Amplitudes in High-Energy-Factorization via BCFW recursion

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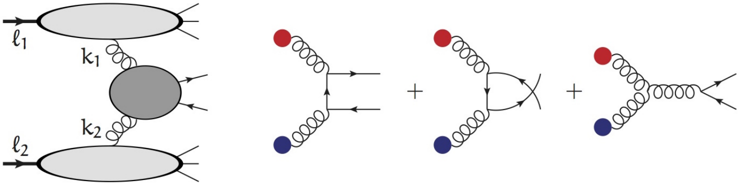
High-Energy-Factorization

High-Energy-Factorization *(Catani,Ciafaloni,Hautmann, 1991 / Collins,Ellis, 1991)*



$$\sigma_{h_1,h_2\to q\bar{q}} = \int d^2k_{1\perp}\, d^2k_{2\perp}\, \frac{dx_1}{x_1}\, \frac{dx_2}{x_2}\, f_g(x_1,k_{1\perp})\, f_g(x_2,k_{2\perp})\, \hat{\sigma}_{gg}\left(\frac{m^2}{x_1x_2s}, \frac{k_{1\perp}}{m}, \frac{k_{2\perp}}{m}\right)$$

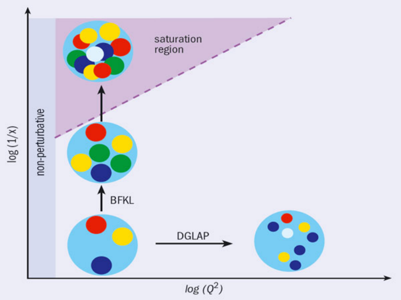
where the f_g's are the gluon densities, obeying BFKL, BK, CCFM evolution equations.

Non neglectable transverse momentum is associated to small-x physics.

Momentum parameterization:

$$k_1^\mu = x_1\, p_1^\mu + k_{1\perp}^\mu \quad , \quad k_2^\mu = x_2\, p_2^\mu + k_{2\perp}^\mu \quad \text{for} \quad p_i \cdot k_i = 0 \quad k_i^2 = -k_{i\perp}^2 \quad i = 1,2$$

To be applied in the regime: $s >> M^2 \sim k_\perp^2$



Possible applications: $\begin{cases} \text{production of forward dijets initiated with gluons : } gg^* \to gg \\ \text{production of forward dijets initiated with quarks : } q\bar{q}^* \to gg \\ \text{production of a } Z \text{ boson by a quark-antiquark pair : } q\bar{q}^* \to Z \end{cases}$

Problem: general partonic processes must be described by **gauge invariant amplitudes**
$\Rightarrow$ ordinary Feynman rules are not enough !

Is there a general method to compute such gauge-invariant amplitudes ?

Just for the formalism: Weyl spinors

High energy limit $\Rightarrow$ massless particles $\Rightarrow$ Weyl basis for spinors.
If $p^2 = 0$, it can be cast in the Pauli matrices language,

$$p \cong p^\mu \, \sigma_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + i\, p^2 \\ -p^1 - i\, p^2 & p^0 + p^3 \end{pmatrix} = |p]\langle p|$$

$$|p] \quad = \quad \begin{pmatrix} L(p) \\ \mathbf{0} \end{pmatrix} \qquad L(p) = \frac{1}{\sqrt{|p^0 + p^3|}} \begin{pmatrix} -p^1 + i\, p^2 \\ p^0 + p^3 \end{pmatrix}$$

$$|p\rangle \quad = \quad \begin{pmatrix} \mathbf{0} \\ R(p) \end{pmatrix} \qquad R(p) = \frac{\sqrt{|p^0 + p^3|}}{p^0 + p^3} \begin{pmatrix} p^0 + p^3 \\ p^1 + i\, p^2 \end{pmatrix}$$

and the charge-conjugated spinors

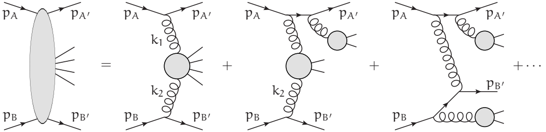
$$[p| = \left((\mathcal{E} L(p))^T \,, \mathbf{0} \right) \qquad \langle p| = \left(\mathbf{0} \; (\mathcal{E}^T R(p))^T \right) \qquad \text{where} \quad \mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Prescription for off-shell gluons

ONE IDEA:

on-shell amplitudes are gauge invariant, so off-shell gauge-invariant amplitudes could be got by embedding them into on-shell processes...

...first result...: 1) For off-shell gluons: represent g^* as coming from a $\bar{q}qg$ vertex, with the quarks taken to be on-shell



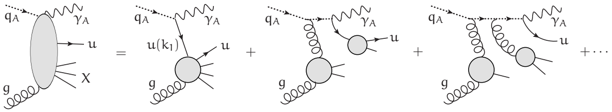
- embed the scattering of the off-shell gluons in the scattering of two quark pairs carrying momenta $p_A^\mu = k_1^\mu$, $p_B^\mu = k_2^\mu$, $p_{A'}^\mu = 0$, $p_{B'}^\mu = 0$
- Assign the spinors $|p_1\rangle, |p_1]$ to the A-quark and the propagator $\frac{i\,\not{p}_1}{p_1 \cdot k}$ instead of $\frac{i\not{k}}{k^2}$ to the propagators of the A-quark carrying momentum k; the same goes for the B-quark line.
- multiply the amplitude by $g_s^{-1} x_1 \sqrt{-2\,k_{1\perp}^2} \times g_s^{-1} x_2 \sqrt{-2\,k_{2\perp}^2}$.
- ordinary Feynman rules must be used everywhere else and the procedure holds for any number of off-shell gluons (including 1).

K. Kutak, P. Kotko, A. van Hameren, JHEP 1301 (2013) 078

Prescription for off-shell quarks

... and second result:

2) for off-shell quarks: represent q^* as coming from a $\gamma\bar{q}q$ vertex, with a 0 momentum and $\bar{q}$ on shell (and vice-versa)



- embed the scattering of the quark with whatever set of particles in the scattering of an auxiliary quark-photon pair, q_A and γ_A carrying momenta $p^\mu_{q_A} = k^\mu_1$, $p^\mu_{\gamma_A} = 0$
- Let q_A-propagators of momentum k be $\frac{i\,\not{p}_1}{p_1 \cdot k}$ and assign the spinors $|p_1\rangle, |p_1]$ to the A-quark.
- Assign the polarization vectors $\epsilon^\mu_+ = \frac{\langle q|\gamma^\mu|p_1]}{\sqrt{2}\langle p_1 q\rangle}$, $\epsilon^\mu_- = \frac{\langle p_1|\gamma^\mu|q]}{\sqrt{2}[p_1 q]}$ to the auxiliary photon, with q a light-like auxiliary momentum.
- Multiply the amplitude by $x_1\sqrt{-k^2_{1\perp}/2}$ and use ordinary Feynman rules everywhere else.

K. Kutak, T. Salwa, A. van Hameren, Phys.Lett. B727 (2013) 226-233

Prescription for off-shell gluons: derivation 1

Auxiliary vectors (complex in general):
$$\begin{cases} p_3^\mu = \frac{1}{2}\langle p_2|\gamma^\mu|p_1] \\ p_4^\mu = \frac{1}{2}\langle p_1|\gamma^\mu|p_2] \\ p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0 \\ p_{1,2}\cdot p_{3,4} = 0\,, \quad p_1\cdot p_2 = -p_3\cdot p_4 \end{cases}$$

Auxiliary momenta:
$$\begin{cases} p_A^\mu = (\Lambda + x_1)p_1^\mu - \frac{p_4\cdot k_{1\perp}}{p1\cdot p_2}p_3^\mu\,, \quad p_{A'}^\mu = \Lambda p_1^\mu + \frac{p_3\cdot k_{1\perp}}{p1\cdot p_2}p_4^\mu \\ p_B^\mu = (\Lambda + x_2)p_2^\mu - \frac{p_3\cdot k_{2\perp}}{p1\cdot p_2}p_4^\mu\,, \quad p_{B'}^\mu = \Lambda p_2^\mu + \frac{p_4\cdot k_{2\perp}}{p1\cdot p_2}p_3^\mu \end{cases}$$

For any Λ:
$$\begin{cases} p_A^\mu - p_{A'}^\mu = x_1\, p_1^\mu + k_{1\perp}^\mu \\ p_B^\mu - p_{B'}^\mu = x_2\, p_2^\mu + k_{2\perp}^\mu \\ p_A^2 = p_{A'}^2 = p_B^2 = p_{B'}^2 = 0 \end{cases}$$

Prescription for off-shell gluons: derivation 2

Momentum flowing through a propagator of an auxiliary quark line:

$$k^\mu = (\Lambda + x_k) p_1^\mu + y_k\, p_2^\mu + k_\perp$$

Final step: remove complex components taking the $\Lambda \to \infty$ limit.

$$\frac{\not{k}}{k^2} = \frac{(\Lambda + x_k)\not{p}_1 + y_k\,\not{p}_2 + \not{k}}{2(\Lambda + x_k) y_k\, p_1 \cdot p_2 + k_\perp^2} \xrightarrow{\Lambda \to \infty} \frac{\not{p}_1}{2\, y_k\, p_1 \cdot p_2} = \frac{\not{p}_1}{2 p_1 \cdot k}$$

...and the factor $x_1 \sqrt{-k_\perp^2/2}$ is to match the collinear limit.

In agreement with other approaches (e.g. Lipatov's effective action)

One left issue: huge slowness

The diagrammatic approach is too slow to allow for the computation of amplitudes containing more than 4 particles in a reasonable time.

Computing scattering amplitudes in Yang-Mills theories via ordinary Feynman diagrams: soon overwhelming !

Number of Feynman diagrams at tree level on-shell:

# of gluons	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

And there are even more with the proposed method for amplitudes with off-shell particles due to the gauge-restoring terms.

A method to efficiently compute helicity amplitudes: **BCFW recursion relation**

Britto, Cachazo, Feng, Nucl.Phys. B715 (2005) 499-522
Britto, Cachazo, Feng, Witten, Phys.Rev.Lett. 94 (2005) 181602

BCFW recursion relation

Two very simple ideas for tree level amplitudes:

1. Cauchy's residue theorem: if the amplitude is formally treated as a function of a complex variable z and if it is rational and vanishes for $z \to \infty$, then the integral extended to an infinite contour enclosing all poles vanishes

$$\lim_{z\to\infty} \mathcal{A}(z) = 0 \Rightarrow \frac{1}{2\pi i} \oint dz \, \frac{\mathcal{A}(z)}{z} = 0$$

   implying that the value at $z = 0$ (physical amplitude) can be determined as a sum of the residues at the poles:

$$\mathcal{A}(0) = -\sum_i \frac{\lim_{z\to z_i}[(z - z_i)\, f(z)]}{z_i}$$

   where z_i is the location of the i-th pole

2. Unitarity: Poles in Yang-Mills tree level amplitudes can only be due to gluon propagators dividing the n-point amplitude into two on-shell sub-amplitudes with $k+1$ and $n-k+1$ gluons $\Rightarrow$ it is all about finding the proper way to "complexify" an amplitude.

To properly "complexify" $\mathcal{A}$: for helicities $(h_1, h_n) = (-,+)$ (no loss of generality...)

$$\begin{array}{ccl} |1] & \to & |\hat{1}] \equiv |1] - z\,|n] \Rightarrow p_1 \to \hat{p}_1 = |1]\langle 1| - z|1]\langle n| \\ |n\rangle & \to & |\hat{n}\rangle \equiv |n\rangle + z|1\rangle \Rightarrow p_n \to \hat{p}_n = |n]\langle n| + z|1]\langle n| \end{array}$$

With such a choice

- On-shellness, gauge invariance and momentum conservation preserved throughout.
- the most serious issue is the behaviour for $z \to \infty$, but either a result derived with twistor methods (*Cachazo,Svrcek and Witten JHEP 0409 (2004) 006*) or a smart choice of reference lines always allow to overcome the problem, so that $\lim_{z\to\infty} \mathcal{A}(z) = 0$ holds

BCFW applies to* color-ordered partial amplitudes*, for which the kinematics and gauge structure are factorised like

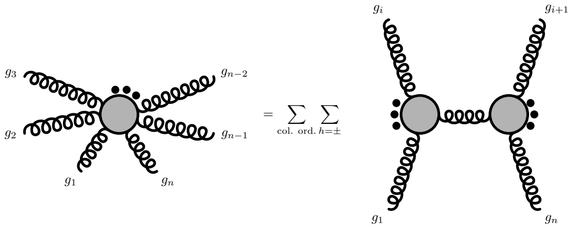
$$\mathcal{M}_n = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T_{\sigma(1)} \dots T_{\sigma(n)})\, \mathcal{A}(g_{\sigma(1)}, \dots, g_{\sigma(n)})$$

The result is an amazingly simple recursive relation:

any tree-level color-ordered amplitude is the sum of residues of the poles it develops when it is made dependent on a complex variable as above.
Such residues are simply products of color-ordered lower-point amplitudes evaluated at the pole times an intermediate propagator. Shifted particles are always on opposite sides of the propagator.

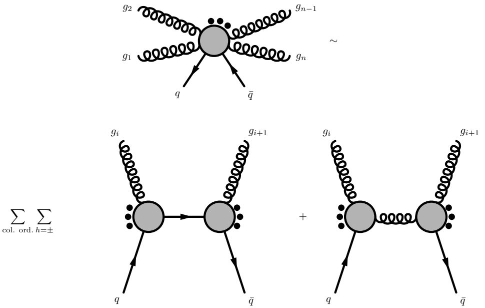
$$\mathcal{A}(g_1,\ldots,g_n) = \sum_{i=2}^{n-2}\sum_{h=+,-}\mathcal{A}(g_1,\ldots,g_i,\hat{P}^h)\,\frac{1}{(p_1+\cdots+p_i)^2}\,\mathcal{A}(-\hat{P}^{-h},g_{i+1},\ldots,g_n)$$

$z_i = \dfrac{(p_1+\cdots+p_i)^2}{[1|p_1+\cdots+p_i|n\rangle}$ location of the pole corresponding for the "i-th" partition



The inclusion of fermions and MHV amplitudes

The BCFW recursion was promptly extended to Yang-Mills theories with fermions:
M. Luo, C. Wen, JHEP 0503 (2005) 004



A couple of MHV amplitudes:

$$\mathcal{A}(g_1^+, g_2^+, \ldots, g_i^-, \ldots, g_j^-, \ldots, g_n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n-1\; n\rangle\langle n1\rangle}$$

$$\mathcal{A}(q^-, g_1^-, g_2^+, \ldots, g_n^+, \bar{q}^+) = \frac{\langle q1\rangle^3\langle \bar{q}1\rangle}{\langle \bar{q}q\rangle\langle q1\rangle\langle 12\rangle \ldots \langle n\bar{q}\rangle}$$

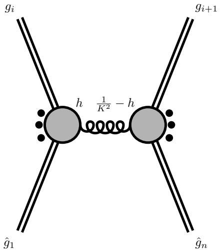
It is natural to ask whether something like a BCFW recursion relation exists with off-shell particles. For off shell, gluons, the answer was first found in *A. van Hameren, JHEP 1407 (2014) 138*

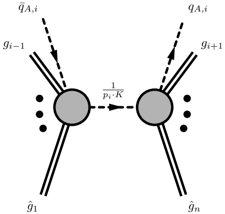
$$\mathcal{A}(0) = \sum_{s=g,f}\left(\sum_{p}\sum_{h=+,-}\mathrm{A}^{s}_{p,h} + \sum_{i}\mathrm{B}^{s}_{i} + \mathrm{C}^{s} + \mathrm{D}^{s}\right) ,$$

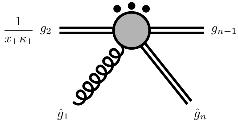
- $\mathrm{A}^{g/f}_{p,h}$ are due to the poles which appear in the original BCFW recursion for on-shell amplitudes. The pole appears because one of the intermediate virtual gluon, whose shifted momentum squared $K^2(z)$ goes on-shell.
- $\mathrm{B}^{g/f}_{i}$ are due to the poles appearing in the propagator of auxiliary eikonal quarks. This means $p_i \cdot \hat{K}(z) = 0$ for $z = -\frac{2\,p_i\cdot K}{2\,p_i\cdot e}$. $\hat{K}$ is the momentum flowing through the eikonal propagator.
- $\mathrm{C}^{g/f}$ and $\mathrm{D}^{g/f}$ show up us the first/last shifted particle is off-shell and their external propagator develops a pole.
  The external propagator for off-shell particles is necessary to ensure

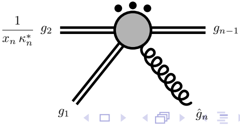
$$\lim_{z\to\infty}\mathcal{A}(z) = 0$$

Classification of poles in the gluon case

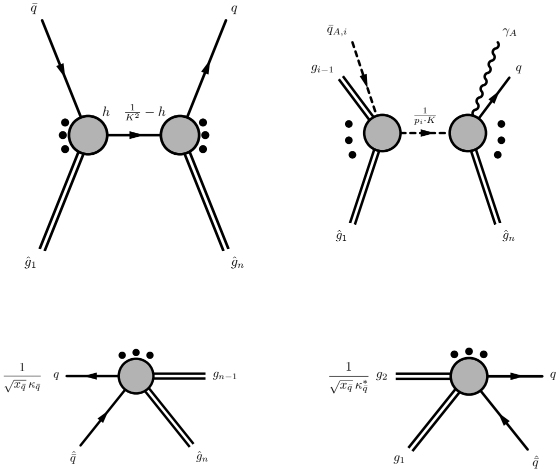








Classification of poles in the fermion case



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General outline of the results

- It is necessary to understand which shifts are legitimate in the off-shell case, i.e. for which choices $\lim_{z\to\infty}\mathcal{A}(z)=0$. We provide a full classification of the possibilities.
- It turns out that amplitudes which are MHV in the on-shell case (2 of the partons have different helicity sign w.r.t. all the others) preserve a similar structure in the off-shell case.
- 5-point amplitudes exhibit some non-MHV structures, which have been calculated for the first time
- Numerical cross-checks are always successful. They were performed cross checked with a program implementing Berends-Giele recursion relation, *A. van Hameren, M. Bury, arXiv:1503.08612*

So 4-point amplitudes are always MHV , juts as in the on-shell case.
On the other hand 5-point amplitudes are not always MHV , differently both from the on-shell case and the purely gluonic case with some of the gluons off the mass-shell

Explicit results are presented and discussed thoroughly in

A. van Hameren, M.S. http://arxiv.org/abs/1504.00315.

MHV amplitudes

Transverse momentum parameterization:
$$\begin{cases} k_{T\,i}^{\mu} = -\frac{\kappa_i}{2}\,\frac{\langle p_i|\gamma^\mu|q]}{[p_i q]} - \frac{\kappa_i^*}{2}\,\frac{\langle q|\gamma^\mu|p_i]}{\langle q p_i\rangle} \\ \kappa_i \equiv \frac{\langle q|\not{k}_i|p_i]}{\langle q p_i\rangle} \qquad \kappa_i^* \equiv \frac{\langle p_i|\not{k}_i|q]}{[p_i q]} \\ q^2 = 0 \quad \text{auxiliary momentum} \end{cases}$$

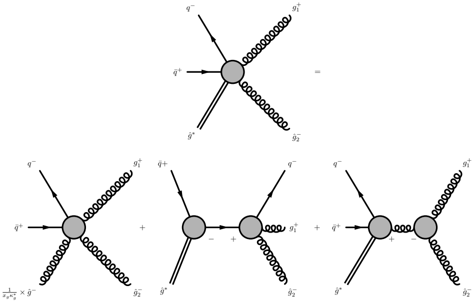
Subleading contribution: it is zero in the on-shell case !

$$\mathcal{A}(g_1^+, g_2^+, \ldots, g_{n-1}^+, \bar{q}, q, g_n^+) = \frac{\langle \bar{q} q\rangle^3}{\langle 12\rangle\langle 23\rangle \ldots \langle \bar{q} q\rangle\langle q n\rangle\langle n1\rangle}$$

Structure of MHV amplitudes

$$\begin{aligned} \mathcal{A}(g_1^+, g_2^+, \ldots, g_{n-1}^+, \bar{q}^*, q^+, g_n^-) &= \frac{1}{\kappa_{\bar{q}}^*}\,\frac{\langle \bar{q} n\rangle^3\langle q n\rangle}{\langle 12\rangle\langle 23\rangle \ldots \langle \bar{q} q\rangle\langle q n\rangle\langle n1\rangle} \\ \mathcal{A}(g^*, \bar{q}^+, q^-, g_1^+, g_2^+, \ldots, g_n^+) &= \frac{1}{\kappa_g^*}\,\frac{\langle g q\rangle^3\langle g \bar{q}\rangle}{\langle g \bar{q}\rangle\langle \bar{q} q\rangle \ldots \langle n-1|n\rangle\langle n g\rangle} \end{aligned}$$

But not everything is so smooth...



$$
\begin{aligned}
\mathcal{A}(g^*, \bar{q}^+, q^-, g_1^+, g_2^-) \quad = \quad & \frac{1}{\kappa_g^*} \frac{[\bar{q}1]^3 \langle 2g \rangle^4}{[\bar{q}q]\langle g|\not{p}_2 + \not{k}_g|1]\langle 2|\not{k}_g \, (\not{k}_g + \not{p}_2)\,|g]\langle 2|\not{k}_g|\bar{q}]} \\
& + \frac{1}{\kappa_g} \frac{1}{(k_g + p_{\bar{q}})^2} \frac{[g\bar{q}]^2 \langle 2q \rangle^3 \langle 2|\not{k}_g + \not{p}_{\bar{q}}|g]}{\langle 1q \rangle \langle 12 \rangle \left\{ (k_g + p_{\bar{q}})^2 [\bar{q}g]\langle 2q \rangle - \langle 2|\not{k}_g + \not{p}_{\bar{q}}|g]\langle q|\not{k}_g|\bar{q}] \right\}} \\
& + \frac{\langle gq \rangle^3 [g1]^4}{\langle \bar{q}q \rangle [12][g2]\langle q|\not{p}_1 + \not{p}_2|g]\langle g|\not{p}_1 + \not{p}_2|g]\langle g|\not{k}_g + \not{p}_2|1]}
\end{aligned}
$$

Conclusions and perspectives

- High-energy factorisation requires gauge invariant scattering amplitudes with off-shell partons.
- "Embedding tricks" to provide such amplitudes have been devised recently: they work, but suffer from the same kind of computational slow-down when it comes to amplitudes with many (≥ 5) particles.
- the BCFW construction was extended to Yang Mills with fermions with off-shell particles. This implies identifying a new set of poles in the auxiliary complex variable. A complete set of 5 point amplitudes with 1 off-shell parton has been obtained...**and cross-checked numerically**
- It is possible, in the same way, to obtain the scattering amplitudes with more off-shel partons...just some more work.
- Applications of these results to Multi Parton Scattering processes are being studied right now...more on this topic in the next few months.
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