Solving the NLO BK equation in coordinate space

T. Lappi

University of Jyväskylä, Finland

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Outline

This talk describes the results of the paper
"Direct numerical solution of the coordinate space Balitsky-Kovchegov equation at next to leading order,"
T. L., **H. Mäntysaari**, Phys. Rev. D **91** (2015) 7, 074016, [arXiv:1502.02400 [hep-ph]]

Outline of the talk:

- NLO BK equation
- Numerical result: $\ln r$ divergence
- Conformal dipole

Approach

Brute force solution of the coordinate space equation as it is written down by Balitsky & Chirilli in
Phys. Rev. D **77** (2008) 014019, [arXiv:0710.4330 [hep-ph]].

Motivation

Many ingredients available for NLO small-*x* calculation

- NLO BK equation
- NLO JIMWLK equation
- NLO γ^* impact factor for DIS
- NLO single inclusive cross section for forward pA

Want to start doing NLO phenomenology with these!

But first need to solve the equations and calculate the cross sections!

The equation

Equation

$$\partial_y S(r) = \frac{\alpha_s N_c}{2\pi^2} K_1 \otimes [S(X)S(Y) - S(r)] + \frac{\alpha_s^2 N_F N_c}{8\pi^4} K_f \otimes S(Y)[S(X') - S(X)]$$
$$+ \frac{\alpha_s^2 {N_c}^2}{8\pi^4} K_2 \otimes [S(X)S(z - z')S(Y') - S(X)S(Y)]$$

Notations & approximations

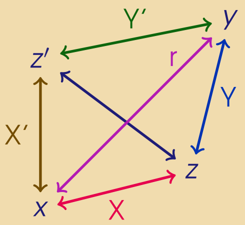
$$S(x - y) \equiv \left\langle \operatorname{Tr} U^\dagger(x) U(y) \right\rangle$$

$$\otimes = \int d^2 z \quad \Big/ \quad \int d^2 z \, d^2 z'$$

- Large N_c
- Mean field:

$$\left\langle \operatorname{Tr} U^\dagger U \operatorname{Tr} U^\dagger U \right\rangle \to \left\langle \operatorname{Tr} U^\dagger U \right\rangle \left\langle \operatorname{Tr} U^\dagger U \right\rangle$$

Coordinates



Kernels

Expressions are frightening, the derivation even more.

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2}\right.\right.$$
$$\left.\left. + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_\mathrm{F}}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2Y'^2 - X'^2Y^2)}\right.$$
$$\left. + \frac{r^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{r^2}{X^2Y'^2(z-z')^2}\right]\ln\frac{X^2Y'^2}{X'^2Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2Y^2 + Y'^2X^2 - r^2(z-z')^2}{(z-z')^4(X^2Y'^2 - X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2}$$

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- Leading order

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- Leading order
- Running coupling (Terms with β function coefficient)

Kernels

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- Leading order
- Running coupling (Terms with β function coefficient)
- Conformal logs $\Longrightarrow$ vanish for $r = 0$ ($X = Y$ & $X' = Y'$)

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- Leading order
- Running coupling (Terms with β function coefficient)
- Conformal logs $\Longrightarrow$ vanish for $r = 0$ ($X = Y$ & $X' = Y'$)
- Nonconformal double log $\Longrightarrow$ blows up for $r = 0$

Running coupling

Absorb the β-terms into

- "Balitsky" running for LO term
- Parent dipole running for NLO terms

Now:

$$\frac{\alpha_s N_c}{2\pi^2} K_1 = \frac{\alpha_s(r) N_c}{2\pi^2} \left[\frac{r^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X)}{\alpha_s(Y)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y)}{\alpha_s(X)} - 1 \right) \right]$$

$$+ \frac{\alpha_s(r)^2 {N_c}^2}{8\pi^3} \frac{r^2}{X^2 Y^2} \left[\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{N_F}{N_c} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right]$$

Initial condition

$$N(r) \equiv 1 - S(r) = 1 - \exp\left[-\frac{(r^2 Q_{s0}^2)^\gamma}{4} \ln\left(\frac{1}{r\Lambda_{QCD}} + e\right)\right],$$

2 tunable parameters

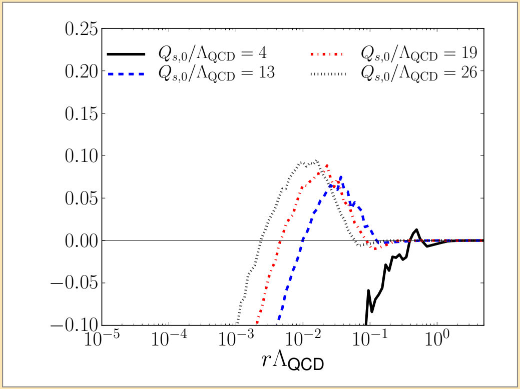
- $\frac{Q_{s0}}{\Lambda_{QCD}}$ $\Longrightarrow$ basically determines value of α_s
- γ: anomalous dimension: shape
  - LO phenomenology prefers $\gamma \gtrsim 1$
  - This eventually evolves into $\gamma \sim 0.8$ (running α_s)

Evolution speed at initial condition

ln r divergence

$\gamma = 1$ (MV model)

- Small Q_s/Λ_{QCD} $\Longrightarrow$ large α_s NLO corrections big, amplitude decreases at all r
- For smaller α_s region around $r \sim 1/Q_s$ is ok.
- For small dipoles $\partial_y N/N \sim \ln r$



$\partial_y N/N$

$N < 0$ as a practical problem

LO equation

$$\partial_y S(r) = \frac{\alpha_s N_c}{2\pi^2} \int_z \frac{r^2}{X^2 Y^2} [N(X) + N(Y) - N(r) - N(X)N(Y)]$$

Consider a small but finite dy (as in a numerical solution)

A diverging $\partial_y N/N$ makes $N < 0$ in one rapidity step

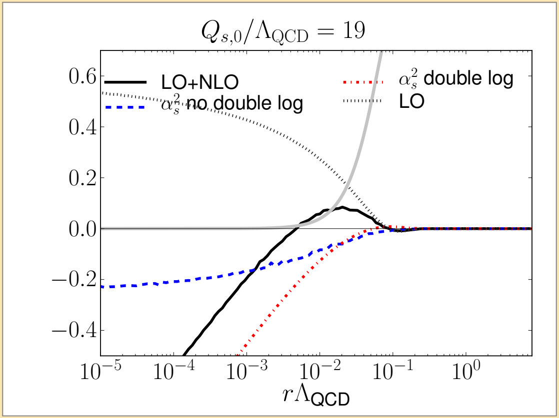
Convergence of the z-integral on the r.h.s. of the (LO!) BK equation requires $N(r) \to 0$ for $r \to 0$ (Limit $X \to 0$, $Y \to r$ in integral)

$\Longrightarrow$ If $N < 0$ for $r \to 0$:

- The equation blows up mathematically
- Solution is inconsistent with the definition $N(\mathbf{x}_T - \mathbf{y}_T) = 1 - \frac{1}{N_c} \operatorname{Tr} U^\dagger(\mathbf{x}_T) U(\mathbf{y}_T)$

In the numerics, we enforce $N(r) \geq 0$ by hand.

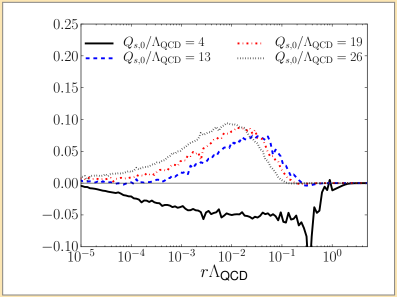
Origin of the negativity

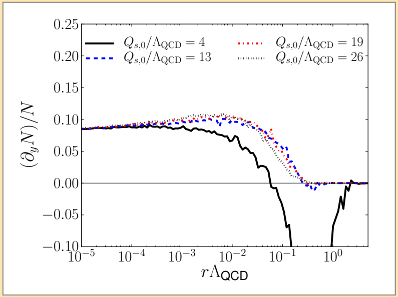


In *r* behavior caused by the nonconformal double log term

Changing the initial condition

Decrease γ





$\gamma = 0.8$

- Initally looks ok, if α_s small enough
- But γ gradually increases, blows up by $y \gtrsim 20$

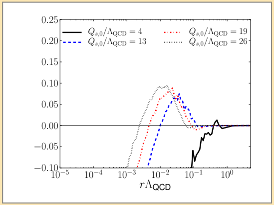
$\gamma = 0.6$

- Requires α_s small enough
- Solution behaves ok at least up to $y = 30$

Interpreting the $\partial_y N/N$ plot

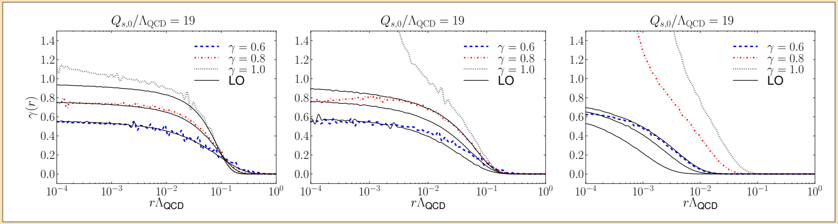
- LO: roughly $N(r) \sim (Q_s r)^{2\gamma}$ & $Q_s^2 \sim e^{\lambda y}$
  $\Longrightarrow \partial_y N/N \approx 2\gamma\lambda > 0$
- If $\partial_y N/N \to -c < 0$ for $r \to 0$,
  $\Longrightarrow N \sim e^{-cy} \Longrightarrow$ still ok

  (But why would $N(y)$ decrease with y?)



Parametrizing $N(r) \sim (Q_s r)^{2\gamma(y)}$ $\quad \partial_y N/N \sim c \ln r \Longrightarrow \gamma(y) \sim y$
$\Longrightarrow$ front gets steeper $\quad \Longrightarrow$ eventually $N(r) \sim \theta(r - 1/Q_s)$
(This is assuming c =const, but in numerics c grows $\Longrightarrow$ divergence worse)

Plot $\gamma(r) \equiv \mathrm{d}\ln N(r)/\,\mathrm{d}\ln r^2$:



$y = 1$ $\qquad y = 5$ $\qquad y = 30$

Conformal composite dipole

Proposal by Balitsky & Chirilli,
Nucl. Phys. B **822** (2009) 45 `[arXiv:0903.5326 [hep-ph]]` :
absorb logarithm into redefinition of dipole operator:

$$S(r)^{\mathrm{conf}} = S(r) - \frac{\alpha_s N_c}{4\pi^2} \int \mathrm{d}^2 z \frac{r^2}{X^2 Y^2} \ln \frac{a r^2}{X^2 Y^2} [S(X)S(Y) - S(r)].$$

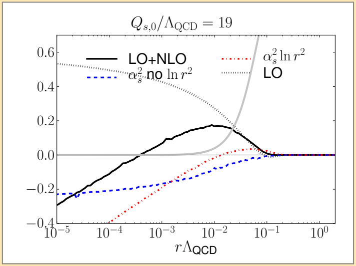
- a: dimensionful constant, cancels out in the end
- Double log $\ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2}$ drops out from K_1
- New term $\frac{2r^2}{X^2 Y'^2 (z-z')^2} \ln \frac{r^2 (z-z')^2}{X'^2 Y^2}$ appears in kernel K_2

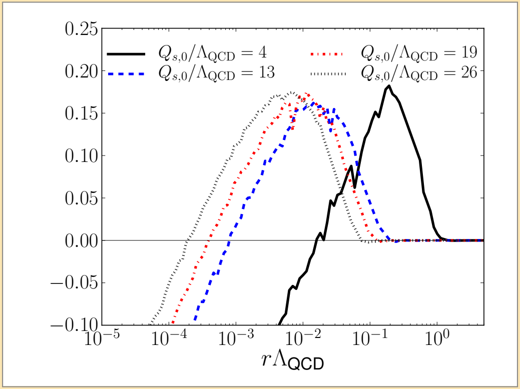
Evolution speed at $y = 0$: conformal dipole

$\gamma = 1$ (MV model)

- $r \sim 1/Q_s$ ok, even for large α_s
- For small dipoles $\partial_y N/N \sim \ln r$ still

As expected: reason is new $\ln r$ term:





$\partial_y N/N$

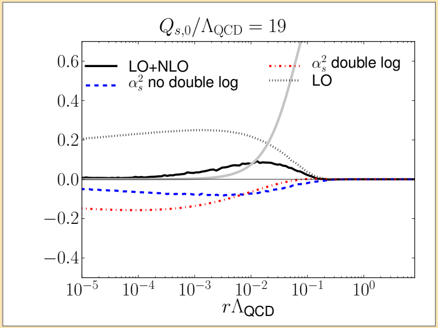
$\gamma < 1$ similar as for original equation.

Conclusions

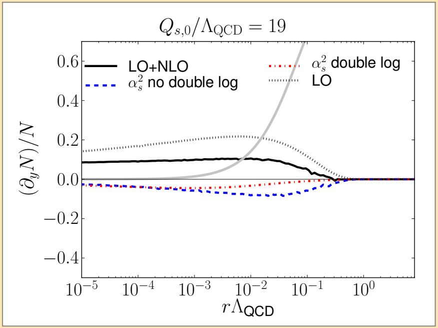
- NLO corrections mostly negative: slow down evolution
  - This is good for phenomenology
- Divergence $\sim \ln r$ for small r problematic
  - Makes solving the equation for $\gamma \sim 1$ impossible
  - Problem at small $r \implies$ presumably related to large Q^2 logs $\implies$ resummation needed?

Changing the initial condition

Decrease γ

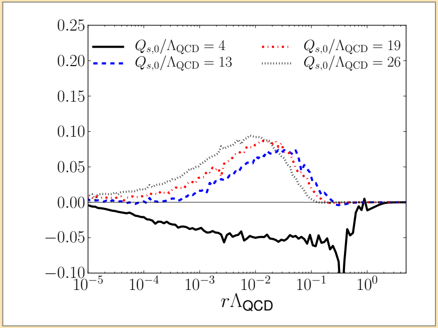


$\gamma = 0.8$

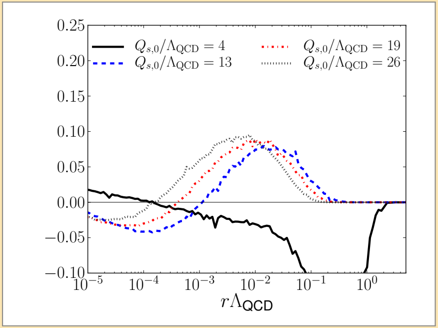


$\gamma = 0.6$

Evolving instability for $\gamma = 0.8$



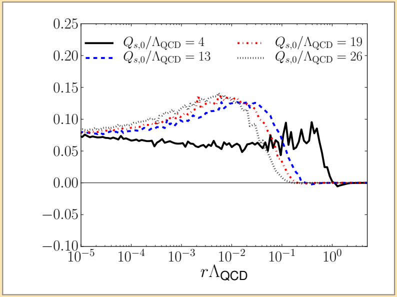
$y = 0$

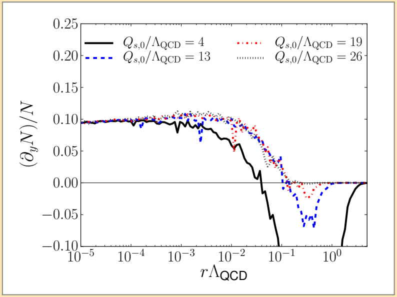


$y = 5$

Changing the initial condition: conformal dipole

Decrease γ





$\gamma = 0.8$

- Kind of ok $y \gtrsim 20$

$\gamma = 0.6$

- Small r ok, large r starts to be erratic