Eikonal gluon radiation at finite-N$_c$ beyond two-loops

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DIS 2015

27 April - 1 May 2015

Dallas, Texas

Introduction

- Calculations of QCD scattering amplitudes only feasible at high energies $\rightsquigarrow$ asymptotic freedom $\rightsquigarrow$ perturbation theory (PT).
- Calculations of even the first few orders in QCD PT is delicate $\rightsquigarrow$ use approximations (specific phase space regions. e.g, soft & collinear).
- A particularly interesting approx. is the *Eikonal approx.* *(soft insertion rules)*.

Levy & Sucher '69, Abarbanel & Itzykson '69, Wallace '73

  - Valid in the limit where the *momenta* of the radiated gluons become *soft*
  - Standard Feynman rules $\rightarrow$ *eikonal Feynman rules* (calculations greatly simplified)
  - Eikonal amplitudes shown to *factorise & exponentiate* both for *abelian* and *non-abelian* $\rightsquigarrow$ allow for resummation of large logs

Yennie et al '61, Sterman '81, Gatheral '83, Frenkel & Taylor '84

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Two main issues that have so far jeopardised calculations of QCD eikonal amplitudes :

1. Colour space in QCD is matrix-valued and thus non-commutative in general.
2. Number of Feynman diagrams that need to be considered at the nth order in PT increases factorially.

✓Partial solution: large-N$_c$ limit

1. Colour-space is scalar-valued.
2. Only planar diagrams contribute $\rightsquigarrow$ hugely reduces number of Feynman diagrams.

$\rightsquigarrow$ If impose (strong) energy ordering $\rightsquigarrow$ all-orders amplitude is obtained.

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If restore the full/finite-N_c
then how do we proceed ? main goal of this talk

1. Use ColorMath, a Mathematica package that automatically performs colour summed calculations in $SU(N_c)$ $\rightsquigarrow$ solves colour struct. prob ✓ Sjödahl '13, Sjödahl & Keppeler '13

2. Write a Mathematica program that loops over all possible dipoles at a given order in PT $\rightsquigarrow$ solves non-linear branching (all possible Feyn diags) ✓

Combining (1) & (2) yields : EikAmp, *a Mathematica package that automatically calculates eikonal amplitudes at finite-N_c at "any" given order in PT* [still under development!]

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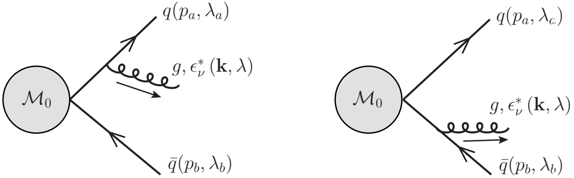
Eikonal Approximation

Background & review

- Better illustrate eikonal approx. with a simple example $e^+e^- \to q + \bar{q} + g$



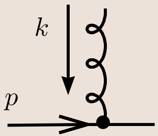
- Employing standard Feynman rules, the amplitude reads

$$\imath\mathcal{M}^\mu \propto \gamma^\mu \frac{(\not{p}_a + \not{k})}{(p_a + k)^2}\, \imath\mathcal{M}_0\,(p_a + k, p_b)$$

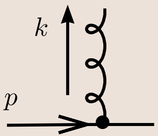
- Eikonal approx. $\Rightarrow |\mathbf{k}| \ll |\mathbf{p}_a - \mathbf{p}_b|$ equiv. $k \to 0$ & $p_a \sim p_b$

$$\imath\mathcal{M}^\mu \propto \frac{p_a^\mu}{2(p_a \cdot k)}\, \imath\mathcal{M}_0\,(p_a, p_b)$$

Eikonal Feynman rules



$$-\frac{p^\mu}{p.k}$$

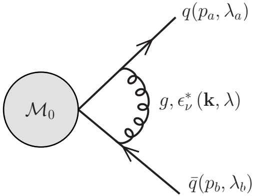


$$\frac{p^\mu}{p.k}$$

The eikonal cross section

$$\sigma_{\rm eik} = \sigma_0 \cdot g_s^2\, C_F \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2|\mathbf{k}|} \frac{2\,(p_a\cdot p_b)}{(p_a\cdot k)\,(k\cdot p_b)}$$

For on-shell partons, only vertex correction is non-zero



The one-loop virtual cross section

$$\sigma_{\mathrm{1loop,eik}} = \sigma_0 \cdot \mathbf{Re}\left\{ - g_s^2 \mathrm{C_F} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2|\mathbf{k}|} \frac{2\,(p_a \cdot p_b)}{(p_a \cdot k)\,(k \cdot p_b)} + \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^3\,|\mathbf{k}_\perp^2|} \cdot \imath\pi \right\}$$

Eikonal approx.
Virtual emission: one gluon

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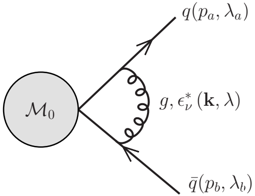

Eikonal approx

Eik. amp. factorises into Born $\times$ sum of dipole antennae functions
virtual corr. $=$ - real emission

Eikonal amplitudes

$$e^+e^- \to q + \bar{q} + g_1 + g_2 + \cdots + g_n$$

Assuming

- Eikonal approx. (Implicitly (Strong) energy ordering : $\omega_n \ll \omega_{(n-1)} \ll \cdots \ll \omega_1$)

Eikonal amp. for the emission of the $(m+1)^{\text{th}}$ gluon by an ensemble of m harder partons

$$|m+1\rangle = g_s \sum_{i=1}^{m} \frac{p_i \cdot \epsilon^{*\,a_{m+1}}}{(p_i \cdot p_{m+1})} \mathbf{T}_i^{a_{m+1}} |m\rangle$$

Dokshitzer et al '91, Forshaw et al '08

Iterating down to Born level

$$|m+1\rangle = g_s^{m-1} \left(\prod_{n=2}^{m} \sum_{i_n=1}^{n} \frac{p_{i_n} \cdot \epsilon^{*a_{n+1}}}{(p_{i_n} \cdot p_{n+1})} \right) \mathbf{T}_{i_m}^{a_{m+1}} \cdots \mathbf{T}_{i_2}^{a_3} \left| \mathcal{M}_0^{(2)}(p_1, p_2) \right\rangle$$

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The diff. xsec for the emission of m energy-ordered soft gluons normalised to the Born xsec for a given *gluonic configuration* X reads

$$\frac{d\sigma_m^X}{\sigma_0} = \sum_m \frac{1}{m!} d\Phi_m \, \langle m+2|m+2\rangle_X$$

where the phase space factor is

$$d\Phi_m = \prod_{i=1}^{m} \frac{d^3\mathbf{k}_i}{(2\pi)^3 2\omega_i} = \prod_{i=1}^{m} \frac{\omega_i d\omega_i}{4\pi^2}\frac{d\Omega_i}{4\pi} = \prod_{i=1}^{m} \frac{k_{ti} dk_{ti}}{4\pi^2}\frac{d\eta_i d\phi_i}{2\pi}$$

Re-arranging one may write

$$\frac{d\sigma_m^X}{\sigma_0} = \sum_m \frac{\bar{\alpha}_s^m}{m!} \left(\prod_{i=1}^{m} \frac{dk_{ti}}{k_{ti}}\frac{d\eta_i d\phi_i}{2\pi}\right) \mathcal{W}_{12\ldots m}^X$$

where $X = \mathrm{x}_1\mathrm{x}_2\ldots\mathrm{x}_m$ where $\mathrm{x}_i = \{\mathrm{R},\mathrm{V}\}$ corresponds to whether the i^{th} gluon is real (R) or virtual (V)

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where $X = \mathrm{x_1 x_2 \ldots x_m}$ where $\mathrm{x_i} = \{\mathrm{R}, \mathrm{V}\}$ corresponds to whether the i^{th} gluon is real (R) or virtual (V)

The final form of the eikonal amplitude squared

$$\mathcal{W}^X_{12\ldots m} = \left(\prod_{n=1}^{m} \sum_{i_{n-1},i'_{n-1}\in U_{m-1}} w^n_{i_{n-1}i'_{n-1}}\right) \mathcal{C}^{i_0 i_1\ldots i_{m-1}}_{i'_0 i'_1\ldots i'_{m-1}}$$

where $U_m = \{a, b, 1, 2, \ldots, m\}$ (and $U_0 = \{a, b\}$) is the set of all possible emitters/dipoles of the m^{th} gluon, and the antenna function w^ℓ_{ij} is

$$w^\ell_{ij} = p^2_{t\ell}\frac{(p_i\cdot p_j)}{(p_i\cdot p_\ell)(p_j\cdot p_\ell)} \equiv \frac{(ij)}{(i\ell)(\ell j)}$$

with $(ij) = 1 - \cos\theta_{ij} = \cosh(\eta_{ij}) - \cos\phi_{ij}$.

The colour factor is

$$\mathcal{C}^{i_0\ldots i_{m-1}}_{i'_0\ldots i'_{m-1}} = \mathbf{T}^{a_1}_{i_0}\cdots\mathbf{T}^{a_m}_{i_{m-1}} \cdot \mathbf{T}^{a_1}_{i'_0}\cdots\mathbf{T}^{a_m}_{i'_{m-1}}$$

the dot "·" means "trace"

Properties of the above eikonal amplitude squared :

1. Totally symmetric under the interchange of the two hardest partons; quark & antiquark ($a \leftrightarrow b$)
2. Totally symmetric under the interchange of the legs of the dipole emitting the softest gluon.
3. Not symmetric under the interchange of the legs of each and every single dipole. *This symmetry breaking is primarily due to the associated colour factor* $\rightsquigarrow$ *see five-loops example*
4. For the softest gluon, it is always true that

$$\mathcal{W}_{12\ldots m}^{\mathrm{x}_1\ldots\mathrm{R}} = -\mathcal{W}_{12\ldots m}^{\mathrm{x}_1\ldots\mathrm{V}}$$

The above eikonal amplitude squared is computed using a `Mathematica` program: "`EikAmp`"

Main algorithm :

1. **Determine** loop order
2. **Begin** with the bra amplitude
3. **For** each emitted gluon ℓ (multiplicative process)
   i. **Find** the set of all possible dipoles
   ii. **Pick up** the first leg of each dipole
   iii. **Determine** the corresp. colour matrix (fund. or Adj. rep)
4. **Move** to the ket amplitude
   i. **Pick up** the second leg of each dipole
   ii. **Determine** the corresponding conj. colour matrix
5. **Call** the `CSimplify` function of `ColorMath`
6. **Decompose** the output of `CSimplify` in terms of $\mathrm{C_F}, \mathrm{C_A}$
7. **Multiply** by the appropriate antenna w_{ij}^{ℓ}.
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In large-N_c limit all the Feynman diagrams that have a *non-planar topology* are discarded as their corresponding cross-sections are suppressed by $1/N_c^2$.

't Hooft '74

Both : number of Feynman diagrams + colour structure simplify considerably:

$$\mathcal{C}^{i_1\ldots i_m}_{i'_1\ldots i'_m} = \left(\frac{N_c}{2}\right)^m$$

$$\widetilde{\mathcal{W}}^{X}_{12\ldots m} = \sum_{\pi_m} \frac{(ab)}{(ai_1)(i_1 i_2)\ldots(i_n b)}$$

where the colour factor has been absorbed into $\bar{\alpha}_s = N_c \alpha_s/\pi$

Bassetto et al '83, Fiorani et al '88, Banfi '02

Easily get large-N$_c$ limit result from `EikAmp` by setting $\mathrm{C_F} \to \mathrm{C_A}/2$

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Examples

- The one-loop eikonal amplitudes squared are quite simple:

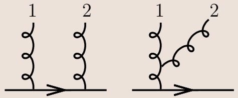
$$\mathcal{W}_1^{\mathrm{R}} = \mathrm{C_F}\left(w_{ab}^1 + w_{ba}^1\right) = 2\,\mathrm{C_F}\,w_{ab}^1$$
$$\mathcal{W}_1^{\mathrm{V}} = -\mathcal{W}_1^{\mathrm{R}}$$

- The two-loop amplitudes are given by

$$\mathcal{W}_{12}^{\mathrm{RR}} = \widetilde{\mathcal{W}}_{12}^{\mathrm{RR}} + \overline{\mathcal{W}}_{12}^{\mathrm{RR}}$$

where *reducible* and *irreducible* contributions are

$$\widetilde{\mathcal{W}}_{12}^{\mathrm{RR}} = \mathcal{W}_1^{\mathrm{R}}\mathcal{W}_2^{\mathrm{R}}$$
$$\left[\frac{1}{2^2}\right]\ \overline{\mathcal{W}}_{12}^{\mathrm{RR}} = \frac{1}{2}\,\mathrm{C_F C_A}\,\mathcal{A}_{ab}^{12}$$



Remarks:

- The antenna function $\mathcal{A}_{ab}^{ij} = w_{ab}^{i}(w_{ai}^{j} + w_{bi}^{j} - w_{ab}^{j})$ represents a two-parton *cascade* emission.
- $\mathcal{A}_{ab}^{ij}$ is both
  - *friable*: no singular dependence on angles $\Leftrightarrow$ no angular logs in σ.
  - *ideal*: fully integrable over the directions of all gluons involved.
- $\mathcal{A}_{ab}^{ij}$ provides next-to-leading log contrib. to σ.

Examples
Three-loops

- $2^3 = 8$ (configs.) amplitudes need to be evaluated.
- Due to symmetry, *only one* is actually to be calculated! $\rightsquigarrow$ the all-real amplitude (all other amplitudes can be deduced from it!)
- The all-real Eikonal amp. squared

$$\mathcal{W}_{123}^{\mathrm{RRR}} = \widetilde{\mathcal{W}}_{123}^{\mathrm{RRR}} + \overline{\mathcal{W}}_{123}^{\mathrm{RRR}}$$

where the reducible contrib. is

$$\widetilde{\mathcal{W}}_{123}^{\mathrm{RRR}} = \prod_{i=1}^{3} \mathcal{W}_i^{\mathrm{R}} + \sum_{ijk=1}^{3} \mathcal{W}_i^{\mathrm{R}}\, \overline{\mathcal{W}}_{jk}^{\mathrm{RR}}$$

and the irreducible contrib. is

$$\left[\frac{1}{2^3}\right] \overline{\mathcal{W}}_{123}^{\mathrm{RRR}} = \frac{1}{4}\, \mathrm{C_F C_A^2} \left(\mathcal{A}_{ab}^{12} \bar{\mathcal{A}}_{ab}^{13} + \mathcal{B}_{ab}^{123}\right)$$

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$$\mathcal{W}_{123}^{\mathrm{RRR}} = \widetilde{\mathcal{W}}_{123}^{\mathrm{RRR}} + \overline{\mathcal{W}}_{123}^{\mathrm{RRR}}$$

where the reducible contrib. is

$$\widetilde{\mathcal{W}}_{123}^{\mathrm{RRR}} = \prod_{i=1}^{3} \mathcal{W}_i^{\mathrm{R}} + \sum_{ijk=1}^{3} \mathcal{W}_i^{\mathrm{R}} \, \overline{\mathcal{W}}_{jk}^{\mathrm{RR}}$$

and the irreducible contrib. is

$$\left[\frac{1}{2^3}\right] \overline{\mathcal{W}}_{123}^{\mathrm{RRR}} = \frac{1}{4} \, \mathrm{C_F C_A^2} \left(\mathcal{A}_{ab}^{12} \bar{\mathcal{A}}_{ab}^{13} + \mathcal{B}_{ab}^{123} \right)$$

Remarks :

- The term $\mathcal{W}_i^{\mathrm{R}}\,\overline{\mathcal{W}}_{jk}^{\mathrm{RR}}$ is an *interference* between the one- and two-loops amplitudes.

- The antenna function $\mathcal{B}_{ab}^{123}$ represents three-parton cascade emission $\rightsquigarrow$ friable and ideal

$$\mathcal{B}_{ab}^{ijk} = w_{ab}^{i}\left(\mathcal{A}_{ai}^{jk} + \mathcal{A}_{ib}^{jk} - \mathcal{A}_{ab}^{jk}\right)$$

- The irreducible contrib. $\overline{\mathcal{W}}_{123}^{\mathrm{RRR}}$ is *not completely* reducible!
  resembles two-loop pattern $\leftrightsquigarrow$ $\mathcal{B}$ is related to $\mathcal{A}$

- No finite-N$_c$ corrections up to this order.

- 2^4 (configs.) amplitudes $\Rightarrow$ Only one is to be calculated!
- The all-real Eikonal amp. squared

$$\mathcal{W}_{1234}^{\mathrm{RRRR}} = \widetilde{\mathcal{W}}_{1234}^{\mathrm{RRRR}} + \overline{\mathcal{W}}_{1234}^{\mathrm{RRRR}}$$

where reducible contrib. is easy to find!
The irreducible contrib. is

$$\overline{\mathcal{W}}_{1234}^{\mathrm{RRRR}} = \widetilde{\overline{\mathcal{W}}}_{1234}^{\mathrm{RRRR}} + \overline{\overline{\mathcal{W}}}_{1234}^{\mathrm{RRRR}}$$

with the "irreducible-reducible" contrib.

$$\left[\frac{1}{2^4}\right] \widetilde{\overline{\mathcal{W}}}_{1234}^{\mathrm{RRRR}} = \frac{\mathrm{C_F C_A^3}}{8} \left(\mathcal{A}_{ab}^{12} \bar{\mathcal{A}}_{ab}^{13} \bar{\mathcal{A}}_{ab}^{14} + \sum_{jk\ell=1}^{4} \mathcal{A}_{ab}^{1j} \bar{\mathcal{B}}_{ab}^{1k\ell} + \mathcal{C}_{ab}^{1234} \right)$$

The new *genuinely* “irreducible-irreducible” contribution that contains the first finite N_c correction,

$$\left[\frac{1}{2^4}\right]\,\overline{\overline{\mathcal{W}}}^{\mathrm{RRRR}}_{1234} = \frac{1}{4}\mathrm{C_F}\mathrm{C_A^2}\left(\mathrm{C_F} - \frac{\mathrm{C_A}}{2}\right)\mathbf{A}^{1234}_{ab} + \frac{1}{8}\mathrm{C_F}\mathrm{C_A^3}\,\mathfrak{A}^{1234}_{ab}$$

where $\mathbf{A}$ and $\mathfrak{A}$ are pseudo-antenna functions related to $\mathcal{A}^{k\ell}_{ij}$.

Remarks :

- All antenna and pseudo-antenna functions in the four-loop amplitude are friable and ideal $\rightsquigarrow$ $\overline{\mathcal{W}}_{1234}^{\mathrm{RRRR}}$ is friable and ideal.

  KKK and Delenda '15

- Dokshitzer et al stated that $\overline{\mathcal{W}}_{1234}^{\mathrm{RRRR}}$ is friable but not deal! They named it "colour monster" with a colour factor equals to $\mathrm{C_F}\, N_c$.

  Dokshitzer et al '91

- Easily see from $\overline{\overline{\mathcal{W}}}_{1234}^{\mathrm{RRRR}}$ that

$$\mathrm{C_F C_A^2}\left(\mathrm{C_F} - \frac{\mathrm{C_A}}{2}\right) = -\mathrm{C_F}\, N_c$$

- First genuine finite-N_c correction appears at this order.

The all-order five-loop eikonal amp. squared may be cast in the form

$$\mathcal{W}_{12345}^{\mathrm{RRRRR}} = \widetilde{\mathcal{W}}_{12345}^{\mathrm{RRRRR}} + \overline{\mathcal{W}}_{12345}^{\mathrm{RRRRR}} + \mathcal{N}_{12345}^{\mathrm{RRRRR}}$$

where the reducible and irreducible contributions follow pattern seen at previous loop order.

The new *non-symmetric* contribution $\mathcal{N}_{12345}^{X}$ reads

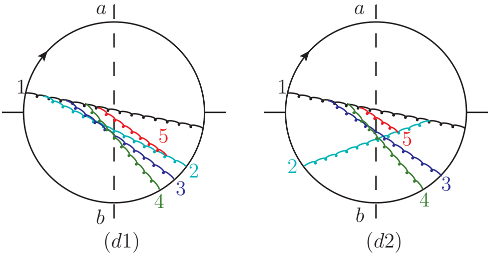
$$\begin{aligned}\mathcal{N}_{12345}^{X} &= \mathcal{W}_{12345}^{X,\,\texttt{EikAmp}} - \left(\widetilde{\mathcal{W}}_{12345}^{X} + \overline{\mathcal{W}}_{12345}^{X}\right) \\ &= \frac{1}{2}\mathrm{C_F^2}\mathrm{C_A^3}\,\mathrm{I}_{12345}^{X} + \frac{1}{8}\mathrm{C_F}\mathrm{C_A^4}\,\mathrm{J}_{12345}^{X}\end{aligned}$$

noting that $\mathcal{W}_{12345}^{X} = \mathcal{W}_{12345}^{X,\,\texttt{EikAmp}}$. The terms I^{X} and J^{X} have not been simplified !

Remarks :

- Unlike previous loop-order, the five-loop amp. squared contains non-symmetric contributions.
- Only 8 out 32 amplitudes contain non-symmetric contribs.
- These non-symmetric contribs. originate from the fact that there are (eikonal) Feynman diagrams which are not symmetric under the interchange of the emitters of one or more gluons (or equivalently, interchange of the legs of one or more emitting dipoles).

Examples of non-symmetric Feynman diagrams



$$(d1): \qquad -\frac{3\,\mathrm{C_F C_A^3}}{8}\left(\mathrm{C_F}-\frac{\mathrm{C_A}}{2}\right) w^1_{ab} w^2_{1b} w^3_{1b} w^4_{1b} w^5_{12}$$

$$(d2): \qquad -\frac{\mathrm{C_F C_A^3}}{4}\left(\mathrm{C_F}-\frac{\mathrm{C_A}}{2}\right) w^1_{ab} w^2_{b1} w^3_{1b} w^4_{1b} w^5_{12}$$

Inspecting all five eikonal amplitudes, one observes:

- All amplitudes may be written in terms of the two-parton/loop antenna function $\mathcal{A}_{ij}^{k\ell} \rightsquigarrow$ *web* (two-eikonal irreducible diagram)

  Laenen et al '11

- There seems to be a pattern of *exponentiation* of the eikonal amplitude

  non-symmetric contributions may be considered as corrections!

$$\mathcal{M} = \mathcal{M}_0 \exp\left[\sum_{i=1}^{n} \widetilde{\mathcal{C}}_i \, \widetilde{\mathcal{M}}_i\right] ???$$

Summary & outlook

Done !

- ✓ Computed eikonal amplitude at finite-N$_c$ for the first time in literature up to five-loops
- ✓ Calculations are automated $\rightsquigarrow$ easily extended to higher orders
- ✓ Symmetry breaking seen at five-loops (and perhaps persists at higher loops too)
- ✓ Eikonal amplitudes seem to : (i) all build up from the two-parton antenna function, (ii) exponentiate

Waiting . . .

- ? Finalise the `EikAmp` `Mathematica` package
- ? Write `EikAmp` using `c++` and/or `Fortran` speed + interface with other MC program
- ? Solve BMS equ and compare to ours at large-N$_c$. Banfi et al '02
- ? Move onto next-to-eikonal approx. next-to-leading log $(\alpha_s^n L^{n-1})$ resum.