Solving the NLO BK equation in coordinate space

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This talk describes the results of the paper
“Direct numerical solution of the coordinate space
Balitsky-Kovchegov equation at next to leading order,”

Outline of the talk:
- NLO BK equation
- Numerical result: \( \ln r \) divergence
- Conformal dipole

Approach
Brute force solution of the coordinate space equation as it is
written down by Balitsky & Chirilli in
Motivation

Many ingredients available for NLO small-x calculation

- NLO BK equation
- NLO JIMWLK equation
- NLO $\gamma^*$ impact factor for DIS
- NLO single inclusive cross section for forward $pA$

Want to start doing NLO phenomenology with these!

But first need to solve the equations and calculate the cross sections!
The equation

**Equation**

\[
\partial_y S(r) = \frac{\alpha_s N_C}{2\pi^2} K_1 \otimes [S(X)S(Y) - S(r)] + \frac{\alpha_s^2 N_F N_C}{8\pi^4} K_f \otimes S(Y)[S(X') - S(X)] \\
+ \frac{\alpha_s^2 N_C^2}{8\pi^4} K_2 \otimes [S(X)S(z - z') S(Y') - S(X)S(Y)]
\]

**Notations & approximations**

\[S(x - y) \equiv \langle \text{Tr} \, U^\dagger(x) U(y) \rangle\]

\[\otimes = \int d^2 z / \int d^2 z \, d^2 z'\]

- Large \(N_c\)
- Mean field:

\[\langle \text{Tr} \, U^\dagger U \, \text{Tr} \, U^\dagger U \rangle \rightarrow \langle \text{Tr} \, U^\dagger U \rangle \langle \text{Tr} \, U^\dagger U \rangle\]

**Coordinates**
Kernels

Expressions are frightening, the derivation even more.

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_C}{4\pi} \left( \frac{\beta}{N_c} \ln r^2 \mu^2 - \frac{\beta}{N_c} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} \right. \right. \]
\[ \left. \left. + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_C} - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \right] \]

\[ K_2 = -\frac{2}{(Z - Z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \right. \]
\[ \left. + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(Z - Z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

\[ K_f = \frac{2}{(Z - Z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]
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+ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_C} - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right] \]

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\left. r^4 \frac{X^2 Y'^2}{X'^2 Y^2} + \frac{r^2}{X^2 Y'^2(Z - Z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

\[ K_f = \frac{2}{(Z - Z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \]

- Leading order
Kernels

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\]

\[
\left. \quad + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{N_F}{N_C} - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \]

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K_2 = -\frac{2}{(Z - Z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4 r^2 (Z - Z')^2}{(Z - Z')^4 (X^2 Y'^2 - X'^2 Y^2)} \right.
\]

\[
\left. \quad + \frac{r^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2 (Z - Z')^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2}
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\]

▶ Leading order
▶ Running coupling (Terms with \( \beta \) function coefficient)
Kernels

Expressions are frightening, the derivation even more.

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_C}{4\pi} \left( \frac{\beta}{N_C} \ln r^2 \mu^2 - \frac{\beta}{N_C} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} \right) 
+ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_C} - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right] \]

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+ \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z - z')^2} \left] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \]

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▶ Leading order
▶ Running coupling (Terms with $\beta$ function coefficient)
▶ Conformal logs $\Rightarrow$ vanish for $r = 0 (X = Y \& X' = Y')$
Kernels

Expressions are frightening, the derivation even more.

\[ K_1 = \frac{r^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{\beta}{N_c} \ln r^2 \mu^2 - \frac{\beta}{N_c} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} \right) + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_c} - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right] \]

\[ K_2 = -\frac{2}{(Z - Z')^4} + \left[ \frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(Z - Z')^2}{(Z - Z')^4(X^2 Y'^2 - X'^2 Y^2)} \right. \]

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▶ Leading order

▶ Running coupling (Terms with \( \beta \) function coefficient)

▶ Conformal logs \( \Rightarrow \) vanish for \( r = 0 \) (\( X = Y \) & \( X' = Y' \))

▶ Nonconformal double log \( \Rightarrow \) blows up for \( r = 0 \)
Running coupling

Absorb the $\beta$-terms into

- “Balitsky” running for LO term
- Parent dipole running for NLO terms

Now:

$$\frac{\alpha_s N_c}{2\pi^2} K_1 = \frac{\alpha_s(r) N_c}{2\pi^2} \left[ \frac{r^2}{X^2 Y^2} + \frac{1}{X^2} \left( \frac{\alpha_s(X)}{\alpha_s(Y)} - 1 \right) + \frac{1}{Y^2} \left( \frac{\alpha_s(Y)}{\alpha_s(X)} - 1 \right) \right]$$

$$+ \frac{\alpha_s(r)^2 N_c^2}{8\pi^3} \frac{r^2}{X^2 Y^2} \left[ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 N_F}{9 N_c} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right]$$
Initial condition

\[ N(r) \equiv 1 - S(r) = 1 - \exp \left[ -\frac{(r^2 Q_{s0}^2)^\gamma}{4} \ln \left( \frac{1}{r \Lambda_{QCD}} + e \right) \right], \]

2 tunable parameters

- \( \frac{Q_{s0}}{\Lambda_{QCD}} \) \( \implies \) basically determines value of \( \alpha_s \)
- \( \gamma \): anomalous dimension: shape
  - LO phenomenology prefers \( \gamma \gtrsim 1 \)
  - This eventually evolves into \( \gamma \sim 0.8 \) (running \( \alpha_s \))
Evolution speed at initial condition

\[ \ln r \] divergence

\[ \gamma = 1 \] (MV model)

- Small \( Q_s/\Lambda_{QCD} \)
  \[ \implies \] large \( \alpha_s \)
  NLO corrections big, amplitude decreases at all \( r \)

- For smaller \( \alpha_s \) region around \( r \sim 1/Q_s \) is ok.

- For small dipoles
  \[ \partial_y N/N \sim \ln r \]
$N < 0$ as a practical problem

LO equation

$$\partial_y S(r) = \frac{\alpha_s N_c}{2\pi^2} \int_z \frac{r^2}{X^2 Y^2} [N(X) + N(Y) - N(r) - N(X)N(Y)]$$

Consider a small but finite $\Delta y$ (as in a numerical solution)

A diverging $\partial_y N/N$ makes $N < 0$ in one rapidity step

Convergence of the $z$-integral on the r.h.s. of the (LO!) BK equation requires $N(r) \to 0$ for $r \to 0$ (Limit $X \to 0$, $Y \to r$ in integral)

$\implies$ If $N < 0$ for $r \to 0$:

- The equation blows up mathematically
- Solution is inconsistent with the definition

$$N(x_T - y_T) = 1 - \frac{1}{N_c} \text{Tr} \ U(x_T)^\dagger U(y_T)$$

In the numerics, we enforce $N(r) \geq 0$ by hand.
In $r$ behavior caused by the nonconformal double log term
Changing the initial condition

Decrease $\gamma$

$\gamma = 0.8$
- Initially looks ok, if $\alpha_s$ small enough
- But $\gamma$ gradually increases, blows up by $\gamma \gtrsim 20$

$\gamma = 0.6$
- Requires $\alpha_s$ small enough
- Solution behaves ok at least up to $y = 30$
Interpreting the $\partial_y N/N$ plot

- **LO**: roughly $N(r) \sim (Q_s r)^{2\gamma}$ & $Q_s^2 \sim e^{\lambda y}$
  \[ \Rightarrow \partial_y N/N \approx 2\gamma \lambda > 0 \]

- If $\partial_y N/N \to -c < 0$ for $r \to 0$,
  \[ \Rightarrow N \sim e^{-cy} \quad \Rightarrow \text{still ok} \]
  (But why would $N(y)$ decrease with $y$?)

Parametrizing $N(r) \sim (Q_s r)^{2\gamma(y)}$ \[ \partial_y N/N \sim c \ln r \quad \Rightarrow \gamma(y) \sim y \]
\[ \Rightarrow \text{front gets steeper} \quad \Rightarrow \text{eventually } N(r) \sim \theta(r - 1/Q_s) \]
(This is assuming $c = \text{const}$, but in numerics $c$ grows $\Rightarrow$ divergence worse)

Plot $\gamma(r) \equiv d \ln N(r)/d \ln r^2$:

\[ \begin{align*}
\text{LO} & \quad 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4 \\
\gamma = 0.6 & \quad \gamma = 0.8 & \quad \gamma = 1.0 \\
Q_{s,0}/\Lambda_{\text{QCD}} = 19 & \quad Q_{s,0}/\Lambda_{\text{QCD}} = 19 & \quad Q_{s,0}/\Lambda_{\text{QCD}} = 19
\end{align*} \]

$y = 1$ \quad $y = 5$ \quad $y = 30$
Conformal composite dipole

Proposal by Balitsky & Chirilli,


absorb logarithm into redefinition of dipole operator:

\[
S(r)^{\text{conf}} = S(r) - \frac{\alpha_s N_c}{4\pi^2} \int d^2 z \frac{r^2}{X^2 Y^2} \ln \frac{ar^2}{X^2 Y^2} [S(X)S(Y) - S(r)].
\]

- \( \alpha \): dimensionful constant, cancels out in the end
- Double log \( \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \) drops out from \( K_1 \)
- New term \( \frac{2r^2}{X^2 Y^2 (z-z')^2} \ln \frac{r^2(z-z')^2}{X'^2 Y'^2} \) appears in kernel \( K_2 \)
Evolution speed at $y = 0$: conformal dipole

$\gamma = 1$ (MV model)

- $r \sim 1/Q_s$ ok, even for large $\alpha_s$
- For small dipoles $\partial_y N/N \sim \ln r$ still

As expected: reason is new $\ln r$ term:

$\gamma < 1$ similar as for original equation.
Conclusions

- NLO corrections mostly negative: slow down evolution
  - This is good for phenomenology
- Divergence \( \sim \ln r \) for small \( r \) problematic
  - Makes solving the equation for \( \gamma \sim 1 \) impossible
  - Problem at small \( r \) \( \implies \) presumably related to large \( Q^2 \) logs
    \( \implies \) resummation needed?
Changing the initial condition

Decrease $\gamma$

$Q_{s,0}/\Lambda_{QCD} = 19$

$\gamma = 0.8$

$\gamma = 0.6$
Evolving instability for $\gamma = 0.8$

$y = 0$

$y = 5$
Changing the initial condition: conformal dipole
Decrease $\gamma$

$\gamma = 0.8$
- Kind of ok $y \gtrsim 20$

$\gamma = 0.6$
- Small $r$ ok, large $r$ starts to be erratic