

# High-energy evolution of Wilson lines at the next-to-leading order

Giovanni Antonio Chirilli

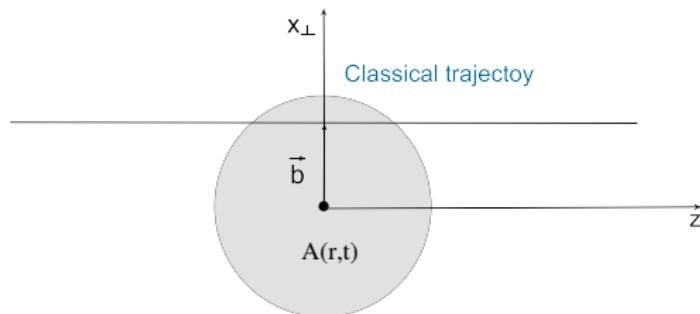
The Ohio State University

XXIII International Workshop on Deep-Inelastic Scattering and  
Related Subjects

Dallas - TX April 27 - May 1, 2015

- High-energy QCD scattering processes and Wilson lines.
- High-energy Operator Product Expansion: factorization in rapidity space.
- Evolution equation and background field method.
- NLO BK equation.
- Hierarchy of Wilson lines evolution at NLO: The Balitsky-JIMWLK evolution equation at NLO.
- Conclusions.

# High-energy scattering in QCD



phase factor for the high-energy scattering: Wilson-line operator

$$U(x_{\perp}, v) = \text{Pe}^{\frac{-ig}{c\hbar} \int_{-\infty}^{+\infty} dt \dot{x}_{\mu} A^{\mu}(x(t))}$$

$$\text{Pe}^{\int_{-\infty}^{+\infty} dt A(t)} = 1 + \int_{-\infty}^{+\infty} dt A(t) + \int_{-\infty}^{+\infty} dt A(t) \int_{-\infty}^t dt' A(t') + \dots$$

# Propagation in the shock wave: Wilson line (Spectator frame)



Each path is weighted with the gauge factor  $P e^{ig \int dx_\mu A^\mu}$ . Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction  $\Rightarrow$  we can replace the gauge factor along the actual path with the one along the straight-line path.



# Propagation in the shock wave: Wilson line (Spectator frame)



$$[z', z] = P e^{ig \int_0^1 du (z' - z)^\mu \textcolor{red}{A}_\mu (uz' + (1-u)z)} \quad U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

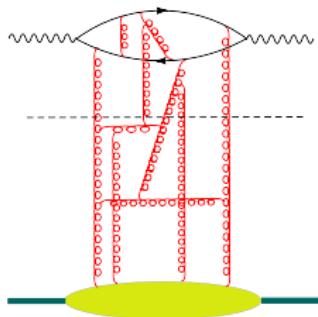
$$p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu, \quad p_1^\mu = \sqrt{s/2}(1, 0, 0, 1), \quad p_2^\mu = \sqrt{s/2}(1, 0, 0, -1)$$

$s$  center-of-mass energy.

# Propagation in the shock wave: Wilson line (Spectator frame)



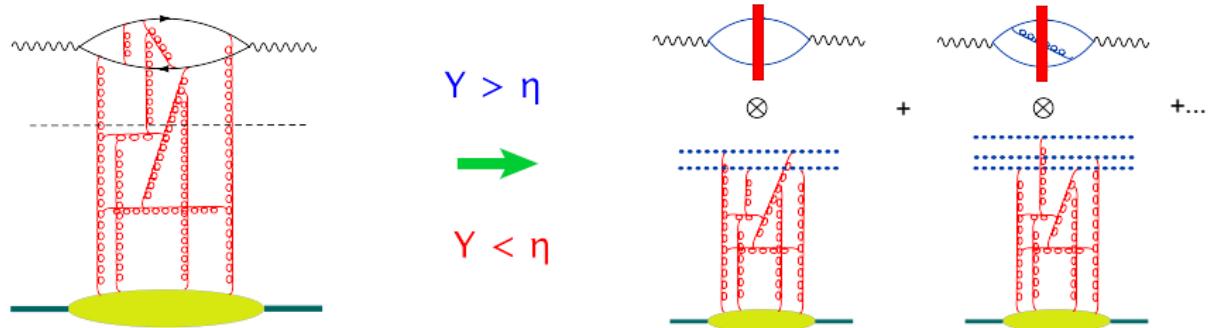
- At high energy all fields are ordered in rapidity  $\Rightarrow$  rapidity  $\eta$  is a suitable factorization parameter.



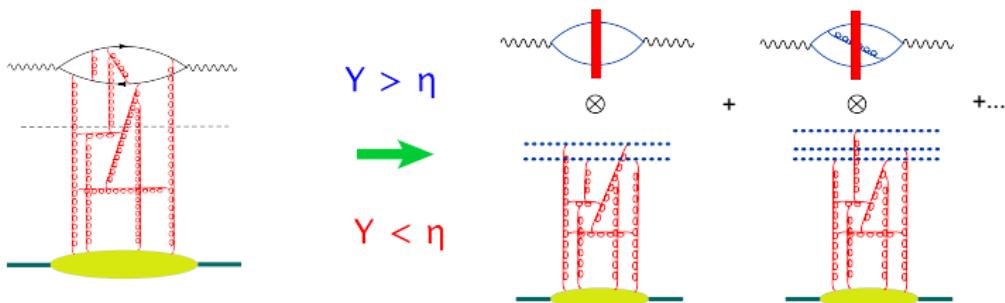
# Propagation in the shock wave: Wilson line (Spectator frame)



- At high energy all fields are ordered in rapidity  $\Rightarrow$  rapidity  $\eta$  is a suitable factorization parameter.



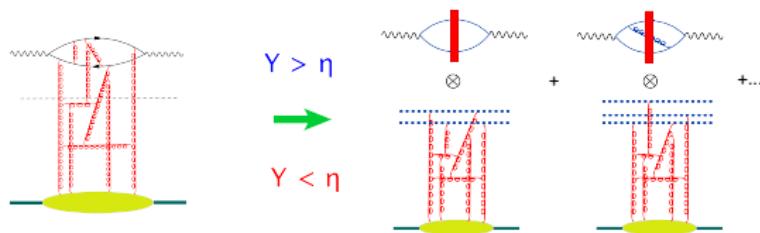
# High-energy Operator Product Expansion



$$\begin{aligned} \langle B | T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} | B \rangle &\simeq \int d^2 z_1 d^2 z_2 I_{\mu\nu}^{LO}(z_1, z_2; x, y) \langle B | \text{tr}\{U_{z_1}^\eta U_{z_2}^{\dagger\eta}\} | B \rangle \\ &+ \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 I_{\mu\nu}^{NLO}(z_1, z_2, z_3; x, y) \langle B | \text{tr}\{U_{z_1}^\eta U_{z_3}^{\dagger\eta}\} \text{tr}\{U_{z_3}^\eta U_{z_2}^{\dagger\eta}\} | B \rangle \end{aligned}$$

- $\eta = \ln \frac{1}{x_B}$
- $|B\rangle$  Target state

# High-energy Operator Product Expansion



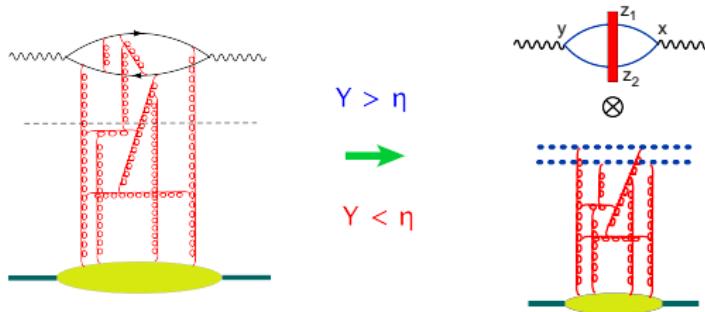
$$\langle B | T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} | B \rangle \simeq \int d^2 z_1 d^2 z_2 I_{\mu\nu}^{LO}(z_1, z_2; x, y) \langle B | [\text{tr}\{U_{z_1}^\eta U_{z_2}^{\dagger\eta}\}]^{\text{conf.}} | B \rangle + \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 I_{\mu\nu}^{NLO}(z_1, z_2, z_3; x, y) \langle B | \text{tr}\{U_{z_1}^\eta U_{z_3}^{\dagger\eta}\} \text{tr}\{U_{z_3}^\eta U_{z_2}^{\dagger\eta}\} | B \rangle$$

$$[\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf.}} = \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} + \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} \right] + O(\alpha_s^2)$$

- Conformal (energy independent) NLO Impact Factor  $I_{\mu\nu}^{NLO}$ 
  - For large nucleus (non-linear case): Balitsky and G.A.C. (2010).
  - For pomeron exchange (linear case): Balitsky and G.A.C. (2012).
- See also G. Beuf (2012): Light front perturbation theory in mixed space.

# Leading Order

$$[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A]_{\text{LO}} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$

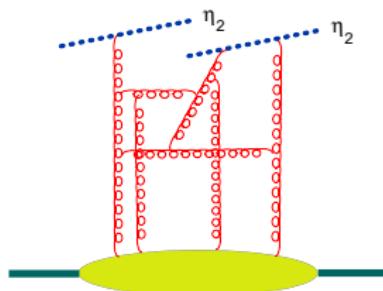


$$\langle B | T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} | B \rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I_{\mu\nu}^{\text{LO}}(x, y; z_1, z_2) \langle B | \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} | B \rangle + \dots$$

- If we use a model to evaluate  $\langle B | \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} | B \rangle$  we can calculate the DIS cross-section.
- If we want to include energy dependence to the DIS cross section, we need to find the evolution of  $\langle B | \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} | B \rangle$  with respect to the rapidity parameter  $\eta$ .

# Regularization of the rapidity divergence

Matrix elements of Wilson lines:  $\langle \text{tr}\{U(x)U^\dagger(y)\} \rangle_A$  are divergent



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

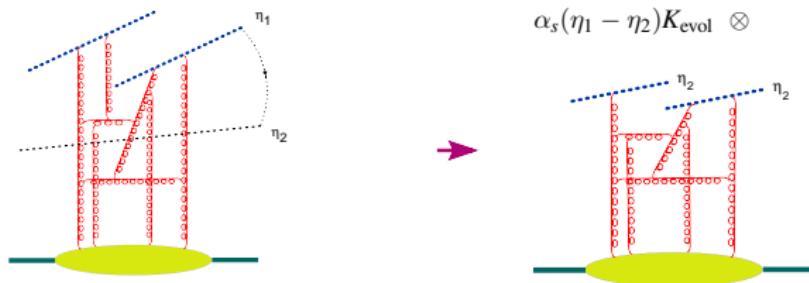
$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

## Regularization by: slope

$$U^\eta(x_\perp) = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} du n_\mu A^\mu (un + x_\perp) \right\} \quad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

- At NLO the regularization by rigid cut-off is more convenient.

# Evolution Equation



- Separate fields in quantum and classical according to low and large rapidity.  
Formally we may write:

$$\langle B | \mathcal{O}^{\eta_1} | B \rangle \rightarrow \langle \mathcal{O}^{\eta_1} \rangle_A \rightarrow \langle \mathcal{O}'^{\eta_2} \otimes \mathcal{O}^{\eta_1} \rangle_A$$

- Integrate over the quantum fields and get one-loop rapidity evolution of the operator  $\mathcal{O}$

$$\langle \mathcal{O}^{\eta_1} \rangle_A = \alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \otimes \langle \mathcal{O}'^{\eta_2} \rangle_A$$

- Where in principle  $\mathcal{O}$  and  $\mathcal{O}'$  are different operators.

# Non-linear evolution equation

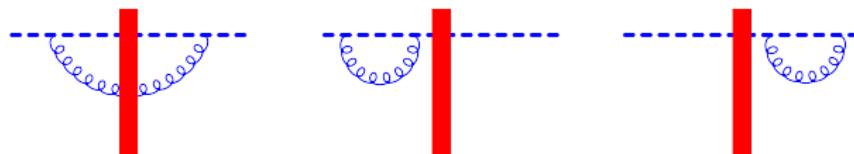
- Linear case  $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \mathcal{O}^{\eta_2}$

## Non-linear evolution equation

- **Linear case**     $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \mathcal{O}^{\eta_2}$
- **Non-linear case**     $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \{\mathcal{O}^{\eta_2} \mathcal{O}^{\eta_2}\}$

# Non-linear evolution equation

- Linear case  $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \mathcal{O}^{\eta_2}$
- Non-linear case  $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \{\mathcal{O}^{\eta_2} \mathcal{O}^{\eta_2}\}$



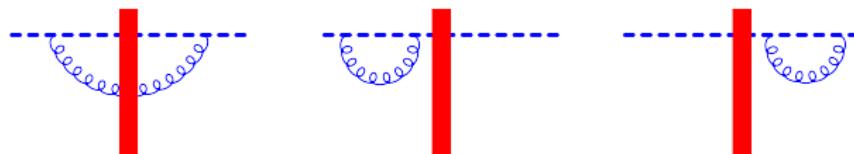
$$\langle \{U_x^{\eta_1}\}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta\eta \int \frac{d^2 z_\perp}{(x-z)_\perp^2} \left[ \langle \text{tr} \{U_x^{\eta_2} U_z^{\eta_2\dagger}\} \{U_z^{\eta_2}\}_{ij} \rangle_A - \langle \frac{1}{N_c} \{U_x^{\eta_2}\}_{ij} \rangle_A \right]$$

$$\Delta = \eta_1 - \eta_2$$

$$\{U_x^{\dagger\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\dagger\eta_1}\}_{ij}, \quad \{U_x^{\dagger\eta_1} U_y^{\dagger\eta_1}\}_{ij}$$

## Non-linear evolution equation

- Linear case  $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \mathcal{O}^{\eta_2}$
- Non-linear case  $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta K_{\text{evol}} \otimes \{\mathcal{O}^{\eta_2} \mathcal{O}^{\eta_2}\}$



$$\langle \{U_x^{\eta_1}\}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta\eta \int \frac{d^2 z_\perp}{(x-z)_\perp^2} \left[ \langle \text{tr} \{U_x^{\eta_2} U_z^{\eta_2\dagger}\} \{U_z^{\eta_2}\}_{ij} \rangle_A - \langle \frac{1}{N_c} \{U_x^{\eta_2}\}_{ij} \rangle_A \right]$$

$$\Delta = \eta_1 - \eta_2$$

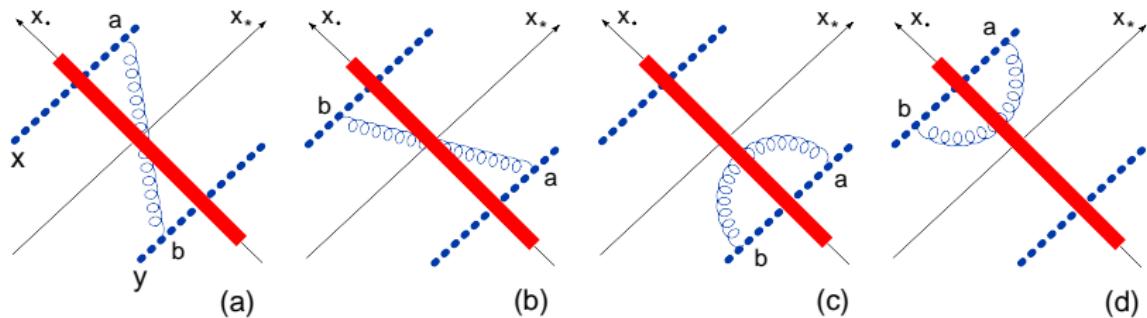
$$\{U_x^{\dagger\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\dagger\eta_1}\}_{ij}, \quad \{U_x^{\dagger\eta_1} U_y^{\dagger\eta_1}\}_{ij}$$

Obtain a set of rules that allow one to get the LO evolution of any trace or product of traces of Wilson lines

# Leading order: BK equation

$$\frac{d}{d\eta} \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$x_\bullet = \sqrt{\frac{s}{2}} x^-$$

$$x_* = \sqrt{\frac{s}{2}} x^+$$

## Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\text{tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

## Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\text{tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x-y)^2}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

Alternative approach: JIMWLK (1997-2000)

# Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\text{tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD  $\Rightarrow$  BFKL

(LLA:  $\alpha_s \ll 1, \alpha_s \eta \sim 1$ )

## Non linear evolution equation: BK equation

$$U_z^{ab} = 2\text{tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD  $\Rightarrow$  BFKL

(LLA:  $\alpha_s \ll 1$ ,  $\alpha_s \eta \sim 1$ )

LLA for DIS in sQCD  $\Rightarrow$  BK eqn

(LLA:  $\alpha_s \ll 1$ ,  $\alpha_s \eta \sim 1$ ,  $\alpha_s^2 A^{1/3} \sim 1$ )

(s for semi-classical)

## Motivation: Why NLO correction?

- How to take higher-order corrections into account (either for BFKL or non-linear evolution equation).
- Higher-order corrections are needed to improve phenomenology:
  - Determine the argument of the coupling constant.
  - Gives precision of LO.
- Check conformal invariance (in  $\mathcal{N}=4$  SYM)

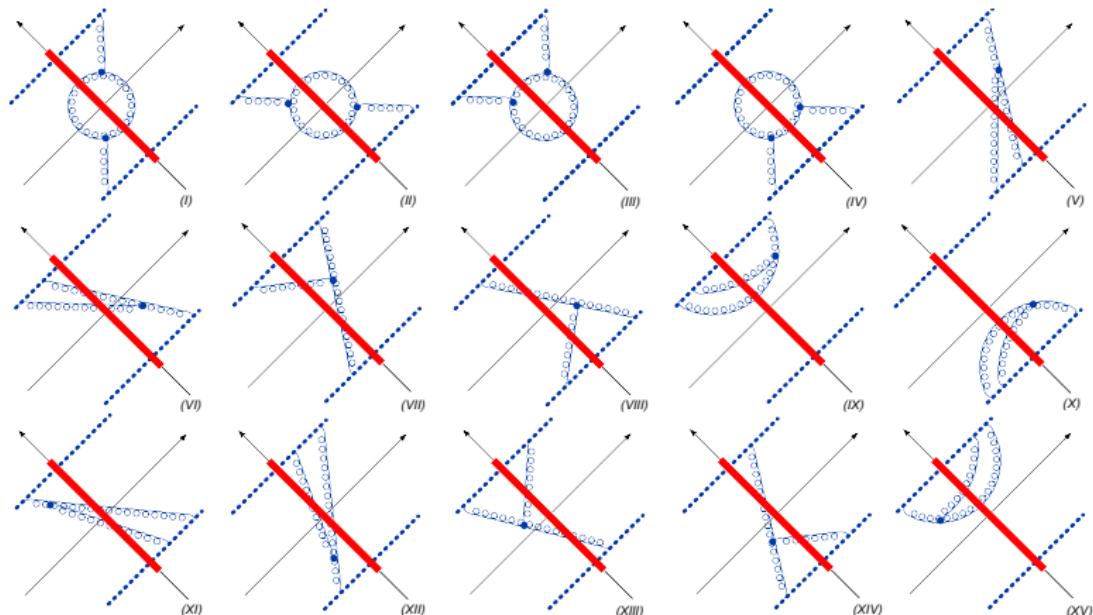
$$\begin{aligned} \frac{d}{d\eta} Tr\{U_x U_y^\dagger\} = & \\ & \int \frac{d^2 z}{2\pi^2} \left( \alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z) \right) [Tr\{U_x U_z^\dagger\} Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] + \\ & \alpha_s^2 \int d^2 z d^2 z' \left( K_4(x,y,z,z') \{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z') \{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\} \right) \end{aligned}$$

$K_{NLO}$  is the next-to-leading order correction to the dipole kernel and  $K_4$  and  $K_6$  are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

- We need to calculate some diagrams analytically (pen and paper).

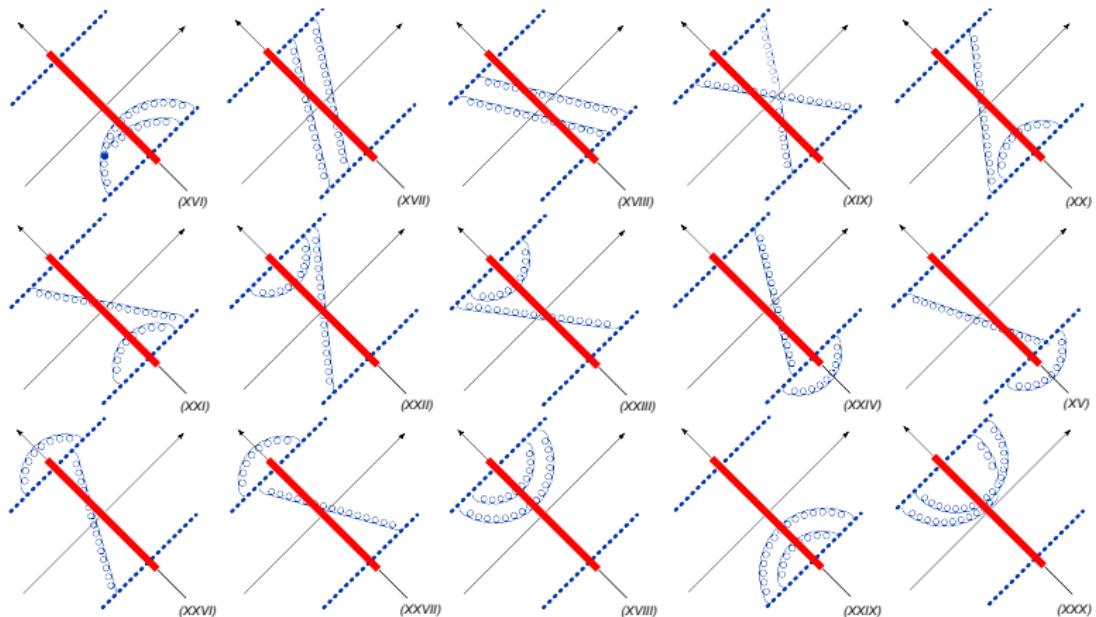
# Diagrams of the NLO gluon contribution

## Diagrams with 2 gluons interaction



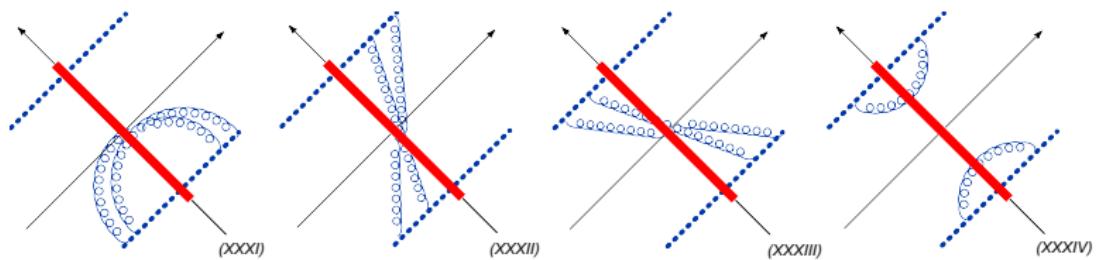
# Diagrams of the NLO gluon contribution

## Diagrams with 2 gluons interaction



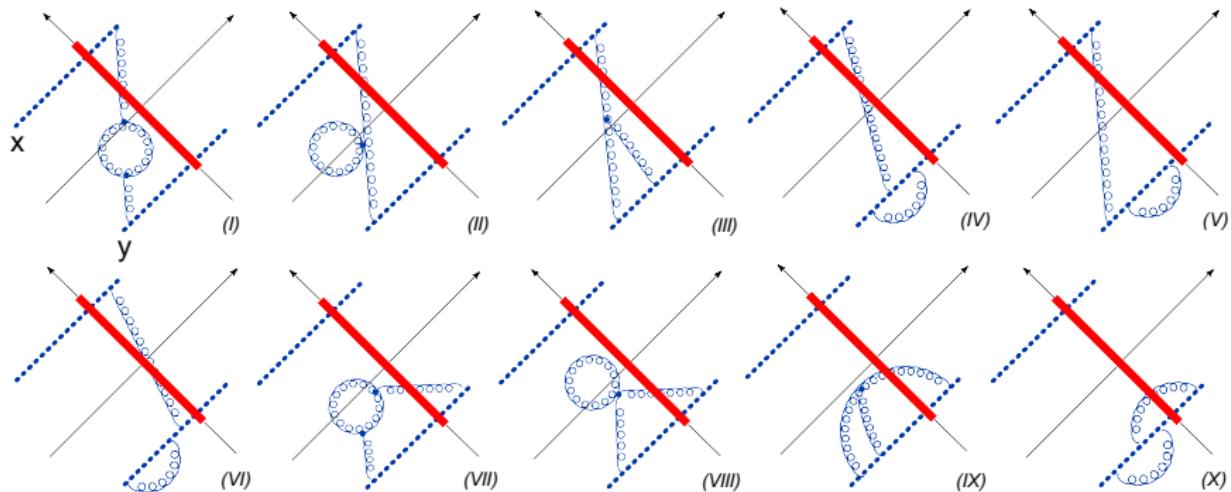
# Diagrams of the NLO gluon contribution

## Diagrams with 2 gluons interaction



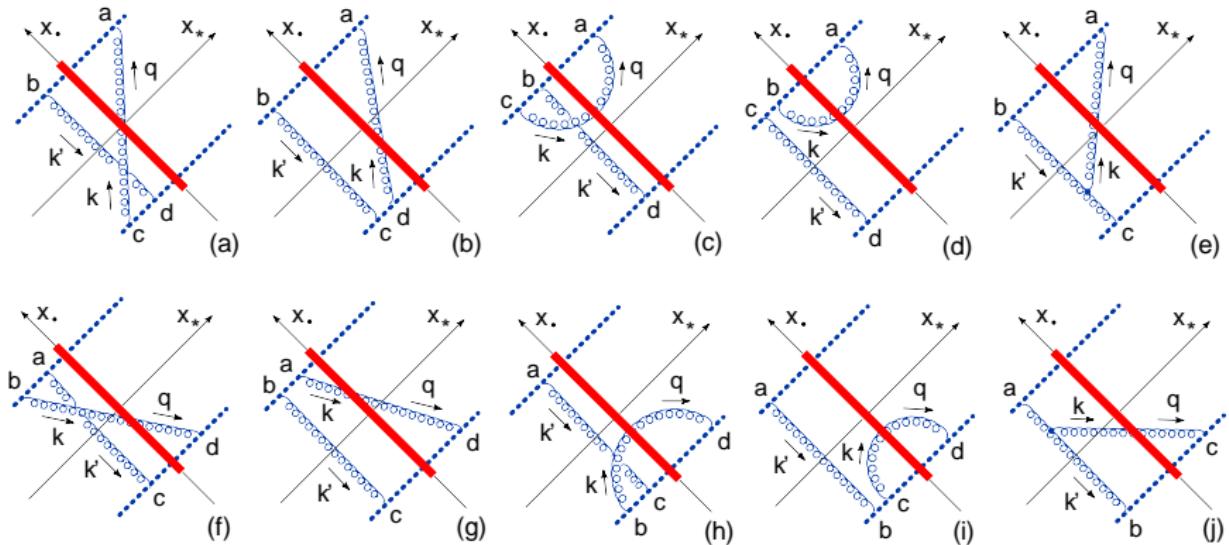
# Diagrams of the NLO gluon contribution

"Running coupling" diagrams



# Diagrams of the NLO gluon contribution

1 → 2 dipole transition diagrams



$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \left( [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 &- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 &+ \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 &\quad -(z' \rightarrow z)] \frac{1}{(z-z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 &+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 &\quad \times \left[ \frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \Big\} \Big)
 \end{aligned}$$

Our result Agrees with NLO BFKL

It respects unitarity

$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \frac{\alpha_s}{2\pi^2} \int d^2 z \left( [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 & \times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 & - \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 & + \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 & -(z' \rightarrow z)] \frac{1}{(z-z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 & + [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 & \times \left. \left[ \frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\}
 \end{aligned}$$

NLO kernel = Running coupling terms + Non-conformal term + Conformal term

# Evolution equation for color dipoles in $\mathcal{N} = 4$

( I. Balitsky and G.A.C. 2009)

$$\begin{aligned} & \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\ & \times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\ & - \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{ad'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'} \end{aligned}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff  $\alpha < \sigma = e^{2\eta}$  in the rapidity of Wilson lines.

# Evolution equation for color dipoles in $\mathcal{N} = 4$

( I. Balitsky and G.A.C. 2009)

$$\begin{aligned} & \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[ \frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\ & \times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\ & - \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{ad'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'} \end{aligned}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff  $\alpha < \sigma = e^{2\eta}$  in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

# Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$ SYM

I. Balitsky and G.A.C (2009)

$$[\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} = \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} + \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

$$\begin{aligned} & \frac{d}{d\eta} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ & - \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[ 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} [(\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta)^{bb'} - (z_4 \rightarrow z_3)] \end{aligned}$$

Now Möbius invariant!

# NLO evolution of composite “conformal” dipoles in QCD

$$\begin{aligned}
 \frac{d}{d\eta} [\text{tr}\{\hat{U}_{z_1} U_{z_2}^\dagger\}]^{\text{conf}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left( [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} \right. \\
 &\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} (b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3}) \right] \\
 &+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_{34}^4} \left\{ \left[ -2 + \frac{z_{14}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4 z_{12}^2 z_{34}^2}{2(z_{14}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{14}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right] \right. \\
 &\times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger \hat{U}_{z_4} \hat{U}_{z_2}^\dagger \hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} - (z_4 \rightarrow z_3)] \\
 &+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left( 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] \\
 &\times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_4}^\dagger \hat{U}_{z_3} \hat{U}_{z_2}^\dagger \hat{U}_{z_4} \hat{U}_{z_3}^\dagger\} - (z_4 \rightarrow z_3)] \left. \right\}
 \end{aligned}$$

$$b = \frac{11}{3}N_c - \frac{2}{3}n_f$$

I. Balitsky and G.A.C

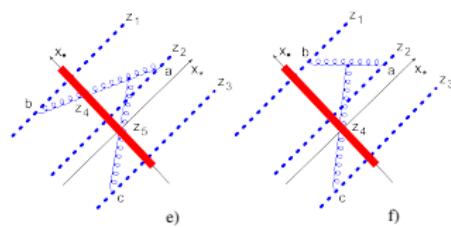
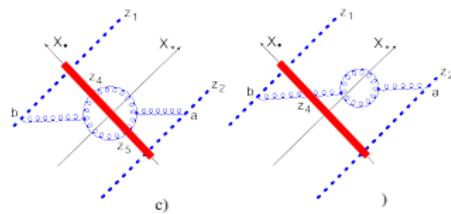
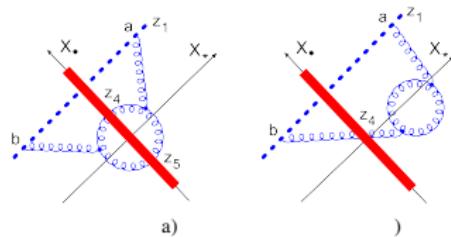
$K_{\text{NLO BK}}$  = Running coupling part + Conformal "non-analytic" (in  $j$ ) part  
 + Conformal analytic ( $\mathcal{N} = 4$ ) part

Linearized  $K_{\text{NLO BK}}$  reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

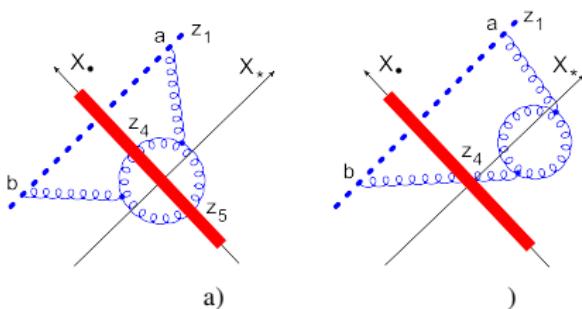
## NLO Balitsky-JIMWLK evolution equation

- Scattering amplitudes for proton-Nucleus and Nucleus-Nucleus collisions are described by matrix elements made of multiple Wilson lines.
- Typical matrix elements in pA and AA collisions is  $\langle \text{tr}\{U_x U_y^\dagger U_w U_z^\dagger\} \rangle$  quadrupole operator.
- To include energy dependence  $\Rightarrow$  need NLO Balitsky-JIMWLK evolution equation.

# NLO Balitsky-JIMWLK evolution equation

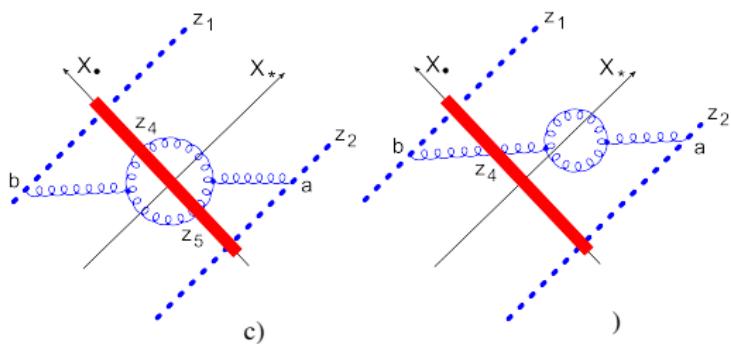


**Sample of diagrams:** a), b) are self-interactions; c), d) are pairwise interactions;  
e), f) are triple interactions.



$$\begin{aligned}
 \frac{d}{d\eta} (U_1)_{ij} = & \frac{\alpha_s^2}{8\pi^4} \int \frac{d^2 z_4 d^2 z_5}{z_{45}^2} \left\{ U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \right. \\
 & \times \left( \left[ 2I_1 - \frac{4}{z_{45}^2} \right] f^{ade} f^{bd'e'} (t^a U_1 t^b)_{ij} + \frac{(z_{14}, z_{15})}{z_{14}^2 z_{15}^2} \ln \frac{z_{14}^2}{z_{15}^2} [if^{ad'e'} (\{t^d, t^e\} U_1 t^a)_{ij} - if^{ade} (t^a U_1 \{t^{d'}, t^{e'}\})_{ij}] \right) \\
 & + \frac{\alpha_s^2 N_c}{4\pi^3} \int d^2 z_4 (U_4^{ab} - U_1^{ab}) (t^a U_1 t^b)_{ij} \times \frac{1}{z_{14}^2} \left[ \frac{11}{3} \ln z_{14}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right]
 \end{aligned}$$

$$I_1 \equiv I(z_1, z_4, z_5) = \frac{\ln z_{14}^2 / z_{15}^2}{z_{14}^2 - z_{15}^2} \left[ \frac{z_{14}^2 + z_{15}^2}{z_{45}^2} - \frac{(z_{14}, z_{15})}{z_{14}^2} - \frac{(z_{14}, z_{15})}{z_{15}^2} - 2 \right]$$



$$\frac{d}{d\eta} (U_1)_{ij} (U_2^\dagger)_{kl} = \frac{\alpha_s^2}{8\pi^4} \int d^2 z_4 d^2 z_5 (\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3) + \frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z_4 (\mathcal{B}_1 + N_c \mathcal{B}_2)$$

$$\begin{aligned}\mathcal{A}_1 = & \left[ (t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl} \right] \\ & \times \left[ f^{ade} f^{bd'e'} U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \left( -K - \frac{4}{z_{45}^4} + \frac{I_1}{z_{45}^2} + \frac{I_2}{z_{45}^2} \right) \right]\end{aligned}$$

$K$  is the NLO BK kernel for  $\mathcal{N}=4$  SYM

$$\begin{aligned}\mathcal{A}_2 = & 4(U_4 - U_1)^{dd'} (U_5 - U_2)^{ee'} \\ & \left\{ i \left[ f^{ad'e'} (t^d U_1 t^a)_{ij} (t^e U_2)_{kl} - f^{ade} (t^a U_1 t^{d'})_{ij} (U_2 t^{e'})_{kl} \right] J_{1245} \ln \frac{z_{14}^2}{z_{15}^2} \right. \\ & \left. + i \left[ f^{ad'e'} (t^d U_1)_{ij} (t^e U_2 t^a)_{kl} - f^{ade} (U_1 t^{d'})_{ij} (t^a U_2 t^{e'})_{kl} \right] J_{2154} \ln \frac{z_{24}^2}{z_{25}^2} \right\}\end{aligned}$$

$$J_{1245} \equiv J(z_1, z_2, z_4, z_5) = \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} - 2 \frac{(z_{15}, z_{45})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{25}^2 z_{45}^2} + 2 \frac{(z_{25}, z_{45})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

$$\begin{aligned}
 \mathcal{A}_3 = & 2U_4^{dd'} \left\{ i \left[ f^{ad'e'} (U_1 t^a)_{ij} (t^d t^e U_2)_{kl} - f^{ade} (t^a U_1)_{ij} (U_2 t^{e'} t^{d'})_{kl} \right] \right. \\
 & \times \left[ \mathcal{J}_{1245} \ln \frac{z_{14}^2}{z_{15}^2} + (J_{2145} - J_{2154}) \ln \frac{z_{24}^2}{z_{25}^2} \right] (U_5 - U_2)^{ee'} \\
 & + i \left[ f^{ad'e'} (t^d t^e U_1)_{ij} (U_2 t^a)_{kl} - f^{ade} (U_1 t^{e'} t^{d'})_{ij} (t^a U_2)_{kl} \right] \\
 & \times \left. \left[ \mathcal{J}_{2145} \ln \frac{z_{24}^2}{z_{25}^2} + (J_{1245} - J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2} \right] (U_5 - U_1)^{ee'} \right\}
 \end{aligned}$$

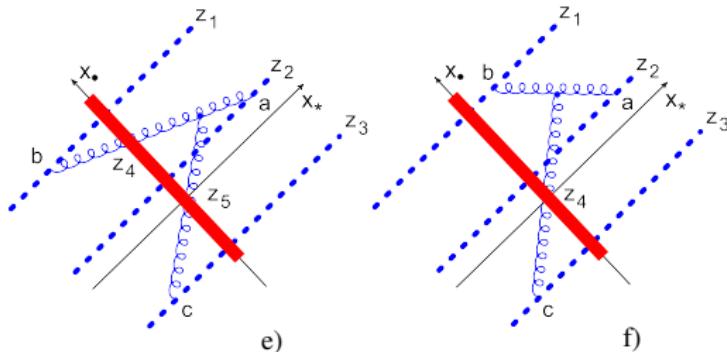
$$\begin{aligned}
 \mathcal{J}_{1245} &\equiv \mathcal{J}(z_1, z_2, z_4, z_5) \\
 &= \frac{(z_{24}, z_{25})}{z_{24}^2 z_{25}^2 z_{45}^2} - \frac{2(z_{24}, z_{45})(z_{15}, z_{25})}{z_{24}^2 z_{25}^2 z_{15}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{24})}{z_{14}^2 z_{24}^2 z_{25}^2 z_{45}^2} - 2 \frac{(z_{14}, z_{24})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{24}^2 z_{25}^2}
 \end{aligned}$$

$$\begin{aligned} \mathcal{B}_1 = & 2 \ln \frac{z_{14}^2}{z_{12}^2} \ln \frac{z_{24}^2}{z_{12}^2} \\ & \times \left\{ (U_4 - U_1)^{ab} i [f^{bde} (t^a U_1 t^d)_{ij} (U_2 t^e)_{kl} + f^{ade} (t^e U_1 t^b)_{ij} (t^d U_2)_{kl}] \left[ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{14}^2} \right] \right. \\ & + (U_4 - U_2)^{ab} i [f^{bde} (U_1 t^e)_{ij} (t^a U_2 t^d)_{kl} + f^{ade} (t^d U_1)_{ij} (t^e U_2 t^b)_{kl}] \left[ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{24}^2} \right] \left. \right\} \end{aligned}$$

$$\begin{aligned} \mathcal{B}_2 = & [2U_4^{ab} - U_1^{ab} - U_2^{ab}] [(t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl}] \\ & \times \left\{ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} \left[ \frac{11}{3} \ln z_{12}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] + \frac{11}{3} \left( \frac{1}{2z_{14}^2} \ln \frac{z_{24}^2}{z_{12}^2} + \frac{1}{2z_{24}^2} \ln \frac{z_{14}^2}{z_{12}^2} \right) \right\} \end{aligned}$$

# Triple interactions

I. Balitsky and G.A.C. (2013) (see also A. Grabovsky (2013))



$$\begin{aligned}\mathcal{J}_{12345} \equiv \mathcal{J}(z_1, z_2, z_3, z_4, z_5) &= -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2} \\ &- \frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{35}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2}\end{aligned}$$

# Triple interactions

I. Balitsky and G.A.C. (2013)

$$\begin{aligned} & \frac{d}{d\eta} (U_1)_{ij} (U_2)_{kl} (U_3)_{mn} \\ &= i \frac{\alpha_s^2}{2\pi^4} \int d^2 z_4 d^2 z_5 \left\{ \begin{aligned} & \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \\ & \times f^{cde} [(t^a U_1)_{ij} (t^b U_2)_{kl} (U_3 t^c)_{mn} (U_4 - U_1)^{ad} (U_5 - U_2)^{be} \\ & - (U_1 t^a)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{da} (U_5 - U_2)^{eb}] \\ & + \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} \\ & \times f^{ade} [(U_1 t^a)_{ij} (t^b U_2)_{kl} (t^c U_3)_{mn} (U_4 - U_3)^{cd} (U_5 - U_2)^{be} \\ & - (t^a U_1)_{ij} \otimes (U_2 t^b)_{kl} (U_3 t^c)_{mn} (U_4^{dc} - U_3^{dc}) (U_5^{eb} - U_2^{eb})] \\ & + \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} \\ & \times f^{bde} [(t^a U_1)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{ad} (U_5 - U_3)^{ce} \\ & - (U_1 t^a)_{ij} (t^b U_2)_{kl} (U_3 t^c)_{mn} (U_4 - U_1)^{da} (U_5 - U_3)^{ec}] \end{aligned} \right\} \end{aligned}$$

# Conclusions

- Dynamics of QCD at high-energy is non-linear.
- Scattering amplitudes at high-energy and high-density QCD are factorized in rapidity space using the high-energy OPE in Wilson lines.
- BK and B-JIMWLK evolution equations include the energy dependence to scattering amplitude at high energy.
- NLO BK and NLO Balitsky hierarchy of evolution equation (NLO B-JIMWLK) has been presented.
  - NLO BK equation: [I. Balitsky and G.A.C. \(2007\)](#)
  - Triple interactions: [A. Grabovsky \(2013\)](#); [I. Balitsky and G.A.C. \(2013\)](#)
  - NLO Balitsky hierarchy (NLO B-JIMWLK): [I. Balitsky and G.A.C. \(2013\)](#)
  - NLO JIMWLK Hamiltonian from triple interaction and NLO BK results: [Kovner and Lublinsky \(2013\)](#); [Kovner, Lublinsky and Yair Mulian \(2014\)](#)