4 Chiral perturbation theory

4.1 Chiral symmetry in QCD

QCD is the accepted theory of the strong interactions. At large momentum transfer, as in deep inelastic scattering processes and the decays of heavy particles such as the Z, the theory is perturbative due to asymptotic freedom. The flip side is that in the infrared, the theory becomes nonperturbative. This is good in the sense that we know that the light hadrons don't look at all like a collection of quarks weakly interacting via gluon exchange. But it does mean that QCD is not of much help in quantitatively understanding this phenomenology without resorting to lattice QCD and a computer. However, there does exist an effective field theory which is very powerful for treating analytically the interactions of the lightest hadrons, the pseudoscalar octet, consisting of the π, K, $\overline{K}$ and η.

The reason that the pseudoscalar octet mesons are lighter is because they are the pseudo-Goldstone bosons (PGBs) that arise from the spontaneous breaking of an approximate symmetry in QCD.

Consider the QCD Lagrangian, keeping only the three lightest quarks, u, d and s:

$$\mathcal{L} = \sum_{i=1}^{3} \left(\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i \right) - \frac{1}{2} \mathrm{Tr} G_{\mu\nu} G^{\mu\nu} \ , \tag{97}$$

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative, $A_\mu = A^a_\mu T_a$ are the eight gluon fields with T_a being $SU(3)$ generators in the 3 representation, and $G_{\mu\nu}$ being the gluon field strength. Note that if I write the kinetic term in terms of right-handed and left-handed quarks, projected out by $(1 \pm \gamma_5)/2$ respectively, then the kinetic term may be written as

$$\sum_i \bar{q}_i i \not{D} q_i = \sum_i \left(\bar{q}_{Li} i \not{D} q_{Li} + \bar{q}_{Ri} i \not{D} q_{Ri} \right) \ . \tag{98}$$

This term by itself evidently respects a $U(3)_L \times U(3)_R$ symmetry, where I rotate the three flavors of left-handed and right-handed quarks by independent unitary matrices. One combination of these transformations, the $U(1)_A$ transformation where $q_i \to e^{i\alpha\gamma_5} q_i$ is in fact not a symmetry of the quantum theory, due to anomalies; it is a symmetry of the action but not of the measure of the path integral. This leaves us with a $U(1)_V \times SU(3)_L \times SU(3)_R$ symmetry. The $U(1)_V$ is just baryon number, under which both left- and right-handed quarks of all flavors pick up a common phase. The remaining $SU(3)_L \times SU(3)_R$ symmetry, under which $q_{Li} \to L_{ij} q_{Lj}$ and $q_{Rj} \to R_{ij} q_{Rj}$, where R and L are independent $SU(3)$ matrices, is called "chiral symmetry".

$SU(3)_L \times SU(3)_R$ is not an exact symmetry of QCD, however. The quark mass terms may be written as

$$\sum_i m_i \bar{q}_i q_i = \sum_{i,j} \bar{q}_{Ri} M_{ij} q_{Lj} + h.c. \ , \qquad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} , \tag{99}$$

where the quark masses m_i are called "current masses", not to be confused with the much bigger constituent quark masses in the quark model. Since the mass term couples left- and right-handed quarks, it is not invariant under the full chiral symmetry. Several observations:

- Note that *if* the mass matrix M were a dynamical field, transforming under $SU(3)_L \times SU(3)_R$ as

$$M \to RML^\dagger \ , \tag{100}$$

then the Lagrangian *would* be chirally invariant. Thinking of the explicit breaking of chiral symmetry as being due to spontaneous breaking due to a field M which transforms as above makes it simple to understand how M must appear in the effective theory, which will have to be chirally invariant given the above transformation. This is called treating M as a "spurion".

- The symmetry is broken to the extent that $M \neq RML^\dagger$. Since m_u and m_d are much smaller than m_s, $SU(2)_L \times SU(2)_R$ is not broken as badly as $SU(3)_L \times SU(3)_R$;
- If all three quark masses were equal but nonzero, then QCD would respect an exact $SU(3)_V \subset SU(3)_L \times SU(3)_R$ symmetry, where one sets $L = R$. This is the $SU(3)$ symmetry of Gell-Mann.
- Since $m_d - m_u$ is small, $SU(2)_V \subset SU(3)_V$, where $L = R$ and they act nontrivially only on the u and d quarks, is quite a good symmetry...also known as isospin symmetry.
- Independent vectorlike phase rotations of the three flavors of quarks are exact symmetries...these three $U(1)$ symmetries are linear combinations of baryon number, I_3 isospin symmetry, and Y (hypercharge). The latter two are violated by the weak interactions, but not by the strong or electromagnetic forces.

We know that this still is not the whole story though. An added complication is that the QCD vacuum spontaneously breaks the chiral $SU(3)_L \times SU(3)_R$ symmetry down to Gell-Mann's $SU(3)_V$ via the quark condensate:

$$\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle = \Lambda^3 \delta_{ij} \ , \tag{101}$$

which transforms as a $(3, \bar{3})$ under $SU(3)_L \times SU(3)_R$. Here Λ has dimensions of mass. If one redefines the quark fields by a chiral transformation, the Kronecker δ-function above gets replaced by a general $SU(3)$ matrix,

$$\delta_{ij} \to (LR^\dagger)_{ij} \equiv \Sigma_{ij} \ . \tag{102}$$

If $L = R$ (an $SU(3)_V$ transformation), $\Sigma_{ij} = \delta_{ij}$ which shows that the condensate leaves unbroken the $SU(3)_V$ symmetry. For $L \neq R$, Σ_{ij} represents a different vacuum from eq. (101), and if it wasn't for the explicit breaking of $SU(3)_L \times SU(3)_R$ by quark masses in the QCD Lagrangian, these two different vacua would be degenerate. By Goldstone's theorem therefore, there would have to be eight exact Goldstone bosons — one for each of the eight broken generators — corresponding to long wavelength, spacetime dependent rotations of the condensate. We will parametrize these excitations by replacing

$$\Sigma \to \Sigma(x) \equiv e^{2i\boldsymbol{\pi}(x)/f} \ , \qquad \boldsymbol{\pi}(x) = \pi_a(x) T_a \tag{103}$$

where the T_a are the $SU(3)$ generators ($a = 1, \ldots, 8$) in the defining representation normalized to

$$\mathrm{Tr} T_a T_b = \frac{1}{2}\delta_{ab} \ , \tag{104}$$

f is a parameter with dimension of mass which we will relate to the pion decay constant f_π, and the π_a are eight mesons transforming as an octet under $SU(3)_V$. These bosons correspond to long wavelength excitations of the vacuum.

If you are somewhat overwhelmed by this amazing mix of symmetries that are gauged, global, exact, approximate, spontaneously broken and anomalous (and usually more than one of these attributes at the same time), rest assured that it took a decade and many physicists to sort it all out (the 1960's).

4.2 Quantum numbers of the meson octet

It is useful to use the basis for $SU(3)$ generators $T_a = \frac{1}{2}\lambda_a$, where λ_a are Gell Mann's eight matrices. The meson matrix $\boldsymbol{\pi} \equiv \pi_a T_a$ appearing in the exponent of Σ is a traceless 3×3 matrix. We know that under and $SU(3)_V$ transformation $L = R = V$,

$$\Sigma \to V\Sigma V^\dagger = e^{2iV\boldsymbol{\pi}V^\dagger/f} , \tag{105}$$

implying that under $SU(3)_V$ the mesons transform as an octet should, namely

$$\boldsymbol{\pi} \to V\boldsymbol{\pi}V^\dagger . \tag{106}$$

Then by restricting V to be an I_3 (T_3) or a Y (T_8) rotation we can read off the quantum numbers of each element of the $\boldsymbol{\pi}$ matrix and identify them with real particles:

$$\boldsymbol{\pi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \tag{107}$$

An easy way to understand the normalization is to check that

$$\mathrm{Tr}(\boldsymbol{\pi}\boldsymbol{\pi}) = \frac{1}{2}\sum_a (\pi_a)^2 = \frac{1}{2}(\pi^0)^2 + \frac{1}{2}\eta^2 + \pi^+\pi^- + K^+K^- + K^0\overline{K}^0 . \tag{108}$$

4.3 The chiral Lagrangian

4.3.1 The leading term and the meson decay constant

We are now ready to write down the effective theory of excitations of the chiral condensate (the chiral Lagrangian), ignoring all the other modes of QCD. This is analogous to the quantization of rotational modes of a diatomic molecule, ignoring the vibrational modes. We are guided by two basic principles of effective field theory: (i) The chiral Lagrangian must exhibit the same approximate chiral symmetry as QCD, which means that it must be invariant under $\Sigma \to L\Sigma R^\dagger$ for arbitrary $SU(3)_L \times SU(3)_R$ matrices L, R. We will also be able to incorporate symmetry breaking effects by including the matrix M, requiring that the chiral Lagrangian be invariant under the chiral symmetry if M were to transform as in eq. (100). (ii) The other principle is that the effective theory be an expansion of local operators suppressed by powers of a cutoff Λ, which is set by the scale of physics we are ignoring, such as the ρ, $K*$, ω, and η' mesons (with masses $m_\rho = 770$ MeV, $m_{K^*} = 892$ MeV,

$m_\omega = 782$ MeV and $m_{\eta'} = 958$ MeV). In practice, the cutoff seems to be at $\Lambda \simeq 1$ GeV in many processes. Our calculations will involve an expansion in powers of momenta or meson masses divided by Λ. This cutoff is to be compared with $m_{\pi^\pm} = 140$ MeV, $m_{K^+} = 494$ MeV and $m_\eta = 548$ MeV. For purely mesonic processes, meson masses always appear squared, which helps. Nevertheless, one can surmise that chiral perturbation theory will work far better for pions than kaons or the η. This is a reflection of the fact that $SU(2)_L \times SU(2)_R$ is a much better symmetry of QCD than $SU(3)_L \times SU(3)_R$.

The lowest dimension chirally symmetric operator we can write down is

$$\mathcal{L}_0 = \frac{f^2}{4}\mathrm{Tr}\partial\Sigma^\dagger\partial\Sigma = \mathrm{Tr}\partial\boldsymbol{\pi}\partial\boldsymbol{\pi} + \frac{1}{3f^2}\mathrm{Tr}[\partial\boldsymbol{\pi},\boldsymbol{\pi}]^2 + \ldots \tag{109}$$

Note that the $f^2/4$ prefactor is fixed by requiring that the mesons have canonically normalized kinetic terms. Thus we have an infinite tower of operators involving a single unknown parameter, f. From the above Lagrangian, it would seem that the only way to determine f is by looking at $\pi\pi$ scattering. However there is a better way: by looking at the charged pion decay $\pi \to \mu\nu$. This occurs through the "semi-leptonic" weak interaction eq. (71), namely the operator

$$\frac{1}{\sqrt{2}}G_F V_{ud}\left(\bar{u}\gamma^\mu(1-\gamma_5)d\right)\left(\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu\right) + \text{h.c.} \tag{110}$$

The matrix element of this operator sandwiched between $|\mu\nu\rangle$ and $\langle\pi|$ factorizes, and the leptonic part is perturbative. We are left with the nonperturbative part,

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^-(p)\rangle \equiv i\sqrt{2}\, f_\pi p^\mu \ . \tag{111}$$

The pion decay constant f_π is determined from the charged pion lifetime to be $f_\pi = 92.4 \pm .25$ MeV.

Even though QCD is nonperturbative, we can easily match this charged current operator onto an operator in the chiral Lagrangian. That is because we can write

$$\bar{u}\gamma^\mu(1-\gamma_5)d = 2\left(j^\mu_{L1} + i j^\mu_{L2}\right) \ , \tag{112}$$

where j^μ_{La} are the eight $SU(3)_L$ currents

$$j^\mu_{La} \equiv \bar{q}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)T_a q \ . \tag{113}$$

To compute these currents in the effective theory is easy, since we know that under $SU(3)_L$ transformations $\Sigma \to L\Sigma$, or $\delta_{La}\Sigma = iT_a\Sigma$, and can just compute the left-handed currents from the Lagrangian eq. (109) using Noethers theorem. The result is:

$$j^\mu_{La} = -i\frac{f^2}{2}\mathrm{Tr}T_a\Sigma^\dagger\partial^\mu\Sigma = f\,\mathrm{Tr}T_a\partial^\mu\boldsymbol{\pi} + O(\boldsymbol{\pi}^2) \ . \tag{114}$$

In particular,

$$2\left(j^\mu_{L1} + i j^\mu_{L2}\right) = 2f\,\mathrm{Tr}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\partial^\mu\boldsymbol{\pi} + O(\boldsymbol{\pi}^2) = \sqrt{2}f\partial^\mu\pi^- + O(\pi^2) \ , \tag{115}$$

were I made use of eq. (107). Comparing this equation with eq. (111) we see that to this order,

$$f = f_\pi = 93\ \text{MeV}\ . \tag{116}$$

In general it is not possible to exactly match quark operators with operators in the chiral Lagrangian; it was possible for the semileptonic decays simply because the weak operator factorized into a leptonic matrix element and a hadronic matrix element of an $SU(3)_L$ symmetry current. For a purely hadronic weak decay, such as $K \to \pi\pi$ the four quark operator cannot be factorized, and matching to operators in the chiral Lagrangian involves coefficients which can only be computed on a lattice. Even for these processes the chiral Lagrangian can be predictive, relating weak decays with different numbers of mesons in the final state.

4.3.2 Explicit symmetry breaking

Up to now, I have only discussed operators in the chiral Lagrangian which are invariant. Note that that there are no chirally invariant operators which do *not* have derivatives (other than the operator 1). For example, one cannot write down a chirally invariant mass term for the pions. Recall that without explicit chiral symmetry breaking in the QCD Lagrangian, there would be an infinite number of inequivalent degenerate vacua corresponding to different constant values of the matrix Σ; therefore the energy (and the Lagrangian) can only have operators which vanish when Σ is constant, up to an overall vacuum energy independent of Σ. In fact, rotating $\Sigma \to \Sigma' = \Sigma + i d\theta_a T_a \Sigma$ is an exact symmetry of the theory ($SU(3)_L$), and corresponds to *shifting* the pion fields $\pi_a \to \pi_a + d\theta_a f/2 + O(\pi^2)$. Derivative interactions are a result of this shift symmetry. (In the literature, this is called a *nonlinearly realized* symmetry, which is to say, a spontaneously broken symmetry). A theory of massless particles with nontrivial interactions at zero momentum transfer (such as QCD) would suffer severe infrared divergences, and so if the interactions had not been purely derivative, the theory would either not make sense, or would become nonperturbative like QCD.

This all changes when explicit chiral symmetry breaking is included. Now not all vacua are equivalent, the massless Goldstone bosons become massive "pseudo-Goldstone bosons" (PGBs), and acquire nonderivative interactions. In pure QCD, the only sources of explicit chiral symmetry breaking are instantons (which explicitly break the $U(1)_A$ symmetry, and the quark mass matrix. Electromagnetic interactions also introduce chiral symmetry breaking, as do weak interactions.

Quark masses. To include the effect of quark masses, we need to include the mass matrix M, recalling that if it transformed as in eq. (100), then the theory would have to be invariant. Just as with derivatives, each power of M will be accompanied by $1/\Lambda$. The leading operator we can write down is

$$\mathcal{L}_M = \Lambda^2 f^2 \left(\frac{c}{2}\frac{1}{\Lambda}\text{Tr} M\Sigma + \text{h.c.} \right) \equiv \frac{1}{2} f^2 \text{Tr}(\widetilde{\Lambda} M)\Sigma + \text{h.c.}\ , \tag{117}$$

where c is an unknown dimensionless coefficient, and I defined

$$c\Lambda \equiv \widetilde{\Lambda} = O(\Lambda)\ . \tag{118}$$

Expanding to second order in the $\boldsymbol{\pi}$, I get

$$\mathcal{L}_M = -m_\pi^2\pi^+\pi^- - m_{K^+}^2K^+K^- - m_{K^0}^2K^0\overline{K}^0 - \frac{1}{2}\begin{pmatrix}\pi^0 & \eta\end{pmatrix}M_0^2\begin{pmatrix}\pi^0\\ \eta\end{pmatrix} , \tag{119}$$

with

$$m_\pi^2 = \widetilde{\Lambda}(m_u+m_d) , \qquad m_{K^+}^2 = \widetilde{\Lambda}(m_u+m_s) , \qquad m_{K^0}^2 = \widetilde{\Lambda}(m_d+m_s) , \tag{120}$$

and

$$M_0^2 = \widetilde{\Lambda}\begin{pmatrix}(m_u+m_d) & (m_u-m_d)\\ (m_u-m_d) & \frac{1}{3}(m_u+m_d+4m_s)\end{pmatrix} \tag{121}$$

Note that (i) the squares of the meson masses are proportional to quark masses; (ii) $\pi^0-\eta$ mixing is isospin breaking and proportional to (m_u-m_d); (iii) expanding in powers of (m_u-m_d), m_η^2 and $m_{\pi^0}^2$ are given by the diagonal entries of M_0^2, up to corrections of $O\left((m_u-m_d)^2\right)$; (iv) we cannot directly relate quark and meson masses because of the unknown coefficient $\widetilde{\Lambda}$.

Ignoring isospin breaking due to electromagnetism and the difference $m_u \neq m_d$, the masses obey the Gell-Mann Okuba formula

$$3m_\eta^2 + m_\pi^2 = 4m_K^2 . \tag{122}$$

The two sides of the above equation are satisfied experimentally to better than 1% accuracy.

Electromagnetism. To include electromagnetism into the chiral Lagrangian, we have to first go back to QCD and ask what currents out of our eight j^μ_{La} and j^μ_{Ra} couple to the photon. That is easy: the photon couples to the electromagnetic current which can be written as

$$J^\mu_{\rm em} = e\bar{q}\gamma^\mu P_L Q_L q + e\bar{q}\gamma^\mu P_R Q_R q , \qquad Q = \begin{pmatrix}\frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3}\end{pmatrix} , \tag{123}$$

a simple linear combination of octet currents. So symmetry determines the covariant derivative in the chiral Lagrangian to be

$$D_\mu\Sigma = \partial_\mu\Sigma - ieA_\mu\left(Q_L\Sigma - \Sigma Q_R\right) \tag{124}$$

since $\Sigma \to L\Sigma R^\dagger$ under $SU(3)_L\times SU(3)_R$. Note that when we set the Σ field to its vacuum value, $\Sigma = 1$, the photon term drops out of the covariant derivative, which is to say that the vacuum does not break electromagnetism spontaneously. Also note that the $Q_{L,R}$ matrices are $SU(3)_L\times SU(3)_R$ spurions: in order to have unbroken chiral symmetry we would need $D_\mu\Sigma$ to have the transformation property $D_\mu\Sigma \to LD_\mu\Sigma R^\dagger$ when $\Sigma \to L\Sigma R^\dagger$, which would require the transformation properties

$$Q_L \to L^\dagger Q_L L^\dagger , \qquad Q_R \to RQ_R R^\dagger , \tag{125}$$

which is to say that Q_L transforms as part of the adjoint (octet) of $SU(3)_L$ and a singlet under $SU(3)_R$, and conversely for Q_R. We would have muddled this if we had not taken care to distinguish $Q_{L,R}$ from each other from the start, even though for the photon then end up being the same matrix.

If we now want to compute the electromagnetic contribution to the $\pi^+ - \pi^0$ mass splitting to order α we naturally look at the two one-loop diagrams we encounter in scalar QED. These are quadratically divergent, which means they need a counter term which would contribute to the pion mass^2 approximately $\sim \alpha/4\pi\Lambda^2 \sim e^2 f^2$. From the transformation properties eq. (125) we see that we can add such a counter term operator to the chiral Lagrangian of the form

$$\mathcal{L}_\alpha = \xi f^4 \frac{\alpha}{4\pi} \mathrm{Tr} Q_L \Sigma Q_R \Sigma^\dagger \tag{126}$$

where would would expect $c = O(1)$, but which needs to be fit to data or computed using lattice QCD. If we use the $\overline{MS}$ scheme in Landau gauge, them the 1-loop diagrams vanish and we are left only with the direct contribution from the above operator in . Expanding it to second order in meson fields we get

$$\mathcal{L}_\alpha = -\xi f^4 e^2 \frac{2}{f^2} \mathrm{Tr} Q_L \left[\boldsymbol{\pi}, [\boldsymbol{\pi}, Q_R]\right] = -2\xi e^2 f^2 \left(\pi^+\pi^- + K^+K^-\right) \tag{127}$$

a simple result which says that the meson mass^2 gets shifted by a constant amount proportional to its charge. We can fit ξ in this scheme and find

$$\xi = \frac{m_{\pi^+}^2 - m_{\pi^0}^2}{2e^2 f^2} = 0.9 \; . \tag{128}$$

Thus to leading order in α and the quark masses, our formula for the meson masses take the form

$$m_{\pi^+}^2 = \widetilde{\Lambda}(m_u + m_d) + \frac{\alpha}{4\pi}\Delta^2 \; , \qquad m_{K^+}^2 = \widetilde{\Lambda}(m_u + m_s) + \frac{\alpha}{4\pi}\Delta^2 \; , \tag{129}$$

with the neutral particle masses unchanged. In the above formula, $\Delta^2 = 2\xi(4\pi f)^2$ is a free parameter as far as we are concerned, but following Weinberg, combine lesson masses in such a way that it drops out and calculate the ratios of quark masses via the formulas

$$\frac{(m_{K^+}^2 - m_{K^0}^2) - (m_{\pi^+}^2 - m_{\pi^0}^2)}{m_{\pi^0}^2} = \frac{m_u - m_d}{m_u + m_d} \; , \qquad \frac{3m_\eta^2 - m_{\pi^0}^2}{m_{\pi^0}^2} = \frac{4m_s}{m_u + m_d} \; . \tag{130}$$

Plugging in the measured meson masses, the result is

$$\frac{m_u}{m_d} \simeq \frac{1}{2} \; , \qquad \frac{m_d}{m_s} \simeq \frac{1}{20} \; . \tag{131}$$

To specify the quark masses themselves, one must perform a lattice QCD calculation and designate a renormalization scheme. Lattice simulations typically find m_s renormalized at $\mu = 2$ GeV in the $\overline{MS}$ scheme lies in the $80 - 100$ MeV range, from which one infers from the above ratios $m_d \sim 5$ MeV, $m_u \sim 2.5$ MeV in the same scheme. Evidently most of the mass of baryons and vector mesons does *not* come from the intrinsic masses of the quarks.

4.4 Loops and power counting

What makes the chiral Lagrangian and EFT and not just another model of the strong interactions is that it consists of all local operators consisted with the symmetries of QCD, and that there exists a power counting scheme that allows one to work to a given order, and to be able to make a reliable estimate of the errors arising from neglecting the subsequent order. As discussed in the second lecture, the power counting scheme is intimately related to how one computes radiative corrections in the theory.

Beyond the leading term is an infinite number of chirally invariant operators one can write down which are higher powers in derivatives, as well as operators with more insertions of the quark mass matrix M. The derivative expansion is in powers of ∂/Λ. This power counting is consistent with the leading operator eq. (109), if you consider the chiral Lagrangian to have a prefactor of $\Lambda^2 f^2$, then even in the leading operator derivatives enter as ∂/Λ. Since we have found that meson octet masses scale as $m_{\boldsymbol{\pi}}^2 \simeq (\widetilde{\Lambda} M)$, and since for on-shell pions $p^2 \sim m^2$, it follows that one insertion of the quark mass matrix is equivalent to two derivatives in the effective field theory expansion. This leads us to write the chiral Lagrangian as a function of (∂/Λ) and $\widetilde{\Lambda} M/\Lambda^2$. Including electromagnetism is straightforward as well: since a derivative $\partial\Sigma$ becomes a covariant derivative $D_\mu \Sigma = \partial_\mu \Sigma - ieA_\mu [Q, \Sigma]$, the photon field enters as eA_μ/Λ. Operators arising from electromagnetic loops involve two insertions of the quark charge matrix Q in the proper way (see problem (III.6), along with a loop factor $\alpha/(4\pi)$. Therefore the chiral Lagrangian takes the form

$$\mathcal{L} = \Lambda^2 f^2 \widehat{\mathcal{L}} \left[\Sigma,\, \partial/\Lambda,\, \widetilde{\Lambda} M/\Lambda^2,\, eA/\Lambda,\, (\alpha/4\pi) Q^2 \right] \ , \tag{132}$$

where $\widehat{\mathcal{L}}$ is a dimensionless sum of all local, chirally invariant operators (treating M and Q as spurions), where the coefficient of each term (except $\mathcal{L}_0$) is preceded by a dimensionless coefficient to be fit to experiment... which we expect to be $O(1)$, but which may occasionally surprise us! That last assumption is what allows one to estimate the size of higher order corrections.

It should be clear now in what sense the u, d and s are light quarks and can be treated in chiral perturbation theory, while the c, b and t quarks are not: whether the quarks are light or heavy is relative to the scale Λ, namely the mass scale of resonances in QCD. Since the c has a mass ~ 1.5 GeV there is no sensible way to talk about an approximate $SU(4) \times SU(4)$ chiral symmetry and include D, D_s and η_c mesons in our theory of pseudo-Goldstone bosons[7] Of course, you might argue that the strange quark is sort of heavy and should be left out as well, but if we don't live dangerously sometimes, life is too boring.

4.4.1 Subleading order: the $O(p^4)$ chiral Lagrangian

It is a straightforward exercise to write down subleading operators of the chiral Lagrangian. These are operators of $O(p^4)$, $O(p^2 M)$ and $O(M^2)$, where M is the quark mass matrix. This was first done by Gasser and Leutwyler, and their choice for the set of operators has

[7] This does not mean that an effective theory for $D - \pi$ interactions is impossible. However, the D mesons must be introduced as heavy matter fields, similar to the way we will introduce baryon fields later, as opposed to approximate Goldstone bosons.

become standard:

$$\begin{aligned}
\mathcal{L}_{p^4} \;=\;& L_1 \left(\mathrm{Tr} \left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right) \right)^2 \\
&+ L_2 \mathrm{Tr} \left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma \right) \mathrm{Tr} \left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma \right) \\
&+ L_3 \mathrm{Tr} \left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma \right) \\
&+ L_4 \mathrm{Tr} \left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right) \mathrm{Tr} \left(\chi \Sigma + \mathrm{h.c.} \right) \\
&+ L_5 \mathrm{Tr} \left(\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right) \left(\chi \Sigma + \mathrm{h.c.} \right) \right) \\
&+ L_6 \left(\mathrm{Tr} \left(\chi \Sigma + \mathrm{h.c.} \right) \right)^2 \\
&+ L_7 \left(\mathrm{Tr} \left(\chi \Sigma - \mathrm{h.c.} \right) \right)^2 \\
&+ L_8 \mathrm{Tr} \left(\chi \Sigma \chi \Sigma + \mathrm{h.c.} \right) \;, \qquad (133)
\end{aligned}$$

where $\chi \equiv 2\widetilde{\Lambda} M$, where $\widetilde{\Lambda}$ entered in eq. (117). Additional operators involving $F_{\mu\nu}$ need be considered when including electromagnetism.

Note that according to our power counting, we expect the L_i to be of size

$$L_i \sim \frac{\Lambda^2 f^2}{\Lambda^4} = \frac{f^2}{\Lambda^2} \sim 10^{-2} \;. \qquad (134)$$

4.4.2 Calculating loop effects

Now consider loop diagrams in the effective theory. These are often divergent, and so the first issue is how to regulate them. It is easy to show that a momentum cutoff applied naively violates chiral symmetry; and while it is possible to fix that, by far the simplest regularization method is dimensional regularization with a mass independent subtraction scheme, such as $\overline{MS}$.

The $\overline{MS}$ scheme introduces a renormalization scale μ, usually chosen to be $\mu = \Lambda$. However, unlike with cutoff regularization, one never gets powers of the renormalization scale μ when computing a diagram; μ can only appear in logarithms. Consider, for example, the $O(\pi^4)$ operator from $\mathcal{L}_0$, of the form $\frac{1}{f^2}(\partial\pi)^2\pi^2$, and contract the two pions in $(\partial\pi)^2$; this one-loop graph will renormalize the pion mass. However, since the diagram is proportional to $1/f^2$, and no powers of the renormalization scale μ can appear, dimensional analysis implies that any shift in the pion mass from this graph must be proportional to $\delta m_\pi^2 \sim (m_\pi^0)^4/(4\pi f)^2$, times a possible factor of $\ln(m_\pi/\mu)$, where $(m_\pi^0)^2 \sim \tilde{\Lambda} M$ is the mass squared of the meson at leading order. Here I have included the factor of $1/(4\pi)^2$ that typically arises from a loop diagram. Ignoring the logarithm, compare with this contribution to the pion mass contribution from the $O(p^4)$ chiral Lagrangian, which yields $\delta m_\pi^2 \sim (\tilde{\Lambda} M)^2/\Lambda^2$. We see that so long as

$$4\pi f_\pi \gtrsim \Lambda \;, \qquad (135)$$

then the contribution from the radiative correction from the lowest order operator is comparable to or smaller than the second order tree-level contribution, up to $\ln m_\pi^2/\mu^2$ corrections.

What about the logarithm? Note that $\ln(\Lambda^2/m_\pi^2) \simeq 4$ for $\mu = 1$ GeV. Therefore a term with a logarithm is somewhat enhanced relative to the higher order tree-level contributions.

It is therefore common to see in the literature a power counting scheme of the form

$$p^2 > p^4 \ln \mu^2/p^2 > p^4 \dots \tag{136}$$

which means that in order of importance, one computes processes in the following order:

1. Tree level contributions from the $O(p^2)$ chiral Lagrangian;
2. Radiative corrections to the $O(p^2)$ chiral Lagrangian, keeping only $O(p^4 \ln p^2)$ terms;
3. Tree level terms from the $O(p^4)$ chiral Lagrangian, as well as $O(p^4)$ radiative contributions from the $O(p^2)$ chiral Lagrangian;

and so forth. Keeping the logs and throwing out the analytic terms in step #2 is equivalent to saying that most of the $O(p^4)$ chiral Lagrangian renormalized at $\mu = m_\pi$ would come from running induced by the $O(p^2)$ Lagrangian in scaling down from $\mu = \Lambda$ to $\mu = m_\pi$, and not from the initial values of couplings in the $O(p^4)$ Lagrangian renormalized at $\mu = \Lambda$. This procedure would *not* be reasonable in the large N_c limit (see problem (III.7)) but seems to work reasonably well in the real world.

4.4.3 Renormalization of $\langle 0|\bar{q}q|0\rangle$

As an example of a simple calculation, consider the computation of the ratios of the quark condensates,

$$x = \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} \ . \tag{137}$$

Since the operator $\bar{q}q$ gets multiplicatively renormalized, $\langle 0|\bar{q}_i q_i|0\rangle$ is scheme dependent, but the ratio x is not. The QCD Hamiltonian density is given by $\mathcal{H} = \ldots + \bar{q}Mq + \ldots$, and so it follows from the Feynman-Hellman theorem[8] that

$$\langle 0|\bar{q}_i q_i|0\rangle = \frac{\partial}{\partial m_i}\langle 0|\mathcal{H}|0\rangle = \frac{\partial \mathcal{E}_0}{\partial m_i} \ , \tag{138}$$

where $\mathcal{E}_0$ is the vacuum energy density. We do not know what is $\mathcal{E}_0$, but we do know its dependence one the quark mass matrix; from eq. (119)

$$\mathcal{E}_0 = \text{const.} - \frac{1}{2} f^2 \text{Tr}(\widetilde{\Lambda} M)\Sigma + \text{h.c.} + O(M^2 \ln M)\Bigg|_{\Sigma_{ij}=\delta_{ij}} = f^2 \widetilde{\Lambda}\text{Tr}M + \dots \ , \tag{139}$$

from which it follows that this scheme

$$\langle 0|\bar{q}_i q_i|0\rangle = \widetilde{\Lambda} f^2 \ , \tag{140}$$

and that in any scheme the leading contributions to x is

$$x = 1 \ . \tag{141}$$

[8] The substance of the Feynman-Hellman theorem is that in first order perturbation theory, the wave function doesn't change while the energy does.

Well good — this is what we started with for massless QCD in eq. (101)! To get the subleading logarithmic corrections, we need to compute the $O(m^2 \ln m^2)$ one-loop correction to the vacuum energy. This loop with no vertices's is the Feynman diagram for which Feynman rules don't work! As easily seen in a Euclidean path integral, the vacuum energy density in a box of 4-volume VT for a real, noninteracting scalar is just

$$\mathcal{E}_0 = -\frac{1}{VT} \ln \left(\det(-\Box + m^2)\right)^{-1/2} = \frac{1}{VT}\frac{1}{2}\mathrm{Tr}\ln(-\Box + m^2) \ . \tag{142}$$

In $d = (4 - 2\epsilon)$ Euclidean dimensions this just involves evaluating for each mass eigenstate the integral

$$\frac{\mu^{4-d}}{2} \int \frac{d^d k}{(2\pi)^d} \ln(k^2 + m^2) \ . \tag{143}$$

where the prefactor of μ^{4-d} was included to keep the mass dimension to equal 4.

Let us first perform the differentiation with respect to quark mass. Then in this scheme we get the correction

$$\delta \langle 0|\bar{q}_i q_i|0\rangle = \frac{1}{2}\sum_a \frac{\partial m_a^2}{\partial m_i}\mu^{4-d} \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2 + m^2} \xrightarrow[\overline{MS}]{} -\sum_a \frac{\partial m_a^2}{\partial m_i}\left(\frac{m_a^2 \ln m_a^2/\mu^2}{32\pi^2}\right) \tag{144}$$

where a is summed over the meson mass eigenstates, and m_i is the mass of the i^{th} flavor of quark. The final result was arrived at after performing the $\overline{MS}$ subtraction (where you only keep the $\ln m^2$ term in the $\epsilon \to 0$ limit; see the appendix **??** for dimensional regularization formulas).

To the order we are working, the quark condensate ratios are therefore given by

$$\frac{\langle 0|\bar{q}_i q_i|0\rangle}{\langle 0|\bar{q}_j q_j|0\rangle} = 1 - \frac{1}{32\pi^2 \widetilde{\Lambda} f^2}\sum_a m_a^2 \ln m_a^2/\mu^2 \left(\frac{\partial m_a^2}{\partial m_i} - \frac{\partial m_a^2}{\partial m_j}\right) \ . \tag{145}$$

Using the masses given in eq. (120) and eq. (121), ignoring $\pi^0 - \eta$ mixing, we find

$$x = \frac{\langle 0|\bar{u}u|0\rangle}{\langle 0|\bar{s}s|0\rangle} = 1 - 3g_\pi + 2g_{K^0} + g_\eta + O(m^4) \ , \tag{146}$$

where

$$g_P \equiv \frac{1}{32\pi^2 f^2} m_P^2 \ln\left(\frac{m_P^2}{\mu^2}\right) \tag{147}$$

with $P = \pi$, K^+, K^0, η. The answer is μ dependent, since I have neglected to include the $O(p^4)$ Lagrangian contributions at tree-level, and in fact it is precisely those operators that serve as counterterms for the $1/\epsilon$ poles subtracted in $\overline{MS}$. However, in the usual practice of chiral perturbation theory, I have assumed that with $\mu = \Lambda$, the contributions from the $O(p^4)$ Lagrangian are small compared to the chiral logs I have included. Plugging in numbers with $\mu = 1$ GeV I find

$$g_\pi \simeq -0.028 \ , \quad g_K \simeq -0.13 \ , \quad g_\eta \simeq -0.13 \tag{148}$$

implying that $x \simeq 0.70$ — a 30% correction from the leading result $x = 1$. This is typical of any chiral correction that involves the strange quark, since $m_K^2/\Lambda^2 \simeq 25\%$. Corrections to $\langle \bar{u}u\rangle / \langle \bar{d}d\rangle$ will be *much* smaller, since they depend on isospin breaking, of which a typical measure is $(m_{K^0}^2 - m_{K^+}^2)/\Lambda^2 \simeq 0.004$.

4.5 Chiral lagrangians for BSM physics

4.5.1 Technicolor

Chiral lagrangians are the tool for studying the physics of Goldstone Bosons. The pions in particular are the ones we have been looking at, but we know that there are at least three other Goldstone bosons in reality: namely the three that are "eaten" via the Higgs mechanism and become the longitudinal degrees of freedom in the $W^\pm$ and the Z^0 gauge bosons in the SM. Let us start by asking what the world would look like if the Higgs doublet was omitted from the SM; to make this toy model simpler, let's also imagine that the world only has one family of fermions, u, d, e, ν_e, and in the following discussion I will ignore the leptons.

Without the Higgs the $SU(2) \times U(1)$ gauge bosons are all massless down at the QCD scale and have to be included in the chiral Lagrangian. Therefore we need only consider an $SU(2) \times SU(2)/SU(2)$ chiral Lagrangian, where the Σ field represents the three pions and is written as

$$\Sigma = e^{2iT_a\pi_a/f_\pi} \ , \qquad T_a = \frac{\sigma_a}{2} \ , \tag{149}$$

but how do we incorporate the gauge fields? Recall that in the SM the gauge charges of the quarks under $SU(2) \times U(1)$ are

$$\begin{pmatrix} u \\ d \end{pmatrix}_L = 2_{\frac{1}{6}} \ , \qquad u_R = 1_{\frac{2}{3}} \ , \qquad d_R = 1_{-\frac{1}{3}} \ . \tag{150}$$

Note that the $SU(2)_L$ of the chiral Lagrangian is exactly the same group as the $SU(2)_L$ that is gauged in the SM; the gauged $U(1)$ charge Y can be written as

$$Y = \frac{1}{2}B + T_{3,R} \tag{151}$$

where B is baryon number, the $U(1)_V$ symmetry that acts the same on all LH and RH quarks, and $T_{3,R} \in SU(2)_R$. The $U(1)_V$ part is uninteresting in the chiral Lagrangian since the pions do not carry that quantum number; however we have to use covariant derivatives for the $SU(2)_L \times U(1)_R$ part of the gauge group:

$$\partial_\mu\Sigma \quad \Longrightarrow \quad D_\mu\Sigma = \partial_\mu\Sigma + igW_\mu^a T_a\Sigma - ig' B_\mu \Sigma T_3 \ , \tag{152}$$

where I used the fact that under $SU(2) \times SU(2)$ the Σ field transforms as $\Sigma \to L\Sigma_R^\dagger$; the dagger on R is what is responsible for the minus sign in front of the $U(1)$ gauge boson piece above.

Something peculiar is going on: now that $SU(2)_L$ is gauged, the matrix L is promoted to a spacetime dependent matrix $L(x)$, and I can choose $L(x) = \Sigma^\dagger(x)$...and a gauge transformation can turn Σ into the unit matrix! This is because the quark condensate $\langle \bar{q}_L q_R \rangle$ has broken the weak interactions spontaneously, the W and B get masses, and we just rediscovered unitary gauge.

Now let's look at the kinetic term for the Σ field in unitary gauge:

$$\frac{f^2}{4}\mathrm{Tr}D_\mu\Sigma(D^\mu\Sigma)^\dagger = \frac{f^2}{4}\left[g^2 W_\mu^a W^{\mu b}\mathrm{Tr}T_aT_b - 2gg'W_\mu^3 B^\mu \mathrm{Tr}T_3T_3 + g'^2\mathrm{Tr}T_3T_3\right] \ . \tag{153}$$

This gives us a mass^2 matrix for the gauge bosons

$$M^2 = \frac{f^2}{4} \begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix} \tag{154}$$

with eigenvalues

$$m^2 = \left\{0,\ \frac{g^2 f^2}{4},\ \frac{g^2 f^2}{4},\ \frac{(g^2 + g'^2) f^2}{4}\right\} \equiv \left\{m_\gamma^2,\ m_w^2,\ m_w^2,\ mz^2\right\} \tag{155}$$

where

$$m_\gamma^2 = 0\ , \qquad m_w^2 = \frac{e^2 f_\pi^2}{4\sin^2\theta_w}\ , \qquad m_z^2 = \frac{m_w^2}{\cos^2\theta_w} \tag{156}$$

with the conventional definitions $g = e/\sin\theta_w$, $g' = e/\cos\theta_w$.

This would look precisely like the SM if instead of $f_\pi = 93$ MeV we took $f_\pi = 250$ GeV! This was the observation of both Weinberg and Susskind [9–11]: that the strong interactions would provide the Higgs mechanism in the SM without the Higgs doublet field. While the W and Z would get massive, as in the SM, in this theory there might not be an actual Higgs boson (it is not in the chiral Lagrangian, and QCD does not have any narrow 0^+ resonance), and the order parameter for the symmetry breaking is a quark bilinear now instead of a fundamental scalar, and so that there is no naturalness problem: the weak scale is set by the QCD scale, which can be naturally much smaller than the GUT or Planck scales due to asymptotic freedom, just as Cooper pairs in superconductivity are so much larger than the crystal lattice spacing.

However, $f_\pi \neq 250$ GeV and we do not want to give away the pion. So instead we posit a new gauge interaction, just like color but at a higher scale called technicolor, and new fermions that carry this gauge quantum number, U, D, called techniquarks. Now the techniquarks condense with a technipion decay constant $F_\Pi = 250$ GeV, and we have explained the SM without the Higgs!

This is great, except for three problems: (i) the Higgs boson has been discovered, long after the invention of technicolor; (ii) it is difficult to construct a technicolor theory where precision radiative corrections to the weak interactions agree with what is measured; (iii) as described here, the techniquark condensate only fulfills one of the roles played by the Higgs serves in the SM: giver of mass to the gauge bosons. To realize the other role — giver of mass to the quarks and leptons — requires new interactions added to be technicolor (e.g., Extended Technicolor), more complications, and potential problems with flavor changing neutral currents. There are more complex versions of the original theory still being explored, but the compelling beauty of the original concept is no longer there.

4.5.2 Composite Higgs

What is so beautiful about technicolor is that by replacing the fundamental Higgs scalar field with a quark bilinear, the fine tuning problem is avoided. Do we have to abandon that idea simply because we have seen a Higgs boson that looks fundamental? No, that has happened before: the pions looked fundamental when discovered, and only over a decade later was it decided that they were composite bound states. Furthermore, they are relativistic bound states – at least in the chiral limit, their compositeness scale is much smaller than their Compton wavelengths. Relativistic bound states are in general baffling things we have no handle on as they always require strong coupling. One of the few exceptions when we understand what is going on exception is when the bound states are Goldstone bosons, like the pions in QCD. So let us return to the QCD chiral Lagrangian for inspiration, this time not looking at the pions as candidates for the longitudinal W and Z, but as candidates for the Higgs doublet [12, 13].

A Higgs doublet field has four real degrees of freedom, so we need to look at a chiral Lagrangian with at least four Goldstone bosons. The $SU(2)\times SU(2)/SU(2)$ chiral Lagrangian only has three pions, so that won't do; however the $SU(3)\times SU(3)/SU(3)$ example has eight mesons. Furthermore, there is a ready made $SU(2)$ doublet in that theory – the kaon. In fact, note that under the unbroken isospin and strong hypercharge $SU(2)\times U(1)$ global symmetry of QCD, the kaon doublet transforms just like the Higgs doublet does under gauged electroweak $SU(2)\times U(1)$ symmetry in the SM. So what if we scaled up a version of QCD with three massless flavors, gauge the vector-like $SU(2)\times U(1)$ symmetry, and identify the kaon doublet with the Higgs doublet?

In such a theory the Higgs would get a mass from $SU(2)\times U(1)$ gauge interactions, just as the π^+ gets a mass contribution from electromagnetic contributions (see problem 6). Unfortunately this contribution to the Higgs mass^2 is positive, and so there is no spontaneous symmetry breaking in the model so far. What is needed is some effect that contributes a negative mass^2. Various examples have been found that will do this: one is an axial gauge group under which the constituents of the composite Higgs are charged [14]; another and more compelling source of destabilization of the composite Higgs potential is the top quark [2]. The basic idea is that operators can be induced in the chiral Lagrangian which favor the vacuum $\Sigma \neq 1$ and prefer a small nonzero value corresponding to the composite Higgs doublet getting a vev v which can be computable in terms of f and coupling constants, such as the top Yukawa coupling. This scale v is set to 250 GeV and is the usual Higgs vev; the theory then predicts at a higher scale $\Lambda \simeq 4\pi f$ the Higgs will reveal itself to be composite. The bigger the ratio f/v, the more fundamental the Higgs will appear at low energy. For a recent discussion of the viability of the composite Higgs idea and accelerator tests, see [15].

4.6 Problems for lecture IV

IV.1) Verify eq. (107).

IV.2) How does Σ transform under P (parity)? What does this transformation imply for the intrinsic parity of the π_a mesons? How does Σ transform under C (charge conjugation)? Which of the mesons are eigenstates of CP, and are they CP even or odd? Recall that under P and C the quarks transform as

$$\begin{aligned} P: &\quad q \to \gamma^0 q \ , \\ C: &\quad q \to C\bar{q}^T \ , \qquad C = C^\dagger = C^{-1} = -C^T \ , \quad C\gamma_\mu C = -\gamma_\mu^T \ , \qquad C\gamma_5 C = \gamma_5 \ . \end{aligned} \tag{157}$$

IV.3) How do we know that c, and hence $\widetilde{\Lambda}$, is positive in eq. (117)? How would the world look different if it were negative? Hint: consider what Σ matrix would minimize the vacuum energy, and its implications for the spectrum of the theory.

IV.4) An axion is a hypothetical particle proposed to explain why the electric dipole moment of the neutron is so small (the strong CP problem). It couples to quarks through the quark mass matrix, where one makes the substitution

$$M \to M e^{iaX/f_a} \tag{158}$$

in eq. (99), where a is the axion field, f_a is the axion decay constant, and X is a 3×3 diagonal matrix constrained to have $\mathrm{Tr}X = 1$. Compute the axion mass in terms of m_π, f_π and f_a, dropping terms of size $m_{u,d}/m_s$. Hint: use the remaining freedom in choosing X to ensure that the axion does not mix with the π^0 or the η mesons.

IV.5) Is the relation eq. (135) obeyed in QCD for large N_c (where N_c is the number of colors, and $N_c = 3$ in the real world)?

IV.6) Consider the operator eq. (127) which accounts for the $\pi^+ - \phi^0$ mass splitting. What would happen if instead of gauging electromagnetism I gauged axial electromagnetism (all quarks have the same Q_R, but now $Q_L \to -Q_L$ in the photon couplings). How do you interpret the effect of this operator in the absence of quark masses, $M = 0$? How could this be used in a composite Higgs model?

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