Eikonal gluon radiation at finite-$N_c$ beyond two-loops

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Calculations of QCD scattering amplitudes only feasible at high energies $\leadsto$ asymptotic freedom $\leadsto$ perturbation theory (PT).

Calculations of even the first few orders in QCD PT is delicate $\leadsto$ use approximations (specific phase space regions. e.g, soft & collinear).

A particularly interesting approx. is the *Eikonal approx.* (soft insertion rules).

- Valid in the limit where the *momenta* of the radiated gluons become *soft*
- Standard Feynman rules $\rightarrow$ *eikonal Feynman rules* (calculations greatly simplified)
- Eikonal amplitudes shown to *factorise & exponentiate* both for *abelian* and *non-abelian* $\leadsto$ allow for resummation of large logs

Levy & Sucher '69, Abarbanel & Itzykson '69, Wallace '73

Yennie et al '61, Sterman '81, Gatheral '83, Frenkel & Taylor '84
Introduction

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Two main issues that have so far jeopardised calculations of QCD eikonal amplitudes:

1. Colour space in QCD is matrix-valued and thus non-commutative in general.

2. Number of Feynman diagrams that need to be considered at the $n^{th}$ order in PT increases factorially.

✓ Partial solution: large-$N_c$ limit

1. Colour-space is scalar-valued.
2. Only planar diagrams contribute → hugely reduces number of Feynman diagrams.

→ If impose (strong) energy ordering → all-orders amplitude is obtained.

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If restore the full/finite-$N_c$ then how do we proceed? main goal of this talk

1. Use ColorMath, a Mathematica package that automatically performs colour summed calculations in $SU(N_c) \rightarrow$ solves colour struct. prob ✓

   Sjödahl '13, Sjödahl & Keppeler '13

2. Write a Mathematica program that loops over all possible dipoles at a given order in PT $\rightarrow$ solves non-linear branching (all possible Feyn diags) ✓

Combining (1) & (2) yields: EikAmp, a Mathematica package that automatically calculates eikonal amplitudes at finite-$N_c$ at “any” given order in PT [still under development!]
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Sjödahl ’13, Sjödahl & Keppeler ’13

K.Khelifa-Kerfa
Eikonal Approximation
Background & review
Better illustrate eikonal approx. with a simple example

\[ e^+ e^- \rightarrow q + \bar{q} + g \]

Employing standard Feynman rules, the amplitude reads

\[ i\mathcal{M}^\mu \propto \gamma^\mu \frac{(p_a + k)}{(p_a + k)^2} i\mathcal{M}_0 (p_a + k, p_b) \]

Eikonal approx. \( \Rightarrow |k| \ll |p_a - p_b| \) equiv. \( k \rightarrow 0 \) & \( p_a \sim p_b \)

\[ i\mathcal{M}^\mu \propto \frac{p_a^\mu}{2(p_a \cdot k)} i\mathcal{M}_0 (p_a, p_b) \]
Eikonal approx.
Real emission: one gluon

Eikonal Feynman rules

The eikonal cross section

$$\sigma_{\text{eik}} = \sigma_0 \cdot g_s^2 C_F \int \frac{d^3k}{(2\pi)^3} \frac{2 (p_a \cdot p_b)}{2|k| (p_a \cdot k) (k \cdot p_b)}$$
Eikonal approx.
Virtual emission: one gluon

For on-shell partons, only vertex correction is non-zero

\[
\mathcal{M}_0
\]

\[
q(p_a, \lambda_a) \quad g, \epsilon^*_\nu(k, \lambda) \quad \bar{q}(p_b, \lambda_b)
\]

The one-loop virtual cross section

\[
\sigma_{1\text{loop}, \text{eik}} = \sigma_0 \cdot \text{Re} \left\{ - g_s^2 C_F \int \frac{d^3k}{(2\pi)^3} \frac{2}{2|k|} \frac{(p_a \cdot p_b)}{(p_a \cdot k) (k \cdot p_b)} \right. \\
+ \left. \int \frac{d^2k_\perp}{(2\pi)^3} \frac{2}{|k_\perp^2|} \cdot \nu \pi \right\}
\]
Eikonal approx.
Virtual emission: one gluon

For on-shell partons, only vertex correction is non-zero

\[ q(p_a, \lambda_a) \quad \bar{q}(p_b, \lambda_b) \quad g, \epsilon^*_\nu(k, \lambda) \]

Eikonal approx
Eik. amp. factorises into Born $\times$ sum of dipole antennae functions
virtual corr. $= -$ real emission
Eikonal amplitudes

\[ e^+e^- \rightarrow q + \bar{q} + g_1 + g_2 + \cdots + g_n \]
Eikonal amplitudes

General form

Assuming

- Eikonal approx. (Implicitly (Strong) energy ordering: 
  \[ \omega_n \ll \omega_{(n-1)} \ll \cdots \ll \omega_1 \] )

Eikonal amp. for the emission of the \((m + 1)^{th}\) gluon by an ensemble of \(m\) harder partons

\[
|m + 1\rangle = g_s \sum_{i=1}^{m} \frac{p_i \cdot \epsilon^* a_{m+1}}{(p_i \cdot p_{m+1})} T_{i} a_{m+1} |m\rangle
\]

Dokshitzer et al '91, Forshaw et al '08

Iterating down to Born level

\[
|m + 1\rangle = g_s^{m-1} \left( \prod_{n=2}^{m} \sum_{i_n=1}^{n} \frac{p_{i_n} \cdot \epsilon^* a_{n+1}}{(p_{i_n} \cdot p_{n+1})} \right) T_{i_m} a_{m+1} \cdots T_{i_2} a_{3} |\mathcal{M}^{(2)}_0(p_1, p_2)\rangle
\]
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\]
Eikonal amplitudes

General form

The diff. xsec for the emission of \( m \) energy-ordered soft gluons normalised to the Born xsec for a given **gluonic configuration** \( X \) reads

\[
\frac{d\sigma^X_m}{\sigma_0} = \sum_m \frac{1}{m!} d\Phi_m \langle m + 2 | m + 2 \rangle_X
\]

where the phase space factor is

\[
d\Phi_m = \prod_{i=1}^m \frac{d^3 k_i}{(2\pi)^3 2 \omega_i} = \prod_{i=1}^m \frac{\omega_i d\omega_i}{4\pi^2} \frac{d\Omega_i}{4\pi} = \prod_{i=1}^m \frac{k_{ti} dk_{ti}}{4\pi^2} \frac{d\eta_i d\phi_i}{2\pi}
\]

Re-arranging one may write

\[
\frac{d\sigma^X_m}{\sigma_0} = \sum_m \bar{\alpha}_s^m \left( \prod_{i=1}^m \frac{dk_{ti}}{k_{ti}} \frac{d\eta_i d\phi_i}{2\pi} \right) \mathcal{W}^X_{12 \ldots m}
\]

where \( X = x_1 x_2 \ldots x_m \) where \( x_i = \{ R, V \} \) corresponds to whether the \( i \)th gluon is real (R) or virtual (V).
The differential cross section for the emission of \( m \) energy-ordered soft gluons normalised to the Born cross section for a given \textit{gluonic configuration} \( X \) reads

\[
\frac{d\sigma^X_m}{\sigma_0} = \sum_m \frac{1}{m!} d\Phi_m \langle m + 2 | m + 2 \rangle_X
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\]

Re-arranging one may write

\[
\frac{d\sigma^X_m}{\sigma_0} = \sum_m \frac{\bar{\alpha}_s^m}{m!} \left( \prod_{i=1}^{m} \frac{d\theta_i d\phi_i}{2\pi} \right) W_{12\ldots m}^X
\]

where \( X = x_1 x_2 \ldots x_m \) where \( x_i = \{R, V\} \) corresponds to whether the \( i^{\text{th}} \) gluon is real (R) or virtual (V).
The final form of the eikonal amplitude squared

\[ W_{12...m}^{X} = \left( \prod_{n=1}^{m} \sum_{i_{n-1},i'_{n-1} \in U_{m-1}} \omega_{n}^{i_{n-1}i'_{n-1}} \right) C_{i_{0}i'_{1}...i_{m-1}}^{i_{0}i'_{1}...i_{m-1}} \]

where \( U_{m} = \{a, b, 1, 2, \ldots, m\} \) (and \( U_{0} = \{a, b\} \)) is the set of all possible emitters/dipoles of the \( m^{th} \) gluon, and the antenna function \( w_{ij}^{\ell} \) is

\[ w_{ij}^{\ell} = p_{\ell i}^{2} \frac{(p_{i} \cdot p_{j})}{(p_{i} \cdot p_{\ell})(p_{j} \cdot p_{\ell})} \equiv \frac{(ij)}{(i\ell)(\ell j)} \]

with \((ij) = 1 - \cos \theta_{ij} = \cosh(\eta_{ij}) - \cos \phi_{ij}\).

The colour factor is

\[ C_{i_{0}...i_{m-1}}^{i'_{0}...i'_{m-1}} = T_{i_{0}}^{a_{1}} \cdots T_{i_{m-1}}^{a_{m}} \cdot T_{i'_{0}}^{a_{1}} \cdots T_{i'_{m-1}}^{a_{m}} \]

the dot “\( \cdot \)” means “trace”
Properties of the above eikonal amplitude squared:

1. Totally symmetric under the interchange of the two hardest partons; quark & antiquark \((a \leftrightarrow b)\)

2. Totally symmetric under the interchange of the legs of the dipole emitting the softest gluon.

3. Not symmetric under the interchange of the legs of each and every single dipole. *This symmetry breaking is primarily due to the associated colour factor \(\sim\) see five-loops example*

4. For the softest gluon, it is always true that

\[
\mathcal{W}^{x_1 \ldots R}_{12 \ldots m} = -\mathcal{W}^{x_1 \ldots V}_{12 \ldots m}
\]
Eikonal amplitudes
The EikAmp program

The above eikonal amplitude squared is computed using a Mathematica program: “EikAmp”

Main algorithm:

1. Determine loop order
2. Begin with the bra amplitude
3. For each emitted gluon $\ell$ (multiplicative process)
   - i. Find the set of all possible dipoles
   - ii. Pick up the first leg of each dipole
   - iii. Determine the corresp. colour matrix (fund. or Adj. rep)
4. Move to the ket amplitude
   - i. Pick up the second leg of each dipole
   - ii. Determine the corresponding conj. colour matrix
5. Call the CSimplify function of ColorMath
6. Decompose the output of CSimplify in terms of $C_F, C_A$
7. Multiply by the appropriate antenna $w_{ij}^{\ell}$.
8. Sum up all contributions.
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In large-$N_c$ limit all the Feynman diagrams that have a **non-planar topology** are discarded as their corresponding cross-sections are suppressed by $1/N_c^2$.

Both : number of Feynman diagrams + colour structure simplify considerably:

\[
\mathcal{C}^{i_1\ldots i_m}_{i'_1\ldots i'_m} = \left( \frac{N_c}{2} \right)^m
\]

\[
\tilde{\mathcal{W}}_{12\ldots m}^X = \sum_{\pi_m} \frac{(ab)}{(ai_1)(i_1i_2)\ldots(i_nb)}
\]

where the colour factor has been absorbed into $\tilde{\alpha}_s = N_c\alpha_s/\pi$

Easily get large-$N_c$ limit result from EikAmp by setting $\mathcal{C}_F \to \mathcal{C}_A/2$
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Easily get large-$N_c$ limit result from EikAmp by setting $C_F \rightarrow C_A/2$
Examples
- The one-loop eikonal amplitudes squared are quite simple:

\[ W^R_1 = C_F \left( w_{ab}^1 + w_{ba}^1 \right) = 2 C_F w_{ab}^1 \]
\[ W^V_1 = -W^R_1 \]

- The two-loop amplitudes are given by

\[ W^{RR}_{12} = \tilde{W}^{RR}_{12} + \overline{W}^{RR}_{12} \]

where reducible and irreducible contributions are

\[ \tilde{W}^{RR}_{12} = W^R_1 W^R_2 \]
\[ \left[ \frac{1}{2^2} \right] \overline{W}^{RR}_{12} = \frac{1}{2} C_F C_A A_{ab}^{12} \]
Remarks:

- The antenna function $A_{ab}^{ij} = w_{ab}^i (w_{ai}^j + w_{bi}^j - w_{ab}^j)$ represents a two-parton *cascade* emission.

- $A_{ab}^{ij}$ is both
  - **friable**: no singular dependence on angles $\Leftrightarrow$ no angular logs in $\sigma$.
  - **ideal**: fully integrable over the directions of all gluons involved.

- $A_{ab}^{ij}$ provides next-to-leading log contrib. to $\sigma$. 
Examples

Three-loops

- $2^3 = 8$ (configs.) amplitudes need to be evaluated.

- Due to symmetry, only one is actually to be calculated! \( \rightsquigarrow \) the all-real amplitude (all other amplitudes can be deduced from it!)

- The all-real Eikonal amp. squared

\[
W_{123}^{RRR} = \tilde{W}_{123}^{RRR} + \overline{W}_{123}^{RRR}
\]

where the reducible contrib. is

\[
\tilde{W}_{123}^{RRR} = \prod_{i=1}^{3} W_i^R + \sum_{ijk=1}^{3} W_i^R \overline{W}_{jk}^{RR}
\]

and the irreducible contrib. is

\[
\left[ \frac{1}{2^3} \right] W_{123}^{RRR} = \frac{1}{4} C_F C_A^2 \left( A_{12}^{12} A_{13}^{13} + B_{ab}^{123} \right)
\]
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Three-loops

- $2^3 = 8$ (configs.) amplitudes need to be evaluated.

- Due to symmetry, *only one* is actually to be calculated! $\Rightarrow$ the all-real amplitude (all other amplitudes can be deduced from it!)

- The all-real Eikonal amp. squared

$$\mathcal{W}_{123}^{RRR} = \mathcal{W}_{123}^{RR} + \mathcal{W}_{123}^{RRR}$$

where the reducible contrib. is

$$\mathcal{W}_{123}^{RRR} = \prod_{i=1}^{3} \mathcal{W}_{i}^{R} + \sum_{ijk=1}^{3} \mathcal{W}_{i}^{R} \overline{\mathcal{W}}_{jk}^{RR}$$

and the irreducible contrib. is

$$\left[ \frac{1}{2^3} \right] \overline{\mathcal{W}}_{123}^{RRR} = \frac{1}{4} C_F C_A^2 \left( \mathcal{A}_{ab}^{12} \overline{\mathcal{A}}_{ab}^{13} + B_{ab}^{123} \right)$$
Remarks:

- The term $\mathcal{W}_i^R \mathcal{W}_{jk}^{RR}$ is an interference between the one- and two-loops amplitudes.

- The antenna function $B_{ab}^{123}$ represents three-parton cascade emission $\rightsquigarrow$ friable and ideal

\[ B_{ab}^{ijk} = w_{ab}^i \left( A_{ai}^{jk} + A_{ib}^{jk} - A_{ab}^{jk} \right) \]

- The irreducible contrib. $\mathcal{W}_{123}^{RRR}$ is not completely reducible!
  resembles two-loop pattern $\rightsquigarrow B$ is related to $A$

- No finite-$N_c$ corrections up to this order.
- $2^4$ (configs.) amplitudes ⇒ Only one is to be calculated!

- The all-real Eikonal amp. squared

$$\mathcal{W}_{1234}^{RRRR} = \mathcal{W}_{1234}^{RRRR} + \mathcal{W}_{1234}^{RRRR}$$

where reducible contrib. is easy to find!

The irreducible contrib. is

$$\mathcal{W}_{1234}^{RRRR} = \mathcal{W}_{1234}^{RRRR} + \mathcal{W}_{1234}^{RRRR}$$

with the “irreducible-reducible” contrib.

$$\left[\frac{1}{2^4}\right] \mathcal{W}_{1234}^{RRRR} = \frac{C_F C_A^3}{8} \left( A_{ab}^{12} A_{ab}^{13} A_{ab}^{14} + \sum_{jkl=1}^{4} A_{ab}^{1j} B_{ab}^{1k\ell} + C_{1234}^{ab} \right)$$
The new *genuinely* “irreducible-irreducible” contribution that contains the first finite $N_c$ correction,

\[
\left[ \frac{1}{2^4} \right] \mathcal{W}_{1234}^{RRRR} = \frac{1}{4} C_F C_A^2 \left( C_F - \frac{C_A}{2} \right) A_{ab}^{1234} + \frac{1}{8} C_F C_A^3 \mathcal{A}_{ab}^{1234}
\]

where $A$ and $\mathcal{A}$ are pseudo-antenna functions related to $A_{ij}^{kl}$. 
Remarks:

- All antenna and pseudo-antenna functions in the four-loop amplitude are friable and ideal $\sim W_{1234}^{RRRR}$ is friable and ideal.

  KKK and Delenda '15

- Dokshitzer et al. stated that $W_{1234}^{RRRR}$ is friable but not deal! They named it “colour monster” with a colour factor equals to $C_F N_c$.

  Dokshitzer et al '91

- Easily see from $W_{1234}^{RRRR}$ that

  $$C_F C_A^2 \left(C_F - \frac{C_A}{2}\right) = -C_F N_c$$

- First genuine finite-$N_c$ correction appears at this order.
The all-order five-loop eikonal amp. squared may be cast in the form

\[ W_{12345}^{RRRRR} = \tilde{W}_{12345}^{RRRRR} + W_{12345}^{RRRRR} + N_{12345}^{RRRRR} \]

where the reducible and irreducible contributions follow pattern seen at previous loop order.

The new *non-symmetric* contribution \( N_{12345}^{X} \) reads

\[ N_{12345}^{X} = W_{12345}^{X, \text{EikAmp}} - \left( \tilde{W}_{12345}^{X} + \bar{W}_{12345}^{X} \right) \]

\[ = \frac{1}{2} C_F^2 C_A^3 I_{12345}^{X} + \frac{1}{8} C_F C_A^4 J_{12345}^{X} \]

noting that \( W_{12345}^{X} = W_{12345}^{X, \text{EikAmp}} \). The terms \( I_{12345}^{X} \) and \( J_{12345}^{X} \) have not been simplified!
Remarks:

- Unlike previous loop-order, the five-loop amp. squared contains non-symmetric contributions.

- Only 8 out of 32 amplitudes contain non-symmetric contribs.

- These non-symmetric contribs. originate from the fact that there are (eikonal) Feynman diagrams which are not symmetric under the interchange of the emitters of one or more gluons (or equivalently, interchange of the legs of one or more emitting dipoles).
Examples of non-symmetric Feynman diagrams

\[ (d1) : \quad - \frac{3 C_F C_A^3}{8} \left( C_F - \frac{C_A}{2} \right) w_{1b}^1 w_{1b}^2 w_{1b}^3 w_{1b}^4 w_{1b}^5 \]

\[ (d2) : \quad - \frac{C_F C_A^3}{4} \left( C_F - \frac{C_A}{2} \right) w_{1b}^1 w_{b1}^2 w_{1b}^3 w_{1b}^4 w_{1b}^5 \]
Inspecting all five eikonal amplitudes, one observes:

- All amplitudes may be written in terms of the two-parton/loop antenna function $A_{ij}^{k\ell} \sim \text{web}$ (two-eikonal irreducible diagram)

- There seems to be a pattern of exponentiation of the eikonal amplitude. Non-symmetric contributions may be considered as corrections!

$$
\mathcal{M} = \mathcal{M}_0 \exp \left[ \sum_{i=1}^{n} \tilde{C}_i \tilde{M}_i \right]
$$
Summary & outlook

Done !
✓ Computed eikonal amplitude at finite-$N_c$ for the first time in literature up to five-loops
✓ Calculations are automated \(\implies\) easily extended to higher orders
✓ Symmetry breaking seen at five-loops (and perhaps persists at higher loops too)
✓ Eikonal amplitudes seem to: (i) all build up from the two-parton antenna function, (ii) exponentiate

Waiting . . .

? Finalise the EikAmp Mathematica package
? Write EikAmp using C++ and/or Fortran speed + interface with other MC program
? Solve BMS equ and compare to ours at large-$N_c$.  \cite{Banfi et al '02}
? Move onto next-to-eikonal approx. next-to-leading log \(\alpha_s^n L^{n-1}\) resum.