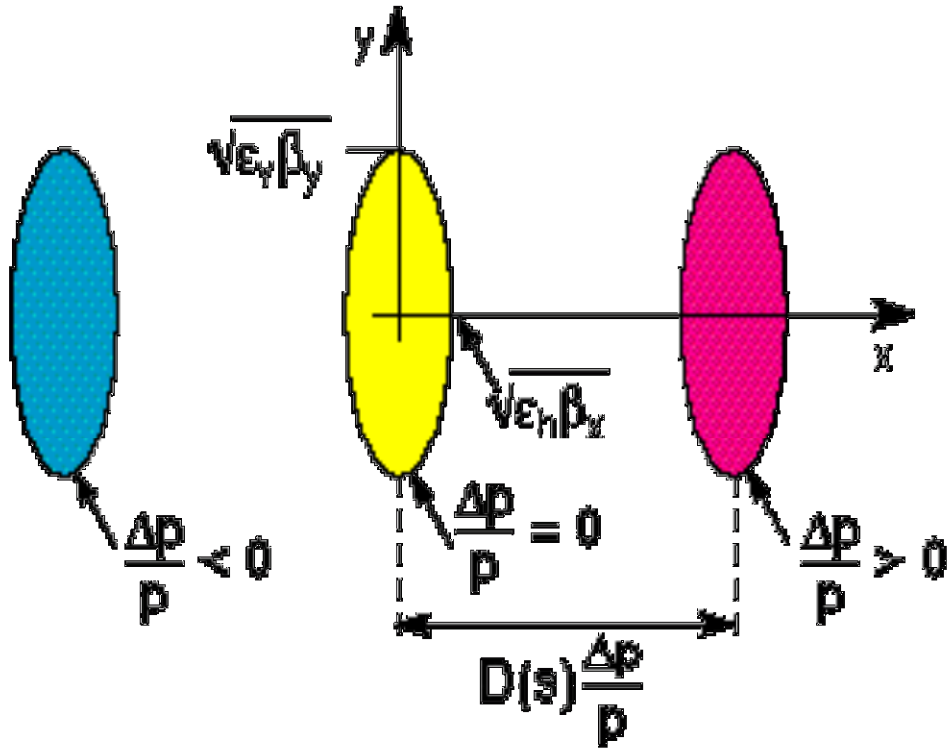
Fig. cas 1.7-7.1C

Dispersion in the SPS



- This is the long straight section where dipoles are omitted to leave room for other equipment - RF - Injection - Extraction, etc
- The pattern of missing dipoles in this region indicated by "0" is chosen to control the Fourier harmonics and make *D(s)* small
- It doesn't matter that it is big elsewhere

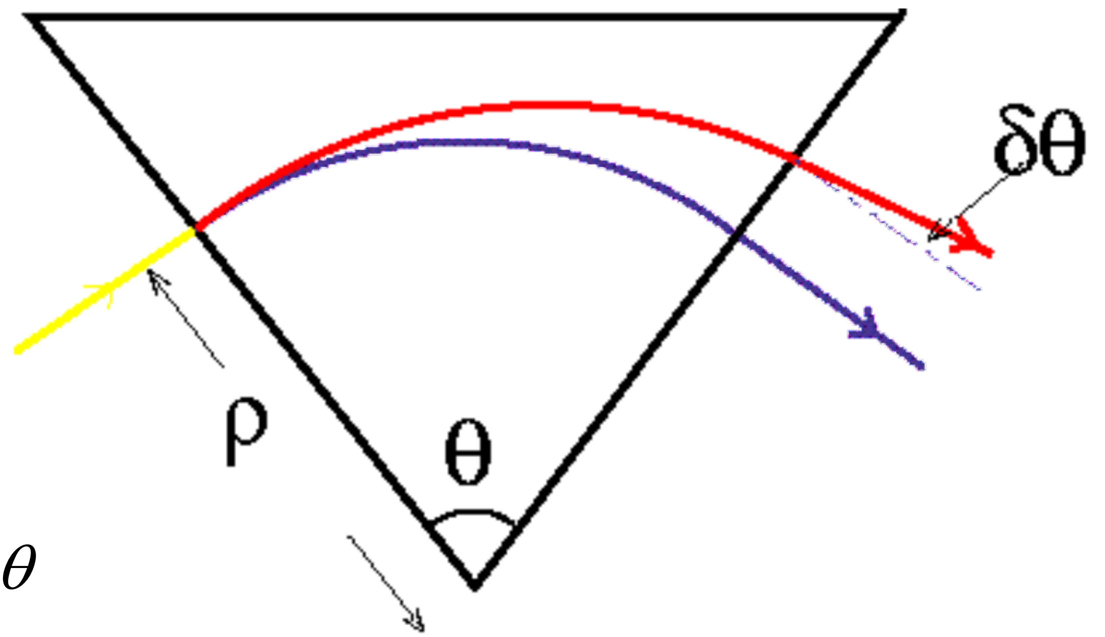
Dispersed beam cross sections



- These are real cross-section of beam
- The central and extreme momenta are shown
- There is of course a continuum between
- The vacuum chamber width must accommodate the full spread
- Half height and half width are:

$$\boxed{a_V = \sqrt{\beta_V \varepsilon_V} \quad , \quad a_H = \sqrt{\beta_H \varepsilon_H} + D(s)\frac{\Delta p}{p} \quad .}$$

Dispersion in a bend (approx)



$$\delta\theta = -\frac{dp}{p}\theta$$

$$x = \int_0^\theta x' ds = \rho\int_0^\theta x' d\theta = \left(\frac{dp}{p}\right)\rho\int_0^\theta \theta d\theta = \left(\frac{dp}{p}\right)\rho\left[\frac{\theta^2}{2}\right]_0^\theta$$

$$\begin{pmatrix} x_2 \\ x_2' \\ \frac{\delta p}{p} \end{pmatrix} = \begin{pmatrix} 1 & \rho\theta & \frac{\rho\theta^2}{2} \\ \frac{-\theta}{\rho} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \\ \frac{\delta p}{p} \end{pmatrix}$$

Bender.adb

Dispersion – from the "sine and cosine" trajectories

- **The combination of diplacement, divergence and dispersion gives:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

- **Expressed as a matrix**

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s_0}$$

- **It can be shown that:**

$$\boxed{D(s) = S(s) \int_{s_0}^{s} \frac{1}{\rho(t)} C(t)\,dt - C(s) \int_{s_0}^{s} \frac{1}{\rho(t)} S(t)\,dt}$$

- **Fulfils the particular solution of Hill's eqn. when forced :**

$$D''(s) + K(s) D(s) = \frac{1}{\rho(s)}$$

Principal trajectories

$$y(s) = C(s)y_0 + S(s)y_0' + D(s)\frac{\Delta p}{p_0}$$

$$y'(s) = C'(s)y_0 + S'(s)y_0' + D'(s)\frac{\Delta p}{p_0}$$

$$\begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} y \\ y' \\ \Delta p/p \end{pmatrix}_s \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \Delta p/p \end{pmatrix}_0$$

From “three by three” matrices

- **Adding momentum defect to horizontal divergence and displacement vector–**

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_2 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_1$$

- **Compute the ring as a product of small matrices and then use:**

$$D'(s) = \frac{m_{13}m_{21} + (1 - m_{11})m_{23}}{(1 - m_{11})(1 - m_{22}) - m_{21}m_{12}}$$

$$D(s) = \left(\frac{m_{13} - m_{12}D'(s)}{1 - m_{11}}\right) D'(s)$$

- **To find the dispersion vector at the starting point**
- **Repeat for other points in the ring**

Lecture 7 - Transverse Dynamics – Summary

- RF Cavity Cells
- Phase stability
- Bucket and pendulum
- Closed orbit of an ideal machine
- Analogy with gravity
- Dispersion
- Dispersion in the SPS
- Dispersed beam cross sections
- Dispersion in a bend (approx)
- Dispersion – from the "sine and cosine" trajectories
- From "three by three" matrices.