1 Effective Quantum Mechanics

1.1 What is an effective field theory?

The uncertainty principle tells us that to probe the physics of short distances we need high momentum. On the one hand this is annoying, since creating high relative momentum in a lab costs a lot of money! On the other hand, it means that we can have predictive theories of particle physics at low energy without having to know everything about physics at short distances. For example, we can discuss precision radiative corrections in the weak interactions without having a grand unified theory or a quantum theory of gravity. The price we pay is that we have a number of parameters in the theory (such as the Higgs and fermion masses and the gauge couplings) which we cannot predict but must simply measure. But this is a lot simpler to deal with than a mess like turbulent fluid flow where the physics at many different distance scales are all entrained together.

The basic idea behind effective field theory (EFT) is the observation that the non analytic parts of scattering amplitudes are due to intermediate process where physical particles can exist on shell (that is, kinematics are such that internal propagators $1/(p^2 - m^2 + i\epsilon)$ in Feynman diagrams can diverge with $p^2 = m^2$...then one is sensitive to the $i\epsilon$ and sees cuts in the amplitude due to logarithms, square roots, etc). Therefore if one can construct a quantum field theory that correctly accounts for these light particles, then all the contributions to the amplitude from virtual heavy particles that cannot be physically created at these energies can be Taylor expanded p^2/M^2, where M is the energy of the heavy particle. (By "heavy" I really mean a particle whose energy is too high to create; this might be a heavy particle at rest, but it equally well applies to a pair of light particles with high relative momentum.) However, the power of of this observation is not that one can Taylor expand parts of the scattering amplitude, but that the Taylor expanded amplitude can be computed directly from a quantum field theory (the EFT) which contains *only* light particles, with local interactions between them that encode the small effects arising from virtual heavy particle exchange. Thus the standard model does not contain X gauge bosons from the GUT scale, for example, but can be easily modified to account for the very small effects such particles could have leading to proton decay, for example.

So in fact, all of our quantum field theories are EFTs; only if there is some day a Theory Of Everything (don't hold your breath) will we be able to get beyond them. So how is a set of lectures on EFT different than a quick course on quantum field theory? Traditionally a quantum field theory course is taught from the point of view that held sway from when it was originated in the late 1920s through the development of nonabelian gauge theories in the early 1970s, while EFT incorporates the ideas of Wilson and others that were developed in the early 1970s and completely turned on its head how we think about UV (high energy) physics and renormalization, and how we interpret the results of calculations. As an example of the extreme reversal of viewpoint, a theory of a massive boson with a ϕ^4 interaction used to be considered one of the few well-defined theories, while the Fermi theory of weak interactions was viewed as useful but "nonrenormalizable" and sick; now the scalar theory is considered sick, while the Fermi theory is a simple prototype of all successful quantum field theories. Is the modern view the last word? Probably not, and I will mention unresolved mysteries at the end of my lectures.

There are three basic uses for effective field theory I will touch on in these lectures:

- Top-down: you know the theory to high energies, but either you do not need all of its complications to arrive at the desired description of low energy physics, or else the full theory is nonperturbative and you cannot compute in it, so you construct an EFT for the light degrees of freedom, constraining their interactions from your knowledge of the symmetries of the more complete theory;
- Bottom-up: you explore small effects from high dimension operators in your low energy EFT to gain cause about what might be going on at shorter distances than you can directly probe;
- Philosophizing: you marvel at how "fine-tuned" our world appears to be, and pondering whether the way our world appears is due to some missing physics, or because we live in a special corner of the universe (the anthropic principle), or whether we live at a dynamical fixed point resulting from cosmic evolution. Such investigations are at the same time both fascinating and possibly an incredible waste of time!

To begin with I will not discuss effective field theories, however, but effective quantum mechanics. The essential issues of approximating short range interactions with point-like interactions have nothing to do with relativity or many-body physics, and can be seen in entirety in non relativistic quantum mechanics. I thought I would try this introduction because I feel that the way quantum mechanics and quantum field theory are traditionally taught it looks like they share nothing in common except for mysterious ladder operators, which is of course not true.

1.2 Scattering in 1D

1.2.1 Square well scattering in 1D

We have all solved the problem of scattering in 1D quantum mechanics, from both square barrier potentials and delta-function potentials. Consider scattering of a particle of mass m from an attractive square well potential of width Δ and depth $\frac{\alpha^2}{2m\Delta^2}$,

$$V(x) = \begin{cases} -\frac{\alpha^2}{2m\Delta^2} & 0 \le x \le \Delta \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

Here α is a dimensionless number that sets the strength of the potential. It is straight forward to compute the reflection and transmission coefficients at energy E (with $\hbar = 1$)

$$R = (1 - T) = \left[\frac{4\kappa^2 k^2 \csc^2(\kappa\Delta)}{(k^2 - \kappa^2)^2} + 1 \right]^{-1} , \tag{2}$$

where

$$k = \sqrt{2mE} , \qquad \kappa = \sqrt{k^2 + \frac{\alpha^2}{\Delta^2}} . \tag{3}$$

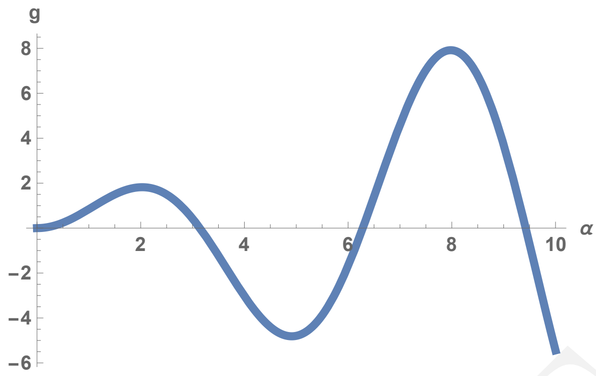

Figure 1: *The matching condition in 1D: the appropriate value of g in the effective theory for a given α in the full theory.*

For low k we can expand the reflection coefficient and find

$$R = 1 - \frac{4}{\alpha^2 \sin^2 \alpha} \Delta^2 k^2 + O(\Delta^4 k^4) \tag{4}$$

Note that $R \to 1$ as $k \to 0$, meaning that the potential has a huge effect at low enough energy, no matter how weak...we can say the interaction is very *relevant* at low energy.

1.2.2 Relevant δ-function scattering in 1D

Now consider scattering off a δ-function potential in 1D,

$$V(x) = -\frac{g}{2m\Delta}\delta(x) \ , \tag{5}$$

where the length scale Δ was included in order to make the coupling g dimensionless. Again one can compute the reflection coefficient and find

$$R = (1 - T) = \left[1 + \frac{4k^2\Delta^2}{g^2}\right]^{-1} = 1 - \frac{4k^2\Delta^2}{g^2} + O(k^4) \ . \tag{6}$$

By comparing the above expression to eq. (4) we see that at low momentum the δ function gives the same reflection coefficient to up to $O(k^4)$ as the square well, provided we set

$$g = \alpha \sin \alpha \ . \tag{7}$$

This matching condition is shown in Fig. 1, and interpreting this figure is one of the problems for the lecture.

1.3 Scattering in 3D

Now let's see what happens if we try the same thing in 3D (three spatial dimensions), choosing the strength of a δ-function potential to mimic low energy scattering off a square well potential [1].

First, a quick review of a few essentials of scattering theory in 3D, focussing only on s-wave scattering.

A scattering solution for a particle of mass m in a finite range potential must have the asymptotic form for large $|\mathbf{r}|$

$$\psi \xrightarrow{r\to\infty} e^{ikz} + \frac{f(\theta)}{r}e^{ikr} \ . \tag{8}$$

representing an incoming plane wave in the z direction, and an outgoing scattered spherical wave. The quantity f is the scattering amplitude, and $|f|^2$ encodes the probability for scattering; in particular, the differential cross section is simply

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2 \ . \tag{9}$$

For scattering off a spherically symmetric potential, both $f(\theta)$ and $e^{ikz} = e^{ikr\cos\theta}$ can be expanded in Legendre polynomials ("partial wave expansion"); I will only be interested in s-wave scattering (angle independent) and therefore will replace $f(\theta)$ simply by f — independent of angle, but still a function of k. For the plane wave we can replace

$$e^{ikz} = e^{ikr\cos\theta} \xrightarrow[s-\text{wave}]{} \int \frac{d\theta}{2\pi} e^{ikr\cos\theta} = j_0(kr) \ , \tag{10}$$

where $j_0(z) = \sin z/z$ is a regular spherical Bessel function. So we are interested in a solution to the Shchr̈dinger equation with asymptotic behavior

$$\psi \xrightarrow[s-\text{wave}]{r\to\infty} j_0(kr) + \frac{f}{r}e^{ikr} = j_0(kr) + kf\,(ij_0(kr) - n_0(kr)) \qquad (s\text{-wave}) \tag{11}$$

where I used the spherical Bessel functions

$$j_0(x) = \frac{\sin z}{z} \ , \qquad n_0(z) = -\frac{\cos z}{z} \tag{12}$$

to rewrite e^{ikr}. Since ψ is an exact s-wave solution to the free Schrödinger equation outside the potential, and the most general solutions to the free radial Schrödinger equation are spherical Bessel functions, the asymptotic form for ψ can also be written as

$$\psi \xrightarrow{r\to\infty} A\,(\cos\delta\, j_0(kr) - \sin\delta\, n_0(kr)) \ . \tag{13}$$

where A and δ (the phase shift) are real constants. Relating these two expressions eq. (11) and eq. (13) we find

$$f = \frac{1}{k\cot\delta - ik} \ . \tag{14}$$

[1] Why this fixation with δ-function potentials? They are not particularly special in non-relativistic quantum mechanics, but in a relativistic field theory they are the *only* instantaneous potential which can be Lorentz invariant. That is why we always formulate quantum field theories as interactions between particles only when they are at the same point in spacetime. All the issues of renormalization in QFT arise from the singular nature of these δ-function interactions. So I am focussing on δ-function potentials in quantum mechanics in order to illustrate what is going on in the relativistic QFT.

So solving for the phase shift δ is equivalent to solving for the scattering amplitude f, using the formula above.

The quantity $k\cot\delta$ is interesting, since one can show that for a finite range potential it must be analytic in the energy, and so has a Taylor expansion in k involving only even powers of k , called "the effective range expansion":

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2}r_0 k^2 + O(k^4)\ . \tag{15}$$

The parameters have names: a is the scattering length and r_0 is the effective range; these terms dominate low energy (low k) scattering. Proving the existence of the effective range expansion is somewhat involved and I refer you to a quantum mechanics text; there is a low-brow proof due to Bethe and a high-brow one due to Schwinger.

And the last part of this lightning review of scattering: if we have two particles of mass M scattering off each other it is often convenient to use Feynman diagrams to describe the scattering amplitude; I denote the Feynman amplitude – the sum of all diagrams – as $i\mathcal{A}$. The relation between $\mathcal{A}$ and f is

$$\mathcal{A} = \frac{4\pi}{M} f\ , \tag{16}$$

where f is the scattering amplitude for a single particle of reduced mass $m = M/2$ in the inter-particle potential. This proportionality is another result that can be priced together from quantum mechanics books, which I won't derive.

1.3.1 Square well scattering in 3D

We consider s-wave scattering off an attractive well in 3D,

$$V = \begin{cases} -\frac{\alpha^2}{m\Delta^2} & r < \Delta \\ 0 & r > \Delta\ . \end{cases} \tag{17}$$

We have for the wave functions for the two regions $r < \Delta$, $r > \Delta$ are expressed in terms of spherical Bessel functions as

$$\psi_<(r) = j_0(\kappa r)\ , \qquad \psi_>(r) = A\left[\cos\delta\, j_0(kr) - \sin\delta\, n_0(kr)\right] \tag{18}$$

where $\kappa = \sqrt{k^2 + \alpha^2/\Delta^2}$ as in eq. (3) and δ is the s-wave phase shift. Equating ψ and ψ' at the edge of the potential at $r = \Delta$ gives

$$k\cot\delta = \frac{k(k\sin\kappa\Delta + \kappa\cot k\Delta\cos\kappa\Delta)}{k\cot k\Delta\sin\kappa\Delta - \kappa\cos\kappa\Delta}\ . \tag{19}$$

With a little help from Mathematica we can expand this in powers of k^2 and find

$$k\cot\delta = \frac{1}{\Delta}\left(\frac{\tan\alpha}{\alpha} - 1\right)^{-1} + O(k^2) \tag{20}$$

where on comparing with eq. (15) we can read off the scattering length from the k^2 expansion,

$$a = -\Delta\left(\frac{\tan\alpha}{\alpha} - 1\right)\ , \tag{21}$$

a relation shown in Fig. 2.

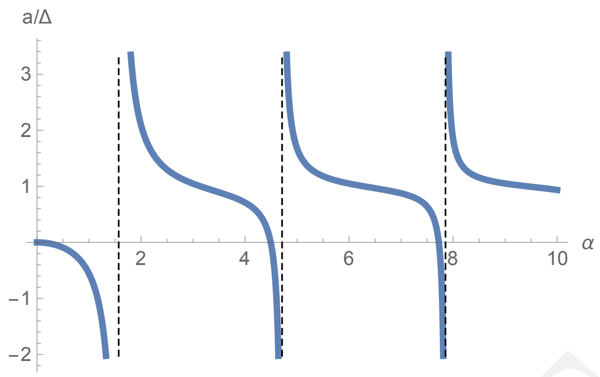

Figure 2: *a/Δ vs. the 3D potential well depth parameter α, from eq. (21).*

1.3.2 Irrelevant δ-function scattering in 3D

Now we look at reproducing the above scattering length from scattering in 3D off a delta function potential. At first look this seems hopeless: note that the result for a square well of width Δ and coupling $\alpha = O(1)$ gives a scattering length that is $a = O(\Delta)$; therefore if you extrapolate to a potential of zero width (a δ function) you would conclude that the scattering length would go to zero, and the scattering amplitude would vanish for low k. This is an example of an *irrelevant* interaction.

On second look the situation is even worse: since $-\delta^3(\mathbf{r})$ scales as $-1/r^3$ while the kinetic $-\nabla^2$ term in the Schrödinger equation only scales as $1/r^2$ you can see that the system does not have a finite energy ground state. For example if you performed a variational calculation, you could lower the energy without bound by scaling the wave function to smaller and smaller extent. Therefore the definition of a δ-function has to be modified in 3D – this is the essence of renormalization.

These two features go hand in hand: typically singular interactions are "irrelevant" and at the same time require renormalization. We can sometimes turn an irrelevant interaction into a relevant one by fixing a certain renormalization condition which forces a fine tuning of the coupling to a critical value, and that is the case here. For example, consider defining the δ-function as the $\rho \to 0$ limit of a square well of width ρ and depth $V_0 = \bar{\alpha}^2(\rho)/(m\rho^2)$, while adjusting the coupling strength $\bar{\alpha}(\rho)$ to keep the scattering length fixed to the desired value of a given in eq. (21). We find

$$a = \rho \left(1 - \frac{\tan \bar{\alpha}(\rho)}{\bar{\alpha}(\rho)}\right) \qquad (22)$$

as $\rho \to 0$. There are an infinite number of solutions, and one of them is

$$\bar{\alpha}(\rho) \xrightarrow{\rho\to 0} \frac{\pi}{2} + \frac{2\rho}{\pi a} + O(\rho^2) \ . \qquad (23)$$

in other words, we have to tune this vanishingly thin square well to have a bound state right near threshold ($\alpha \simeq \pi/2$). However, note that while naively you might think a potential $-g\delta^3(\mathbf{r})$ would be approximated by a square well of depth $V_0 \propto 1/\rho^3$ as $\rho \to 0$, but we

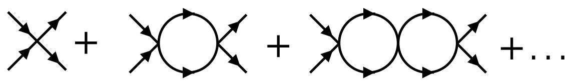

Figure 3: *The sum of Feynman diagrams giving the scattering amplitude for two particles interaction via a δ-function potential.*

see that instead we get $V_0 \propto 1/\rho^2$. This is sort of like using a potential $-r\delta^3(\mathbf{r})$ instead of $-\delta^3(\mathbf{r})$.

We have struck a delicate balance: A naive δ function potential is too strong and singular to have a ground state; a typical square well of depth $\alpha^2/m\rho^2$ becomes irrelevant for fixed α in the $\rho \to 0$ limit; but a strongly coupled potential of form $\alpha^2/m\rho^2$ can lead to a relevant interaction so long as we tune α its critical value $\alpha_\star = \pi/2$ in precisely the right way as we take $\rho \to 0$.

This may all seem more familiar to you if I to use field theory methods and renormalization. Consider two colliding particles of mass M in three spatial dimensions with a δ-function interaction; this is identical to the problem of potential scattering when we identify m with the reduced mass of the two particle system,

$$m = \frac{M}{2}\ . \tag{24}$$

We introduce the field ψ for the scattering particles (assuming they are spinless bosons) and the Lagrange density

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{4} \left(\psi^\dagger \psi \right)^2\ . \tag{25}$$

Here $C_0 > 0$ implies a repulsive interaction. As in a relativistic field theory, ψ annihilates particles and $\psi^\dagger$ creates them; unlike in a relativistic field theory, however, there are no anti-particles.

The kinetic term gives rise to the free propagator

$$G(E, \mathbf{p}) = \frac{i}{E - \mathbf{p}^2/(2M) + i\epsilon}\ , \tag{26}$$

while the interaction term gives the vertex $-iC_0$. The total Feynman amplitude for two particles then is the sum of diagrams in Fig. 3, which is the geometric series

$$i\mathcal{A} = -iC_0 \left[1 + (C_0 B(E)) + (C_0 B(E))^2 + \ldots \right] = \frac{i}{-\frac{1}{C_0} + B(E)}\ , \tag{27}$$

where B is the 1-loop diagram, which in the center of momentum frame (where the incoming particles have momenta $\pm\mathbf{p}$ and energy $E/2 = \mathbf{k}^2/(2M)$) is given by

$$B(E) = -i \int \frac{d^4q}{(2\pi)^4} \frac{i}{\left(\frac{E}{2} + q_0 - \frac{\mathbf{q}^2}{2M} + i\epsilon \right)} \frac{i}{\left(\frac{E}{2} - q_0 - \frac{\mathbf{q}^2}{2M} + i\epsilon \right)} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{E - \frac{\mathbf{q}^2}{M} + i\epsilon}\ . \tag{28}$$

The B integral is linearly divergent and so I will regulate it with a momentum cutoff and renormalize the coupling C_0:

$$
\begin{aligned}
B(E,\Lambda) &= \int^{\Lambda} \frac{d^3q}{(2\pi)^3}\frac{1}{E-\frac{\mathbf{q}^2}{M}+i\epsilon} \\
&= -\frac{M\left(\Lambda-\sqrt{-eM-i\epsilon}\tan^{-1}\left(\frac{\Lambda}{\sqrt{-eM-i\epsilon}}\right)\right)}{2\pi^2} \\
&= -\frac{M\Lambda}{2\pi^2}+\frac{M}{4\pi}\sqrt{-ME-i\epsilon}+O\left(\frac{1}{\Lambda}\right) \\
&= -\frac{M\Lambda}{2\pi^2}-i\frac{Mk}{4\pi}+O\left(\frac{1}{\Lambda}\right) \ .
\end{aligned} \tag{29}
$$

Thus from eq. (27) we get the Feynman amplitude

$$
\mathcal{A}=\frac{1}{-\frac{1}{C_0}+B(E)}=\frac{1}{-\frac{1}{C_0}-\frac{M\Lambda}{2\pi^2}-i\frac{Mk}{4\pi}}=\frac{4\pi}{M}\frac{1}{\left(-\frac{4\pi}{M\overline{C}_0}-ik\right)} \tag{30}
$$

where

$$
\frac{1}{\overline{C}_0}=\frac{1}{C_0}-\frac{M\Lambda}{2\pi^2} \tag{31}
$$

is our renormalized coupling. Since in in 3D we have (eq. (14), eq. (16))

$$
\mathcal{A}=\frac{4\pi}{M}\frac{1}{k\cot\delta-ik}\ , \qquad k\cot\delta=-\frac{1}{a}+\frac{1}{2}r_0k^2+\dots \tag{32}
$$

we see that this theory relates $\overline{C}_0$ to the scattering length as

$$
\overline{C}_0=\frac{4\pi a}{M}\ . \tag{33}
$$

Therefore we can reproduce square well scattering length eq. (21) by taking

$$
\overline{C}_0=-\frac{4\pi\Delta}{M}\left(\frac{\tan\alpha}{\alpha}-1\right)\ . \tag{34}
$$

What have we accomplished? We have shown that one can reproduce low energy scattering from a finite range potential in 3D with a δ-function interaction, with errors of $O(k^2\Delta^2)$ with the caveat that renormalization is necessary if we want to make sense of the theory.

However there is second important and subtle lesson: We can view eq. (31) plus eq. (33) to imply a fine tuning of the inverse bare coupling $1/C_0$ coupling as $\Lambda\to\infty$: $M\Lambda C_0/(2\pi^2)$ must be tuned to $1+O(1/a\Lambda)$ as $\Lambda\to\infty$. This is the same lesson we learned looking at square wells: if C_0 didn't vanish at least linearly with the cutoff, the interaction would be too strong to makes sense; while if ΛC_0 went to zero or a small constant, the interaction would be irrelevant. Only if ΛC_0 is fine-tuned to a critical value can we obtain nontrivial scattering at low k.

1.4 Scattering in 2D

1.4.1 Square well scattering in 2D

Finally, let's look at the intermediary case of scattering in two spatial dimensions, where we take the same potential as in eq. (17). This is not just a tour of special functions — something interesting happens! The analogue of eq. (35) for the two dimensional square well problem is

$$\psi_<(r) = J_0(\kappa r)\ , \qquad \psi_>(r) = A\left[\cos\delta\, J_0(kr) - \sin\delta\, Y_0(kr)\right] \tag{35}$$

where κ is given in eq. (3) and J, Y are the regular and irregular Bessel functions. Equating ψ and ψ' at the boundary $r = \Delta$ gives[2]

$$\begin{aligned}\cot\delta &= \frac{kJ_0(\Delta\kappa)Y_1(\Delta k) - \kappa J_1(\Delta\kappa)Y_0(\Delta k)}{kJ_0(\Delta\kappa)J_1(\Delta k) - \kappa J_1(\Delta\kappa)J_0(k\Delta)} \\ &= \frac{2\left(\frac{J_0(\alpha)}{\alpha J_1(\alpha)} + \log\left(\frac{\Delta k}{2}\right) + \gamma_E\right)}{\pi} + O\left(k^2\right)\end{aligned} \tag{36}$$

This result looks very odd because of the logarithm that depends on k! The interesting feature of this expression is not that $\cot\delta(k) \to -\infty$ for $k \to 0$: that just means that the phase shift vanishes at low k. What is curious is that for our attractive potential, the function $J_0(\alpha)/(\alpha J_1(\alpha))$ is strictly positive, and therefore $\cot\delta$ changes sign at a special value for k,

$$k = \Lambda \simeq \frac{2e^{-\frac{J_0(\alpha)}{\alpha J_1(\alpha)} - \gamma_E}}{\Delta}\ , \tag{37}$$

where the scale Λ is *exponentially* lower than our fundamental scale Δ for weak coupling, since then $J_0(\alpha)/\alpha J_1(\alpha) \sim 2/\alpha^2 \gg 1$. This is evidence in the scattering amplitude for a bound state of size $\sim 1/\Lambda$...exponentially larger than the size of the potential!

On the other hand, if the interaction is repulsive, the $J_0(\alpha)/\alpha J_1(\alpha)$ factor is replaced by $-I_0(\alpha)/\alpha I_1(\alpha) < 0$, I_n being one of the other Bessel functions, and the numerator in eq. (36) is always negative, and there is no bound state.

1.4.2 Marginal δ-function scattering in 2D & asymptotic freedom

If we now look at the Schrödinger equation with a δ-function to mock up the effects of the square well for low k we find something funny: the existence of a solution to the equation

$$\left[-\frac{1}{2m}\nabla^2 + \frac{g}{m}\delta^2(\mathbf{r})\right]\psi(r) = E\psi(r) \tag{38}$$

implies a continuous family of solutions $\psi_\lambda(r) = \psi(\lambda r)$ – the same functional form except scaled smaller by a factor of λ – with energy $E_\lambda = \lambda^2 E$. Thus it seems that all possible energy eigenvalues with the same sign as E exist and there are no discrete eigenstates...which

[2] In the following expressions $\gamma_E = 0.577\ldots$ is the Euler constant.

is OK if only positive energy scattering solutions exist, the case for a repulsive interaction — but not if there are bound states: if there is any negative energy state, then there is an unbounded continuum of negative energy states and no ground state. The problem is that ∇^2 and $\delta^2(\mathbf{r})$ have the same dimension, 1/length^2, and so there is no inherent scale in the equation.

Since the δ-function interaction seems to be scale invariant, we say that it is neither relevant (dominating IR physics, as in 1D) nor irrelevant (unimportant to IR physics, as in 3D) but apparently of equal important at all scales, which we call *marginal.* However, we know that (i) the δ function description appears to be sick, and (ii) from our exact analysis of the square well that the IR description of the full theory is not really scale invariant, due to the logarithm. Therefore it is a reasonable guess that our analysis of the *delta*-function is incorrect due to its singularity, and that we are going to have to be more careful, and renormalize.

We can repeat the Feynman diagram approach we used in 3D, only now in 2D. Now the loop integral in eq. (29) is required in $d=3$ spacetime dimensions instead of $d=4$. It is still divergent, but now only log divergent, not linearly divergent. It still needs regularization, but this time instead of using a momentum cutoff I will use dimensional regularization, to make it look even more like conventional QFT calculations. Therefore we keep the number of spacetime dimensions d arbitrary in computing the integral, and subsequently expand about $d=3$ (for scattering in $D=2$ spatial dimensions)[3]. We take for our action

$$S=\int dt\int d^{d-1}x\left[\psi^\dagger\left(i\partial_t+\frac{\nabla^2}{2M}\right)\psi-\mu^{d-3}\frac{C_0}{4}\left(\psi^\dagger\psi\right)^2\right]\ . \tag{39}$$

where the renormalization scale μ was introduced to keep C_0 dimensionless (see problem). Then the Feynman rules are the same as in the previous case, except for the factor of μ^{d-3} at the vertices, and we find

$$\begin{aligned}B(E)\quad &=\quad \mu^{3-d}\int\frac{d^{d-1}q}{(2\pi)^{d-1}}\frac{1}{E-\frac{\mathbf{q}^2}{M}+i\epsilon}\\ &=\quad -M\left(-ME-i\epsilon\right)^{\frac{d-3}{2}}\Gamma\left(\frac{3-d}{2}\right)\frac{\mu^{3-d}}{(4\pi)^{(d-1)/2}}\\ &\xrightarrow{d\to 3}\quad \frac{M}{2\pi}\frac{1}{(d-3)}+\frac{M}{4\pi}\left(\gamma_E-\ln 4\pi+\ln\frac{k^2}{\mu^2}-i\pi\right)+O(d-3)\end{aligned} \tag{40}$$

and the scattering amplitude is therefore

$$\mathcal{A}=\frac{1}{-\frac{1}{C_0}+B(E)}=\left[-\frac{1}{C_0}+\frac{M}{2\pi}\frac{1}{(d-3)}+\frac{M}{4\pi}\left(\gamma_E-\ln 4\pi+\ln\frac{k^2}{\mu^2}-i\pi\right)\right]^{-1} \tag{41}$$

[3]If you are curious why I did not use dimensional regularization for the $D=3$ case: dim reg ignores power divergences, and so when computing graphs with power law divergences using dim reg you do not explicitly notice that you are fine-tuning the theory. This happens in the standard model with the quadratic divergence of the Higgs mass^2...every few years someone publishes a preprint saying there is no fine-tuning problem since one can compute diagrams using dim reg, where there is no quadratic divergence, which is silly. I used a momentum cutoff in the previous section so we could see the fine-tuning of C_0.

At this point it is convenient to define the dimensionless coupling constant g:

$$C_0 \equiv g \frac{4\pi}{M} \tag{42}$$

so that the amplitude is

$$\mathcal{A} = \frac{4\pi}{M} \left[-\frac{1}{g} - \frac{2}{(d-3)} + \gamma_E - \ln 4\pi + \ln \frac{k^2}{\mu^2} - i\pi \right]^{-1} \tag{43}$$

To make sense of this at $d = 3$ we have to renormalize g with the definition:

$$\frac{1}{g} = \frac{1}{\bar{g}(\mu)} + \frac{2}{(d-3)} + \gamma_E - \ln 4\pi \ , \tag{44}$$

where $\bar{g}(\mu)$ is the renormalized running coupling constant, and so the amplitude is given by

$$\mathcal{A} = \frac{4\pi}{M} \left[-\frac{1}{\bar{g}(\mu)} + \ln \frac{k^2}{\mu^2} - i\pi \right]^{-1} \tag{45}$$

Since this must be independent of μ it follows that

$$\mu \frac{d}{d\mu} \left(-\frac{1}{\bar{g}(\mu)} + \ln \frac{k^2}{\mu^2} \right) = 0 \tag{46}$$

or equivalently,

$$\mu \frac{d\bar{g}(\mu)}{d\mu} = \beta(\bar{g}) \ , \qquad \beta(\bar{g}) = 2\bar{g}(\mu)^2 \ . \tag{47}$$

If we specify the renormalization condition $\bar{g}(\mu_0) \equiv \bar{g}_0$, then the solution to this renormalization group equation is

$$\bar{g}(\mu) = \frac{1}{\frac{1}{\bar{g}_0} + 2 \ln \frac{\mu_0}{\mu}} \ . \tag{48}$$

Note that this solution $\bar{g}(\mu)$ blows up at

$$\mu = \mu_0 e^{-1/(2\bar{g}_0)} \equiv \Lambda \tag{49}$$

so that we can write $\bar{g}(\mu)$ as

$$\bar{g}(\mu) = \frac{1}{\ln \frac{\Lambda^2}{\mu^2}} \ , \tag{50}$$

and the amplitude as

$$\mathcal{A} = \frac{4\pi}{M} \frac{1}{\ln \frac{k^2}{\Lambda^2} + i\pi} \ , \tag{51}$$

or equivalently,

$$\cot\delta = -\frac{1}{\pi}\ln\frac{k^2}{\Lambda^2}\ . \tag{52}$$

Now just have to specify Λ instead of $\bar{g}_0$ to define the theory ("dimensional transmutation").

Finally, we can match this δ-function scattering amplitude to the square well scattering amplitude at low k by equating eq. (52) with our expression eq. (36), yielding the matching condition

$$\ln\frac{k^2}{\Lambda^2} = -2\left(\frac{J_0(\alpha)}{\alpha J_1(\alpha)} + \log\left(\frac{\Delta k}{2}\right) + \gamma_E\right) \tag{53}$$

from which the k dependence drops out and we arrive at and expression for Λ in terms of the coupling constant α of the square well:

If $g_0 < 0$ (attractive interaction) the scale Λ is in the IR ($\mu \ll \mu_0$ if g_0 is moderately small) and we say that the interaction is asymptotically free, with Λ playing the same role as $\Lambda_{\rm QCD}$ in the Standard Model – except that here we are not using perturbation theory, the β-function is exact, and we can take $\mu < \Lambda$ and watch $\bar{g}(\mu)$ change from $+\infty$ to $-\infty$ as we scale through a bound state. If instead $g_0 > 0$ (repulsive interaction) then Λ is in the UV, we say the theory is asymptotically unfree, and Λ is similar to the Landau pole in QED. So we see that while the Schrödinger equation *appeared* to have a scale invariance and therefore no discrete states, in reality when one makes sense of the singular interaction, a scale Λ seeks into the theory, and it is no longer scale invariant.

1.5 Lessons learned

We have learned the following by studying scattering from a finite range potential at low k in various dimensions:

- A contact interaction (δ-function) is more irrelevant in higher dimensions;
- marginal interactions are characterized naive scale invariance, and by logarithms of the energy and running couplings when renormalization is accounted for; they can either look like relevant or irrelevant interactions depending on whether the running is asymptotically free or not; and in either case they are characterized by a mass scale Λ exponentially far away from the fundamental length scale of the interaction, Δ.
- Irrelevant interactions and marginal interactions typically require renormalization; an irrelevant interaction can sometimes be made relevant if its coefficient is tuned to a critical value.

All of these lessons will be pertinent in relativistic quantum field theory as well.

1.6 Problems for lecture I

I.1) Explain Fig. 1: how do you interpret those oscillations? Similarly, what about the cycles in Fig. 2?

I.2) Consider dimensional analysis for the non relativistic action eq. (39). Take momenta p to have dimension 1 by definition in any spacetime dimension d; with the uncertainty principle $[x,p] = i\hbar$ and $\hbar = 1$ we then must assign dimension -1 to spatial coordinate x. Write this as

$$[p] = [\partial_x] = 1 \ , \qquad [x] = -1 \ . \tag{54}$$

Unlike in the relativistic theory we can treat M as a dimensionless parameter under this scaling law. If we do that, use eq. (39) with the factor of μ omitted to figure out the scaling dimensions

$$[t] \ , \quad [\partial_t] \ , \quad [\psi] \ , \quad [C_0] \tag{55}$$

for arbitrary d, using that fact that the action S must be dimensionless (after all, in a path integral we exponentiate $S/\hbar$, which would make no sense if that was a dimensional quantity). What is special about $[C_0]$ at $d = 3$? Confirm that including the factor of μ^{d-3}, where μ has scaling dimension 1 ($[\mu] = 1$) allows C_0 to maintain its $d = 3$ scaling dimension for any d.

I.3) In eq. (52) the distinction between attractive and repulsive interactions seems to have been completely lost since that equation holds for both cases! By looking at how the 2D matching works in describing the square well by a δ-function, explain how the low energy theory described by eq. (52) behaves differently when the square well scattering is attractive versus repulsive. Is there physical significance to the scale Λ in the effective theory for an attractive interaction? What about for a repulsive interaction?