2 EFT at tree level

To construct a relativistic effective field theory valid up to some scale Λ, we will take for our action made out of all light fields (those corresponding to particles with masses or energies much less that Λ) including all possible local operators consistent with the underlying symmetries that we think govern the world. All UV physics that we are not including explicitly is encoded in the coefficients of these operators, in the same way we saw in the previous section that a contact interaction (δ-function potential) was able to reproduce the scattering length for scattering off a square well if its coefficient was chosen appropriately (we "matched" it to the UV physics). However, in the previous examples we just tried matching the scattering lengths; we could have tried to also reproduce $O(k^2\Delta^2)$ effects, and so on, but to do so would have required introducing more and more singular contributions to the potential in the effective theory, such as $\nabla^2\delta(\mathbf{r})$, $\nabla^4\delta(\mathbf{r})$, and so on. Going to all orders in k^2 would require an infinite number of such terms, and the same is true for a relativistic EFT. Such a theory is not "renormalizable" in the historical sense: there is typically no finite set of coupling constants that can be renormalized with a finite pieces of experimental data to render the theory finite. Instead there are an infinite number of counterterms need to make the theory finite, and therefore an infinite number of experimental data needed to fix the finite parts of the counterterms. Such a theory would be unless there existed some sort of expansion that let us deal with only a finite set of operators at each order in that expansion.

Wilson provided such an expansion. The first thing to accept is that the EFT has an intrinsic, finite UV cutoff Λ. This scale is typically the mass of the lightest particles *omitted* from the theory. For example, in the Fermi theory of the weak interactions, $\Lambda = M_W$. With a cutoff in place, all radiative corrections in the theory are finite, even if they are proportional to powers or logarithms of Λ. The useful expansion then is a momentum expansion, in powers of k/Λ, where k is the external momentum in some physical process of interest, such as a particle decay, two particle scattering, two particle annihilation, etc. This momentum expansion is the key tool that makes EFTs useful. To understand how this works, we need to develop the concept of operator dimension. In this lecture we will only consider the EFT at tree level.

2.1 Scaling in a relativistic EFT

As a prototypical example of an EFT, consider the Lagrangian (in four dimensional Euclidean spacetime, after a Wick rotation to imaginary time) for relativistic scalar field with a $\phi \to -\phi$ symmetry:

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \sum_n \left(\frac{c_n}{\Lambda^{2n}}\phi^{4+2n} + \frac{d_n}{\Lambda^{2n}}(\partial\phi)^2\phi^{2+2n} + \dots\right) \tag{56}$$

We are setting $\hbar = c = 1$ so that momenta have dimension of mass, and spacetime coordinates have dimension of inverse mass. I indicate this as

$$[p] = 1\ , \qquad [x] = [t] = -1\ , \qquad [\partial_x] = [\partial_t] = 1\ . \tag{57}$$

Since the action is dimensionless, then from the kinetic term for ϕ we see that ϕ has dimension of mass:

$$[\phi] = 1 \ . \tag{58}$$

That means that the operator ϕ^6 is dimension 6, and the contribution to the action $\int d^4x\, \phi^6$ has dimension 2, and so its coupling constant must have dimension -2. The operator $\phi^2(\partial^2\phi)^2$ is dimension 8 and must have a coefficient which is dimension -4. In eq. (56) I have introduced the cutoff scale Λ explicitly into the Lagrangian in such a way as to make the the couplings λ, c_n and d_n all dimensionless, with no loss of generality. I will assume here that $\lambda \ll 1$, $c_n \ll 1$ and $d_n \ll 1$ so that a perturbative expansions in these couplings is reasonable.

You might ask why we do things this way — why not rescale the ϕ^6 operator to have coefficient 1 instead of the kinetic term, and declare ϕ to have dimension 2/3? The reason why is because the kinetic term is more important and determines the size of quantum fluctuations for a relativistic excitation. To see this, consider the path integral

$$\int D\phi\, e^{-S_E} \ , \qquad S_E = \int d^4x\, \mathcal{L}_E \ . \tag{59}$$

Now consider a particular field configuration contributing to this path integral that looks like the "wavelet" pictured in Fig. 4, with wavenumber $|k_\mu| \sim k$, localized to a spacetime volume of size L^4, where $L \simeq 2\pi/k$, and with amplitude ϕ_k. Derivatives acting on such a configuration give powers of k, while spacetime integration gives a factor of $L^4 \simeq (4\pi/k)^4$. With this configuration, the Euclidean action is given by

$$\begin{aligned} S_E &\simeq \left(\frac{2\pi}{k}\right)^4 \left[\frac{k^2\phi_k^2}{2} + m^2\phi_k^2 + \frac{\lambda}{4!}{\phi_k}^4 + \sum_n \left(\frac{c_n}{\Lambda^{2n}}{\phi_k}^{4+2n} + \frac{d_n k^2}{\Lambda^{2n}}{\phi_k}^{4+2n} + \ldots\right)\right] \\ &= (2\pi)^4 \left[\frac{\widehat{\phi}_k^2}{2} + \frac{m^2}{k^2}\widehat{\phi}_k^2 + \frac{\lambda}{4!}\widehat{\phi}_k^4 + \sum_n \left(c_n\left(\frac{k^2}{\Lambda^2}\right)^n \widehat{\phi}_k^{4+2n} + d_n\left(\frac{k^2}{\Lambda^2}\right)^n \widehat{\phi}_k^{4+2n} + \ldots\right)\right] \ , \end{aligned} \tag{60}$$

where in the second line I have rescaled the amplitude by k,

$$\widehat{\phi}_k \equiv \phi_k/k \ . \tag{61}$$

Now for the path integral, consider ordinary integration over the amplitude $\widehat{\phi}_k$ for this particular mode:

$$\int d\widehat{\phi}_k\, e^{-S_E} \ . \tag{62}$$

The integral is dominated by those values of $\widehat{\phi}_k$ for which $S_E \lesssim 1$, because otherwise $\exp(-S_E)$ is very small. Which are the important terms in S_E in this region? First, assume that the particle is relativistic, $m \ll k \ll \Lambda$. Then, since m^2/k^2 and the couplings λ, c_n, d_n are small, as one increases the amplitude $\widehat{\phi}_k$ from zero, the first term in S_E to become become $O(1)$ is the kinetic term, $(2\pi)^4\widehat{\phi}_k^2$, which occurs for $\phi_k = k\widehat{\phi}_k \sim k/(2\pi)^2$. It

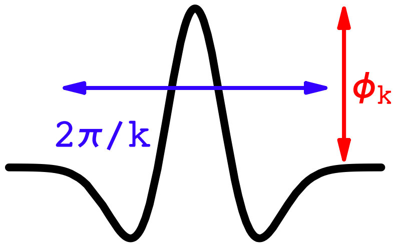

Figure 4: *sample configuration contributing to the path integral for the scalar field theory in eq. (56). Its amplitude is* ϕ_k *and has wave number* $\sim k$ *and spatial extent* $\sim 2\pi/k$.

is because the kinetic term controls the fluctuations of the scalar field that we "canonically normalize" the field such that the kinetic term is $\frac{1}{2}(\partial\phi)^2$, and perturb in the coefficients of the other operators in the theory [4].

What happens as we consider different momenta k? We see from eq. (60) that as k is reduced, the c_n and d_n terms, proportional to $(k^2/\Lambda^2)^n$, get smaller. Such operators are "irrelevant" operators in Wilson's language, because they become unimportant in the infrared (low k). In contrast, the mass term becomes more important; it is called a "relevant" operator. The kinetic term and the $\lambda\phi^4$ interaction do not change; such operators are called "marginal". It used to be thought that the irrelevant operators were dangerous, making the theory nonrenormalizable, while the relevant operators were safe – "superrenormalizable". As we consider radiative corrections later we will see that Wilson flipped this entirely on its head, so that irrelevant operators are now considered safe, while the existence of relevant operators is thought to be a serious problem to be solved.

In practice, when working with a relativistic theory in d spacetime dimensions with small dimensionless coupling constants,the operators with dimension d are the marginal ones, those with higher dimension are irrelevant, and those with lower dimension are relevant. The bottom line is that we can analyze the theory in a momentum expansion, working to a particular order and ignoring irrelevant operators above a certain dimension. The ability to do so will persist even when we include radiative corrections.

2.1.1 Fermi's effective theory of the weak interactions

To see why dimensional analysis has practical consequences, first consider Fermi's theory of the weak interactions. Originally this was a "bottom-up" sort of EFT — Fermi did not have a complete UV description of the weak interactions, and so constructed the theory as a phenomenological modification of QED to account for neutron decay. Now we have the SM, and so we think of the Fermi theory as a "top-down" EFT: not necessary for doing calculations since we have the SM, but very practical.

The weak interactions refer to processes mediated by the $W^\pm$ or Z^0 bosons, whose masses are approximately 80 GeV and 91 GeV respectively. The couplings of these gauge

[4]Note that this is not true if we are interested in a non-relativistic theory where $k \ll m$; in that case the mass term dominates and the scaling behavior changes. The argument I gave could also change if some of the couplings λ, c_n, d_n were sufficiently large.

bosons to quarks and leptons can be written in terms of the electromagnetic current

$$j^\mu_{\text{em}} = \frac{2}{3}\bar{u}_i\gamma^\mu u_i - \frac{2}{3}\bar{d}_i\gamma^\mu d_i - \bar{e}_i\gamma^\mu e_i \tag{63}$$

where $i = 1, 2, 3$ runs over families, and the left-handed $SU(2)$ currents

$$j^\mu_a = \sum_\psi \overline{\psi}\gamma^\mu \left(\frac{1-\gamma_5}{2}\right)\frac{\tau_a}{2}\psi\ , \qquad a = 1, 2, 3\ , \tag{64}$$

where the $\overline{\psi}, \psi$ fields in the currents are either the lepton doublets

$$\begin{pmatrix}\nu_e\\ e\end{pmatrix}\ , \quad \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}\ , \quad \begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}\ , \tag{65}$$

or the quark doublets

$$\psi = \begin{pmatrix}u\\ d'\end{pmatrix}\ , \quad \begin{pmatrix}c\\ s'\end{pmatrix}\ , \quad \begin{pmatrix}t\\ b'\end{pmatrix}\ , \tag{66}$$

with the "flavor eigenstates" d', s' and b' being related to the mass eigenstates d, s and b by the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix[5]:

$$q'_i = V_{ij}q_j\ . \tag{67}$$

The SM coupling of the heavy gauge bosons to these currents is

$$\mathcal{L}_{\text{J}} = \frac{e}{\sin\theta_w}\left(W^+_\mu J^\mu_- + W^-_\mu J^\mu_+\right) + \frac{e}{\sin\theta_w\ \cos\theta_w}Z_\mu\left(j^\mu_3 - \sin^2\theta_w j^\mu_{\text{em}}\right) \tag{68}$$

where

$$J^\mu_\pm = \frac{j^\mu_1 \mp i\, j^\mu_2}{\sqrt{2}}\ . \tag{69}$$

Tree level exchange of a W boson then gives the amplitude at low momentum exchange

$$i\mathcal{A} = \left(-i\frac{e}{\sin\theta_w}\right)^2 J^\mu_- J^\nu_+ \frac{-ig_{\mu\nu}}{q^2 - M_W^2} = -i\frac{e^2}{\sin^2\theta_w M_W^2}J^\mu_- J_{\mu+} + O\left(\frac{q^2}{M_W^2}\right)\ . \tag{70}$$

This amplitude can be reproduced to lowest order in q^2/M_W^2 by a low energy EFT with a contact interaction, Fig. 5,

$$\mathcal{L}_{\text{F}} = -\frac{e^2}{\sin^2\theta_w M_W^2}J^\mu_- J_{\mu+} = \frac{8}{\sqrt{2}}G_F J^\mu_- J_{\mu+}\ , \tag{71}$$

$$G_F \equiv \frac{\sqrt{2}}{8}\frac{e^2}{\sin^2\theta_w M_W^2} = 1.166\times 10^{-5}\ \ \text{GeV}^2\ . \tag{72}$$

This charged current interaction, written in terms of leptons and nucleons instead of leptons and quarks, was postulated by Fermi to explain neutron decay; the numerical factors look

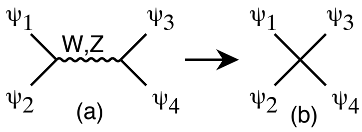

Figure 5: *(a) Tree level W and Z exchange between four fermions. (b) The effective vertex in the low energy effective theory (Fermi interaction).*

funny here because I am normalizing the currents in the way they appear in the SM, while weak currents are historically (pre-SM) normalized differently. Neutral currents were proposed in the 60's and discovered in the 70's.

Since the four-fermion Fermi interaction has dimension 6, it is an irrelevant interaction, according to our previous discussion, explaining why we say the interactions are "weak" and neutrinos are "weakly interacting". Consider, for example, some low energy neutrino scattering cross section σ. Since neutrinos only interact via W and Z exchange, the cross-section σ must be proportional to G_F^2 which has dimension -4. But a cross section has dimensions of area, or mass dimension -2. Since the only other scale around is the center of mass energy $\sqrt{s}$, on purely dimensional grounds σ must scale with energy as

$$\sigma_\nu \simeq G_F^2 s \ , \tag{73}$$

This explains why low energy neutrinos are so hard to detect, and the weak interactions are weak; at LHC energies, however, where the effective field theory has broken down, the weak interactions are marginal and characterized by the $SU(2)$ coupling constant $g \simeq 0.6$, about twice as strong as the electromagnetic coupling. It is a simple result for which one does not need the full machinery of the SM to derive.

It looks like the neutrino cross section grows with s without bound, but remember that this EFT is only valid up to $s \simeq M_W$.

2.1.2 The blue sky

Another top-down application of EFT is to answer the question of why the sky is blue. More precisely, why low energy light scattering from neutral atoms in their ground state (Rayleigh scattering) is much stronger for blue light than red[6] The physics of the scattering process could be analyzed using exact or approximate atomic wave functions and matrix elements, but that is overkill for low energy scattering. Let's construct an "effective Lagrangian" to

[5]The elements of the CKM matrix are named after which quarks they couple through the charged current, namely $V_{11} \equiv V_{ud}$, $V_{12} \equiv V_{us}$, $V_{21} \equiv V_{cd}$, etc.

[6]By "low energy" I mean that the photon energy E_γ is much smaller than the excitation energy ΔE of the atom, which is of course much smaller than its inverse size or mass:

$$E_\gamma \ll \Delta E \ll a_0^{-1} \ll M_{atom} \ .$$

Thus the process is necessarily elastic scattering, and to a good approximation we can ignore that the atom recoils, treating it as infinitely heavy.

describe this process. This means that we are going to write down a Lagrangian with all interactions describing elastic photon-atom scattering that are allowed by the symmetries of the world — namely Lorentz invariance and gauge invariance. Photons are described by a field A_μ which creates and destroys photons; a gauge invariant object constructed from A_μ is the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The atomic field is defined as ϕ_v, where ϕ_v destroys an atom with four-velocity v_μ (satisfying $v_\mu v^\mu = 1$, with $v_\mu = (1,0,0,0)$ in the rest-frame of the atom), while $\phi_v^\dagger$ creates an atom with four-velocity v_μ. In this case we should use relativistic scaling, since we are interested in on-shell photons, and are uninterested in recoil effects (the kinetic energy of the atom):

$$[x] = [t] = -1, \quad [p] = [E] = [A_\mu] = 1 \ , \qquad [\phi] = \frac{3}{2} \ , \tag{74}$$

where the atomic field ϕ destroys an atom with four-velocity v_μ (satisfying $v_\mu v^\mu = 1$, with $v_\mu = (1,0,0,0)$ in the rest-frame of the atom), while $\phi^\dagger$ creates an atom with four-velocity v_μ.

So what is the most general form for $\mathcal{L}_{eff}$? Since the atom is electrically neutral, gauge invariance implies that ϕ can only be coupled to $F_{\mu\nu}$ and not directly to A_μ. So $\mathcal{L}_{eff}$ is comprised of all local, Hermitian monomials in $\phi^\dagger\phi$, $F_{\mu\nu}$, v_μ, and ∂_μ. Certain combinations we needn't consider for the problem at hand — for example $\partial_\mu F^{\mu\nu} = 0$ for radiation (by Maxwell's equations); also, if we define the energy of the atom at rest in it's ground state to be zero, then $v^\mu \partial_\mu \phi = 0$, since $v_\mu = (1,0,0,0)$ in the rest frame, where $\partial_t \phi = 0$. Similarly, $\partial_\mu \partial^\mu \phi = 0$. Thus we are led to consider the interaction Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm eff} &= c_1 \phi^\dagger \phi F_{\mu\nu} F^{\mu\nu} + c_2 \phi^\dagger \phi v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu} \\ &\quad + c_3 \phi^\dagger \phi (v^\alpha \partial_\alpha) F_{\mu\nu} F^{\mu\nu} + \ldots \end{aligned} \tag{75}$$

The above expression involves an infinite number of operators and an infinite number of unknown coefficients! Nevertheless, dimensional analysis allows us to identify the leading contribution to low energy scattering of light by neutral atoms.

With the scaling behavior eq. (**??**), and the need for $\mathcal{L}$ to have dimension 4, we find the dimensions of our couplings to be

$$[c_1] = [c_2] = -3 \ , \qquad [c_3] = -4 \ . \tag{76}$$

Since the c_3 operator has higher dimension, we will ignore it. What are the sizes of the coefficients $c_{1,2}$? To do a careful analysis one needs to go back to the full Hamiltonian for the atom in question interacting with light, and "match" the full theory to the effective theory. We will discuss this process of matching later, but for now we will just estimate the sizes of the c_i coefficients. We note that extremely low energy photons cannot probe the internal structure of the atom, and so the cross-section ought to be classical, only depending on the size of the scatterer. Since such low energy scattering can be described entirely in terms of the coefficients c_1 and c_2, we conclude that

$$c_1 \simeq c_2 \simeq r_0^3 \ .$$

The effective Lagrangian for low energy scattering of light is therefore

$$\mathcal{L}_{eff} = r_0^3 \left(a_1 \phi_v^\dagger \phi F_{\mu\nu} F^{\mu\nu} + a_2 \phi_v^\dagger \phi v^\alpha F_{\alpha\mu} v_\beta F^{\beta\mu} \right) \tag{77}$$

where a_1 and a_2 are dimensionless, and expected to be $\mathcal{O}(1)$. The cross-section (which goes as the amplitude squared) must therefore be proportional to r_0^6. But a cross section σ has dimensions of area, or $[\sigma] = -2$, while $[r_0^6] = -6$. Therefore the cross section must be proportional to

$$\sigma \propto E_\gamma^4 r_0^6 \ , \tag{78}$$

growing like the fourth power of the photon energy. Thus blue light is scattered more strongly than red, and the sky looks blue.

Is the expression eq. (78) valid for arbitrarily high energy? No, because we left out terms in the effective Lagrangian we used. To understand the size of corrections to eq. (78) we need to know the size of the c_3 operator (and the rest we ignored). Since $[c_3] = -4$, we expect the effect of the c_3 operator on the scattering amplitude to be smaller than the leading effects by a factor of E_γ/Λ, where Λ is some energy scale. But does Λ equal M_{atom}, $r_0^{-1} \sim \alpha m_e$ or $\Delta E \sim \alpha^2 m_e$? The latter is the smallest scale and hence the most important. We expect our approximations to break down as $E_\gamma \to \Delta E$ since for such energies the photon can excite the atom. Hence we predict

$$\sigma \propto E_\gamma^4 r_0^6 \left(1 + \mathcal{O}(E_\gamma/\Delta E)\right) . \tag{79}$$

The Rayleigh scattering formula ought to work pretty well for blue light, but not very far into the ultraviolet. Note that eq. eq. (79) contains a lot of physics even though we did very little work. More work is needed to compute the constant of proportionality.

2.2 Accidental symmetry and BSM physics

Now let's switch tactics and talk about bottom-up applications of EFT. We would like to have clues of physics beyond the SM (BSM). Evidence we currently have for BSM physics are the existence of gravity, neutrino masses and dark matter. Hints for additional BSM physics include circumstantial evidence for Grand Unification and for inflation, the absence of a neutron electric dipole moment, and the baryon number asymmetry of the universe. Great puzzles include the origin of flavor and family structure, why the electroweak scale is so low compared to the Planck scale (but not so far from the QCD scale), and why we live in an epoch where matter, dark matter, and dark energy all have have rather similar densities.

In order to make progress we would like to have more data, and looking for subtle effects due to irrelevant operators can in some cases give us a much farther experimental reach than can collider physics. Those cases are necessarily ones where the irrelevant operators violate symmetries that are preserved by the marginal and irrelevant operators in the SM. We call these symmetries "accidental symmetries": they are not symmetries of the UV theory, but they are approximate symmetries of the IR theory.

A simple and practical example of an accidental symmetry is $SO(4)$ symmetry in lattice QCD — the Euclidian version of the Lorentz group. Lattice QCD formulates QCD on a 4d hypercubic lattice, and then looks in the IR on this lattice modes whose wavelengths are so long that they are insensitive to the discretization of spacetime. But why is it obvious that a hypercubic lattice will yield a continuum Lorentz invariant theory? The reason lattice field theory works is because of accidental symmetry: Operators on the lattice are constrained by gauge invariance and the hypercubic symmetry of the lattice. While it is possible to write down operators which are invariant under these symmetries while violating the $SO(4)$ Lorentz symmetry, such operators all have high dimension and are not relevant. For example, if A_μ is a vector field, the $SO(4)$-violating operator $A_1A_2A_3A_4$ is hypercubic invariant and marginal and so could spoil the continuum limit we desire; however, the only vector field in lattice QCD is the gauge potential, and such an operator is forbidden because it is not gauge invariant. In the quark sector the lowest dimension operator one can write which is hypercubic symmetric but Lorentz violating is

$$\sum_{\mu=1}^{4} \bar\psi \gamma_\mu D_\mu^3 \psi \tag{80}$$

which is dimension six and therefore irrelevant. Thus Lorentz symmetry is automatically restored in the continuum limit.

Accidental symmetries in the SM notably include baryon number B and lepton number L: if one writes down all possible dimension ≤ 4 gauge invariant and Lorentz invariant operators in the SM, you will find they all preserve B and L. It is possible to write down dimension five $\Delta L = 2$ operators and dimension six $\Delta B = \Delta L = 1$ operators, however. That means that no matter how completely B and L are broken in the UV, at our energies these irrelevant operators become...irrelevant, and B and L appear to be conserved, at least to high precision. So perhaps B and L are not symmetries of the world at all – they just look like good symmetries because the scale of new physics is very high, so that the irrelevant B and L violating operators have very little effect at accessible energies. We will look at these different operators briefly in turn.

2.2.1 BSM physics: neutrino masses

to write down a dimension five gauge variant $\Delta L = 2$ operator:

$$\mathcal{L}_{\Delta L=2} = -\frac{1}{\Lambda}(LH)(LH)\ , \qquad L = \begin{pmatrix}\nu \\ \ell^-\end{pmatrix}\ , \qquad H = \begin{pmatrix}h^+ \\ h^0\end{pmatrix}\ , \qquad \langle H\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix}0 \\ 1\end{pmatrix}\ , \tag{81}$$

where $v = 250$ GeV. There is only one independent operator (ignoring flavor) since the two Higgs fields (HH) cannot be antisymmetrized and therefore must be in an $SU(2)$-triplet. An operator coupling LL in a weak triplet to HH in a weak triplet can be rewritten in the above form, where the combination (LH) is a weak singlet,

$$(LH) = \left(\nu h^0 - \ell^- h^+\right) \quad \longrightarrow \quad \frac{\nu v}{\sqrt{2}}\ . \tag{82}$$

Therefore after spontaneous symmetry breaking by the Higgs, the operator gives a contribution to the neutrino mass,

$$\mathcal{L}_{\Delta L=2} = -\frac{1}{2} m_\nu \nu\nu\ , \qquad m_\nu = \frac{v^2}{\Lambda}\ , \tag{83}$$

a $\Delta L = 2$ Majorana mass for the neutrino. A mass of $m_\nu = 10^{-2}$ eV corresponds to $\Lambda = 6 \times 10^{15}$ GeV, an interesting scale, being near the scale of GUT models, and far

Figure 6: *Two ways the dimension 5 operator for neutrino masses in eq. (81) could arise from tree level exchange of a heavy particle: either from exchange of a heavy $SU(2) \times U(1)$ singlet fermion N, or else from exchange of a massive $SU(2)$ triplet scalar ϕ.*

beyond the reach of accelerator experiments. Or: if $\Lambda = 10^{19}$ GeV, the Planck scale, then $m_\nu = 10^{-5}$ eV. This operator provides a possible and rather compelling explanation for the smallness of observed neutrino masses: they arise as Majorana masses because lepton number is not a symmetry of the universe, but are very small because lepton number becomes an accidental symmetry below a high scale. Of course, we could have the spectrum of the low energy theory wrong: perhaps there is a light right-handed neutrino and neutrinos only have L-preserving Dirac masses like the charged leptons, small simply because of a very small Yukawa coupling to the Higgs. Neutrinoless double beta decay experiments are searching for lepton number violation in hopes of establishing the Majorana mass scenario.

In any case, it is interesting to imagine what sort of UV physics could give rise to the operator in eq. (81). Two possibilities present themselves for how such an operator could arise from a high energy theory at tree level, shown in Fig. 6 – either through exchange of a heavy $SU(2) \times U(1)$ singlet fermion (a "right handed neutrino"), or else via exchange of a heavy scalar with quantum number 3_1 under $SU(2) \times U(1)$. The fact that the resultant light neutrino mass is inversely proportional to the new scale of physics (called the "see-saw mechanism") simply results from the fact that a neutrino mass operator in the SM is an irrelevant dimension-5 operator.

Note that just as G_F is proportional to g^2/M_W^2, and therefore knowing G_F was not sufficient for predicting the W mass, the scale Λ is not necessarily the mass of a new particle, as it will be inversely proportional to coupling constants about which we know nothing except in the context of some particular UV candidate theory.

2.2.2 BSM physics: proton decay

At dimension 6 one can write down operators in the SM which violate B; as they all consist of three quark fields and a lepton field, they are all $\Delta B = 1$, $\Delta L =$ operators conserving the combination $B - L$. Below the QCD scale one needs to match the three quark operator onto hadron fields. An example of such an operator would be

$$\frac{1}{\Lambda^2}\epsilon_{abc}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}(d_L^{a\alpha}u_L^{b\beta})(u_L^{c\gamma}e_L^{\delta} - d_L^{c\gamma}\nu_L^{\delta}) \ , \tag{84}$$

where a, b, c are color indices and $\alpha, \beta, \gamma, \delta$ are $SU(2)$ Lorentz indices for the left-handed Weyl spinors; the terms in parentheses are weak $SU(2)$ singlets, and the whole operator is neutral under weak hypercharge. Below the QCD scale one has to match the three-quark operator onto hadrons fields. Thus roughly speaking $uud \to Z_1 p + Z_2(p\pi^0 + n\pi^+) + \ldots$. We can assume that the Z factors are made up of pure numbers times the appropriate powers of the strong interaction scale, such as $f_\pi \simeq 100$ MeV, the pion decay constant. The Z_1

term cannot lead to proton decay, but the Z_2 term can via the processes $p \to e^+\pi^0$, or $p \to \pi^+\nu$. We can make a crude estimate of the width (inverse lifetime) to be

$$\Gamma \simeq \frac{M_p^5}{\Lambda^4}\frac{1}{8\pi} \tag{85}$$

where I used dimensional analysis to estimate the M_p^5/Λ^4 factor, assuming that the strong interaction scale in Z_2 as well as powers of momenta from phase space integrals could be approximated by the proton mass M_p , and I inserted a typical 2-body phase space factor of $1/8\pi$. For a bound on the proton lifetime of $\tau_p > 10^{34}$ years, this crude estimate gives us $\Lambda \gtrsim 10^{16}$ GeV, not so far off the bound one finds from a more sophisticated calculation. If proton decay is discovered, that will tell us something about the scale of new physics, and then the task will be to construct the full UV theory from what we learn about proton decay, much as the SM was discovered starting from the Fermi theory.

2.3 BSM physics: "partial compositeness"

This next topic does not have to do with accidental symmetry violation, but instead picks up on an interesting feature of the baryon number violating interaction we just discussed, as it suggests a mechanism for quarks and leptons to acquire masses without a Higgs. In estimating the effects of the dimension six $\Delta B = 1$ operator in the previous section I said that the 3-quark operator could be expanded as $uud \to Z_1 p + Z_2(p\pi^0 + n\pi^+) + \ldots$, and then focussed on the Z_2 term. But what about the Z_1 term? By dimensions, $Z_1 \sim \Lambda_{QCD}^3$, and so that term gives rise to a peculiar mass term of the form

$$\frac{\Lambda_{QCD}^3}{\Lambda^2} pe \tag{86}$$

which allows a proton to mix with a positron. If we imagined eliminating the Higgs doublet from the SM, the proton would still get a mass from chiral symmetry breaking in QCD, and even though there would not be an electron mass, there would be the above contribution allowing positron-proton mixing. For the two component system one would have a mass matrix looking something like

$$\begin{pmatrix} M_p & \frac{\Lambda_{QCD}^3}{\Lambda^2} \\ \frac{\Lambda_{QCD}^3}{\Lambda^2} & 0 \end{pmatrix} \tag{87}$$

and to the extent that $\Lambda \gg \Lambda_{QCD}$ we find the mass eigenvalues to be

$$m_1 \simeq M_p \ , \qquad m_2 \simeq \frac{\Lambda_{QCD}^6}{M_p \Lambda^4} \tag{88}$$

so for $\Lambda = 10^{16}$ GeV the positron gets a mass of $m_e \simeq 10^{-64}$ GeV. Yes, this is a ridiculously small mass of no interest, but it is curious that the positron got a mass at all, without there being any Higgs field! It must be that QCD has spontaneously broken $SU(2) \times U(1)$ without a Higgs, and that this proton decay operator has somehow taken the place of a Higgs Yukawa coupling — the roles the Higgs plays in the SM. Therefore it is worth asking whether this example be modified somehow to obtain more interesting masses for quarks and leptons?

In the last lecture we will examine how QCD breaks the weak interactions, and how a scaled up version called technicolor, with the analogue of the pion decay constant f_π being up at the 250 GeV scale instead of 93 MeV, could properly account for the spontaneous breaking of $SU(1) \times U(1)$ without a Higgs. Here I will just comment that such a theory would be expected to have TeV mass "technibaryons", which could carry color and charge. With an appropriate dimension 6 operator such as our proton decay operator, but with techniquarks in place of quarks, and all the standard model fermions in place of the positron field, in principle one could give masses to all the SM fermions through their mixing with the technibaryons. This is the idea of "partial compositeness", which in its original formulation [1] was not especially useful for model building, but which has become more interesting in the context of composite Higgs [2] – more about composite Higgs later too.

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2.4 Problems for lecture II

II.1) What is the dimension of the operator ϕ^{10} in a $d = 2$ relativistic scalar field theory?

II.2) One defines the "critical dimension" d_c for an operator to be the spacetime dimension for which that operator is marginal. How will that operator behave in dimensions d when $d > d_c$ or $d < d_c$? In a theory of interacting relativistic scalars, Dirac fermions, and gauge bosons, determine the critical dimension for the following operators:

1. A ϕ^3 interaction;
2. A gauge coupling to either a fermion or a boson through the covariant derivative in the kinetic term;
3. A Yukawa interaction, $\phi\bar{\psi}\psi$;
4. An anomalous magnetic moment coupling $\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$ for a fermion;
5. A four fermion interaction, $(\bar{\psi}\psi)^2$.

II.3) Derive the analogue of Fermi's theory in eq. (71) for tree level Z exchange, expressing your answer in terms of G_F using the fact that $M_Z^2 = M_W^2/\cos^2\theta_w$.

II.4) Aside from the dimension 5 lepton number violating operator we discussed, what other interesting dimension 5 operators could be added to the SM, and what experiments could look for their effects?

II.5) Show that the operator

$$\epsilon_{\alpha\beta}\left(L_{\alpha i}(\sigma_2\sigma^a)_{ij}L_{\beta j}\right)\left(H_k(\sigma_2\sigma^a)_{k\ell}H_\ell\right)$$

is equivalent up to a factor of two to

$$\epsilon_{\alpha\beta}(L_{\alpha i}(\sigma_2)_{ij}H_j)(L_{\beta k}(\sigma_2)_{k\ell}H_\ell)$$

where α, β are Weyl spinor indices, and i, j, k, ℓ are the $SU(2)$ gauge group indices. Write down the two high energy theories that could give rise to the neutrino mass operator as in Fig. 6. How do I see that these theories break lepton number by two units?