Initial conditions for the double parton distribution functions

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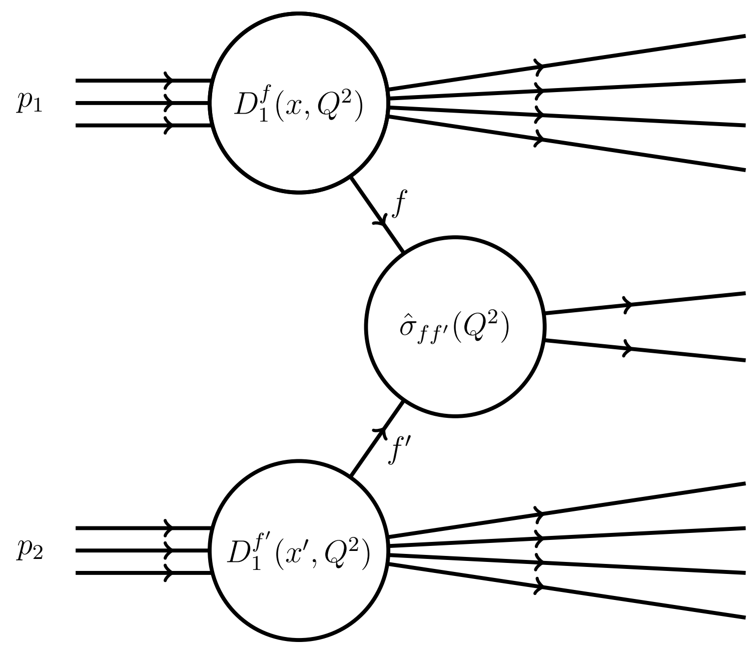
DIS 2015 Workshop, Dallas, Texas, April 29 2015

Outline

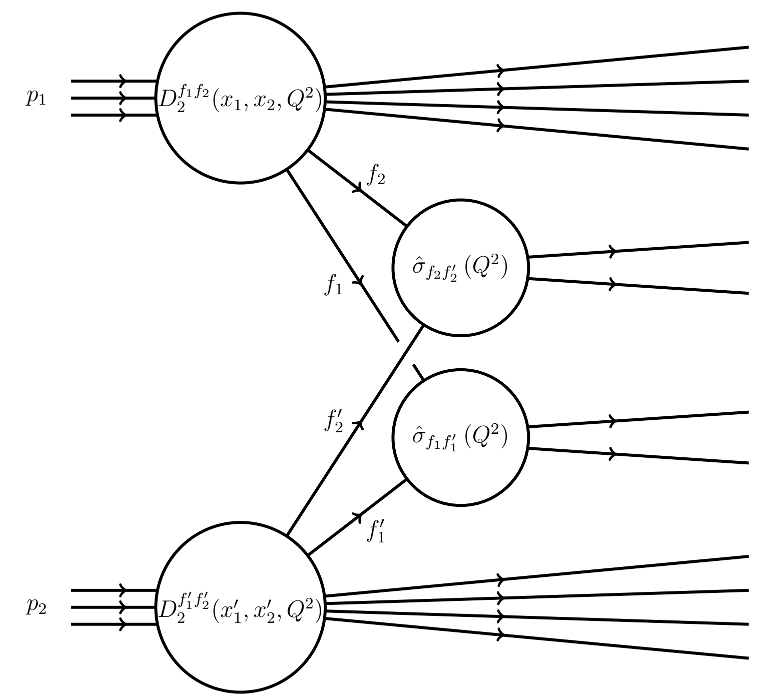
- Motivation
- Evolution equations
- Sum rules
- Initial conditions
- Examples for the single channel: gluons
- Summary and outlook

Single and double PDFs

<u>Single parton scattering: one hard process</u>



<u>Double parton scattering: two hard processes</u>



Single PDF: $D_1^f(x, Q^2)$

Double PDF: $D_2^{f_1, f_2}(x_1, x_2, Q_1^2, Q_2^2; q_T)$

Two types of partons: f_1, f_2 (gluons, quark, antiquarks)

Two momentum fractions: x_1, x_2 $x_1 + x_2 \leq 1$

Two hard scales: $Q_1, Q_2 \gg \Lambda_{QCD}$

Relative transverse momentum: q_T

Evolution of PDFs

Note: in the following we restrict ourselves to the case of equal scales and zero transverse momentum.

$q_T = 0$ $\qquad$ $Q_1 = Q_2 = Q$

Single PDFs evolve through DGLAP equations

$$D_1^f(x, Q_0) \to D_1^f(x, Q)$$

Double PDFs also evolve through DGLAP equations

$$D_2^{f_1 f_2}(x_1, x_2, Q_0) \to D_2^{f_1 f_2}(x_1, x_2, Q)$$

Evolution equations for single PDFs

<u>DGLAP evolution equation for single PDF:</u>

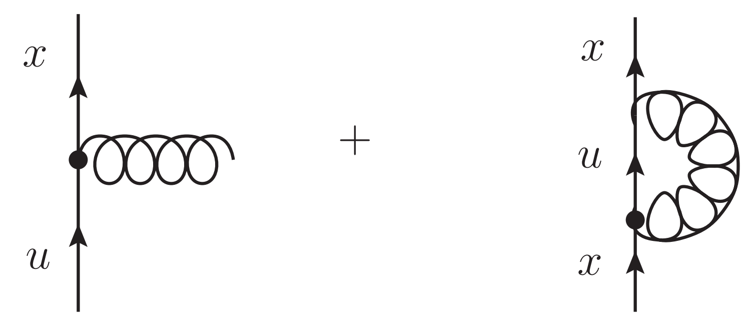
Evolution variable:

$$t = \ln Q^2/Q_0^2$$

$$\boxed{\partial_t D_f(x,t) = \sum_{f'} \int_0^1 du \, \mathcal{K}_{f f'}(x,u,t) \, D_{f'}(u,t)}$$

Real and virtual parts of the kernel:

$$\mathcal{K}_{f f'}(x,u,t) = \mathcal{K}^R_{f f'}(x,u,t) - \delta(u-x) \, \delta_{f f'} \, \mathcal{K}^V_f(x,t)$$



Real emission kernel:

$$\mathcal{K}^R_{f f'}(x,u,t) = \frac{1}{u} P_{f f'}(\frac{x}{u},t) \, \theta(u-x)$$

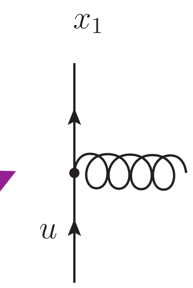
Splitting functions:

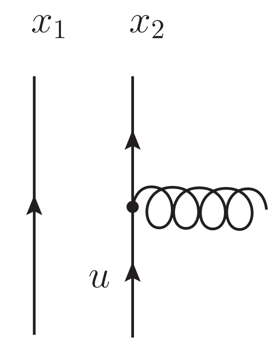
$$P_{f f'}(z,t) = \frac{\alpha_s(t)}{2\pi} P^{(0)}_{f f'}(z) + \frac{\alpha_s^2(t)}{(2\pi)^2} P^{(1)}_{f f'}(z) + \ldots .$$

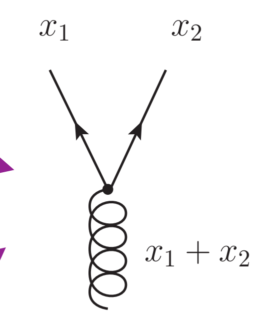
Evolution equations for double PDFs

DGLAP evolution equation for double PDF:

$$
\begin{aligned}
\partial_t \, D_{f_1 f_2}(x_1, x_2, t) \quad &= \sum_{f'} \int_0^{1-x_2} du \, \mathcal{K}_{f_1 f'}(x_1, u, t) \, D_{f' f_2}(u, x_2, t) \\
&+ \sum_{f'} \int_0^{1-x_1} du \, \mathcal{K}_{f_2 f'}(x_2, u, t) \, D_{f_1 f'}(x_1, u, t) \\
&+ \sum_{f'} \mathcal{K}^R_{f' \to f_1 f_2}(x_1, x_2, t) \, D_{f'}(x_1 + x_2, t)
\end{aligned}
$$







Konishi, Ukawa, Veneziano; Snigirev, Zinovev, Shelest

Inhomogeneous term
Splitting term of one parton into two:

$$
\mathcal{K}^R_{f' \to f_1 f_2}(x_1, x_2, t) = \frac{\alpha_s(t)}{2\pi} \frac{1}{x_1 + x_2} P^{(0)}_{f' f_1}\left(\frac{x_1}{x_1 + x_2}\right)
$$

Evolution equation for double PDFs is coupled with single PDFs.

Need to be solved together with suitable initial conditions.

Sum rules for single and double PDFs

Momentum sum rule for single PDFs

$$\sum_f \int_0^1 dx\, x D_f(x,t) = 1$$

Quark number sum rule for single PDFs

$$\int_0^1 dx\, \{D_{q_i}(x,t) - D_{\bar{q}_i}(x,t)\} = N_i$$

Momentum sum rule for double PDFs

$$\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 \frac{D_{f_1 f_2}(x_1,x_2,t)}{D_{f_2}(x_2,t)} = 1 - x_2$$

Conditional probability to find the parton f_1 with the momentum fraction x_1 while keeping fixed the second parton f_2 with momentum x_2 .

Valence quark number sum rule for double PDFs

$$\int_0^{1-x_2} dx_1 \{D_{q_i f_2}(x_1,x_2,t) - D_{\bar{q}_i f_2}(x_1,x_2,t)\}$$
$$= \begin{cases} N_i\, D_{f_2}(x_2,t) & \text{for } f_2 \neq q_i, \bar{q}_i \\ (N_i - 1)\, D_{f_2}(x_2,t) & \text{for } f_2 = q_i \\ (N_i + 1)\, D_{f_2}(x_2,t) & \text{for } f_2 = \bar{q}_i \end{cases}$$

If sum rules hold for initial conditions they will hold for higher scales after the evolution.

How to consistently impose the initial conditions for sPDF and dPDF with sum rules?

Initial conditions: Dirichlet distribution

Consider Beta distribution and gluons only (for now)

$$\boxed{D(x) = N_1\, x^{-\alpha}\,(1-x)^{\beta}}$$

Mellin transform:

$$\tilde{D}(n) = \int_0^1 dx x^{n-1} D(x)$$

Momentum sum rule in Mellin space:

$$\tilde{D}(2) = 1$$

$$\tilde{D}(n) \;=\; \frac{1}{B(2-\alpha, 1+\beta)} \int_0^1 dx\, x^{n-1} x^{-\alpha} (1-x)^{\beta} = \frac{B(n-\alpha, \beta+1)}{B(2-\alpha, \beta+1)}$$

Take the ansatz for double distribution in the form of the Dirichlet distribution:

$$\boxed{D(x_1, x_2) = N_2\, x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1 - x_1 - x_2)^{\tilde{\beta}}}$$

Double Mellin transform:

$$\tilde{D}(n_1, n_2) = \int_0^1 dx_1 x_1^{n_1-1} \int_0^1 dx_2 x_2^{n_2-1} D(x_1, x_2) \longrightarrow \tilde{D}(n_1, n_2) = N_2 \frac{\Gamma(n_1 - \tilde{\alpha})\Gamma(n_2 - \tilde{\alpha})\Gamma(1+\tilde{\beta})}{\Gamma(n_1 + n_2 + 1 + \tilde{\beta} - 2\tilde{\alpha})}$$

Initial conditions: relating the parameters

The momentum sum rule for dPDFs in Mellin space

LHS: Double PDFs in Mellin space $\rightarrow$

$$\boxed{\begin{aligned}\tilde{D}(n_1,2) &= \tilde{D}(n_1) - \tilde{D}(n_1+1)\\ \tilde{D}(2,n_2) &= \tilde{D}(n_2) - \tilde{D}(n_2+1)\end{aligned}}$$

$\leftarrow$ RHS: Single PDFs in Mellin space

RHS:
$$\tilde{D}(n_1) - \tilde{D}(n_1+1) = \frac{1}{B(2-\alpha,\beta+1)}\big(B(n_1-\alpha,\beta+1) - B(n_1+1-\alpha,\beta+1)\big) \quad = \frac{1}{B(2-\alpha,\beta+1)}\frac{\Gamma(n_1-\alpha)\Gamma(2+\beta)}{\Gamma(2+\beta+n_1-\alpha)}$$

Where the following property of Beta function was used:

$$B(a,b) = B(a+1,b) + B(a,b+1)$$

LHS:

$$\tilde{D}(n_1,2) = N_2\frac{\Gamma(n_1-\tilde{\alpha})\Gamma(2-\tilde{\alpha})\Gamma(1+\tilde{\beta})}{\Gamma(n_1+3+\tilde{\beta}-2\tilde{\alpha})}$$

Comparing the functional form of both sides we see that the equality can be satisfied if

$$\boxed{\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1}$$

and

$$N_2 = \frac{1}{B(2-\alpha,\alpha+\beta)B(2-\alpha,\beta+1)}$$

Initial conditions

If the single distribution is given by a Beta distribution

$$\boxed{D(x) = N_1\, x^{-\alpha}\, (1-x)^{\beta}}$$

There is a unique solution in terms of the Dirichlet distribution for the double parton density:

$$\boxed{D(x_1, x_2) = N_2\, x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1 - x_1 - x_2)^{\tilde{\beta}}}$$

With powers of the dPDF being related to the powers of sPDF

$$\boxed{\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1}$$

Normalization for dPDF in this particular case is uniquely determined.

Small x powers for single and double PDFs are the same.

The large x power of the correlating factor in dPDF is related to the sum of large and small x powers of the single distribution.

Initial conditions: quarks and gluons

Momentum sum rule with quarks:

$$\sum_{f_1} \tilde{D}_{f_1 f_2}(2, n_2) \;=\; \tilde{D}_{f_2}(n_2) - \tilde{D}_{f_2}(n_2+1)$$

Quark number sum rule:

$$\tilde{D}_{q_i f_2}(1, n_2) - \tilde{D}_{\bar{q}_i f_2}(1, n_2) \;=\; A_{i f_2} \tilde{D}_{f_2}(n_2)$$

$$A_{i f_2} = N_i - \delta_{f_2 q_i} + \delta_{f_2 \bar{q}_i}$$

Ansatz for dPDF with different flavors:

$$\boxed{D_{f_1 f_2}(x_1, x_2) = N_2\, x_1^{-\tilde{\alpha}^{f_1}}\, x_2^{-\tilde{\alpha}^{f_2}} (1 - x_1 - x_2)^{\tilde{\beta}^{f_1 f_2}}}$$

Ansatz for sPDF :

$$\boxed{D_f(x) = N_1\, x^{-\alpha^f}\, (1-x)^{\beta^f}}$$

- Can perform the same analysis as before.
- Conditions for powers for dPDFs and sPDFs are exactly the same from both momentum and quark sum rules.
- Can satisfy simultaneously both sum rules:

Small x powers are identical:

$$\tilde{\alpha}^{f_2} = \alpha^{f_2}$$

$$\tilde{\alpha}^{f_1} = \alpha^{f_1}$$

Large x powers:

$$\tilde{\beta}^{f_1 f_2} = \beta^{f_2} + \alpha^{f_1} - 1$$

Symmetry with respect to the parton exchange

$$\tilde{\beta}^{f_1 f_2} = \tilde{\beta}^{f_2 f_1}$$

Implies the correlation of powers in sPDFs:

$$\beta^{f_2} + \alpha^{f_1} = \beta^{f_1} + \alpha^{f_2}$$

Initial conditions: expansion

Realistic parametrizations are however more complicated than a single Beta distribution.

Example MSTW2008 gluon PDF: $$xD_1^g\left(x,Q^2\right) = N_1 x^{-\delta_g}\left(1-x\right)^{\eta_g}\left(1+\epsilon_g\sqrt{x}+\gamma_g x\right),$$

However, this parametrization is sum of Beta distributions of the form:

$$\boxed{D(x) = N_1 \sum_{k=1}^{K} a_k x^{-\alpha_k}\left(1-x\right)^{\beta_k}}$$

Assuming that the dPDF is the sum of Dirichlet distributions:

$$\boxed{D(x_1,x_2) = N_2 \sum_{k=1}^{K} c_k x_1^{-\tilde{\alpha}_k} x_2^{-\tilde{\alpha}_k}\left(1-x_1-x_2\right)^{\tilde{\beta}_k}}$$

Performing the same analysis as before (for single channel) one obtains the conditions for each k:

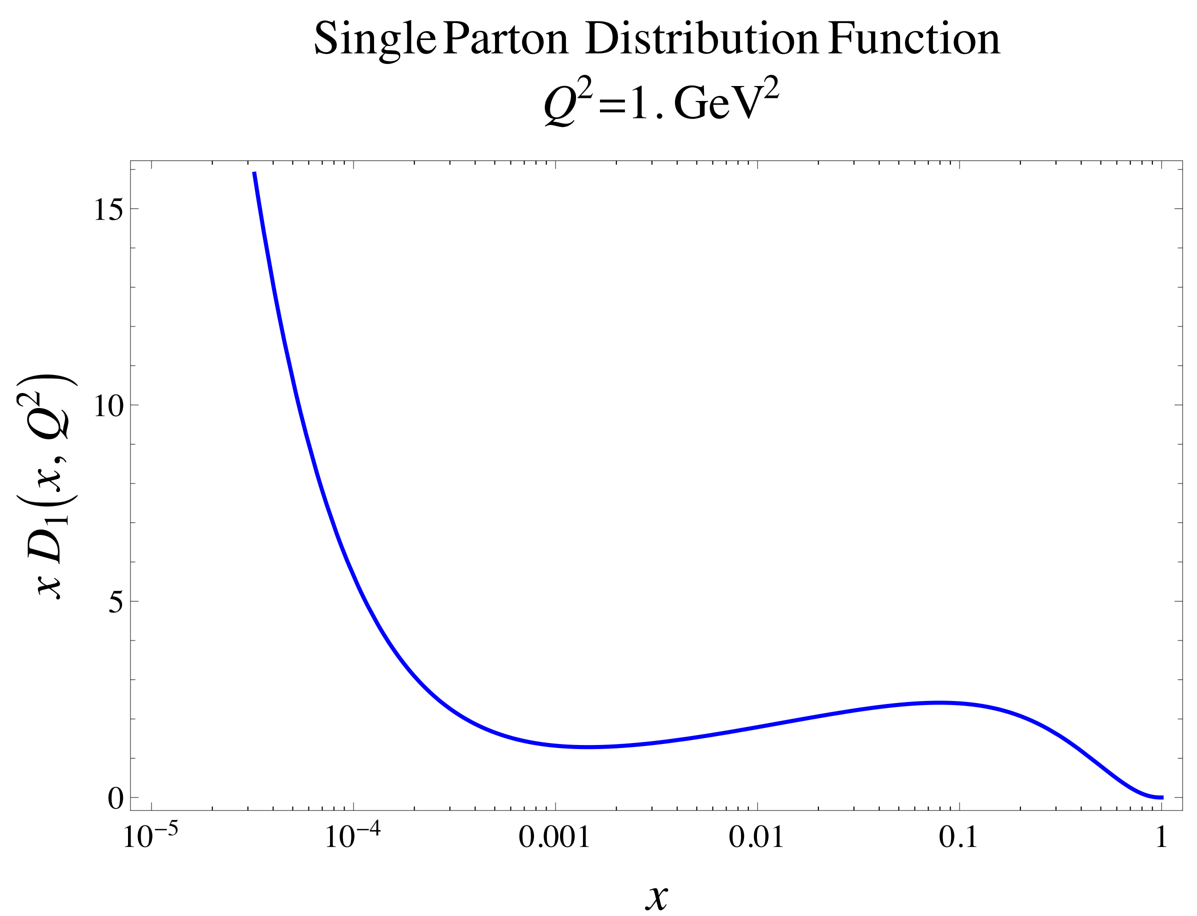
$$\tilde{\alpha}_k = \alpha_k \qquad\qquad \tilde{\beta}_k = \beta_k - 1 + \alpha_k$$

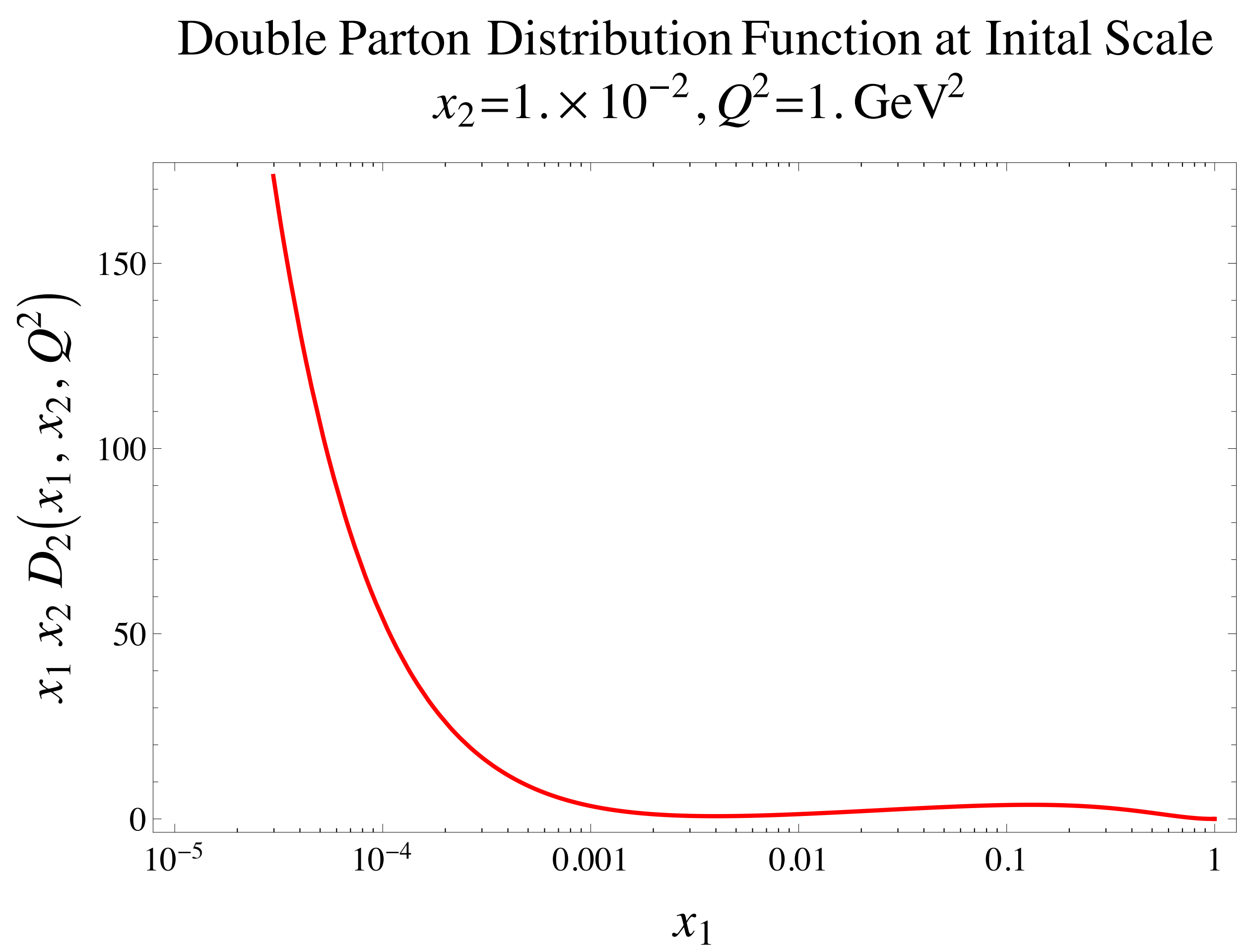
The normalizations:

$$c_k = a_k \frac{B(\alpha_1+\beta_1,2-\alpha_1)}{B(\beta_k+\alpha_k,2-\alpha_k)} \qquad\qquad N_2 = N_1 \frac{1}{B(\alpha_1+\beta_1,2-\alpha_1)}$$

Initial conditions for dPDFs using MSTW2008 gluon

- Use this algorithm, expansion in terms of Beta and Dirichlet distributions, to construct dPDF from MSTW2008 gluon.
- Single channel (gluons) only.
- Using different normalization for the LO MSTW2008 gluon.



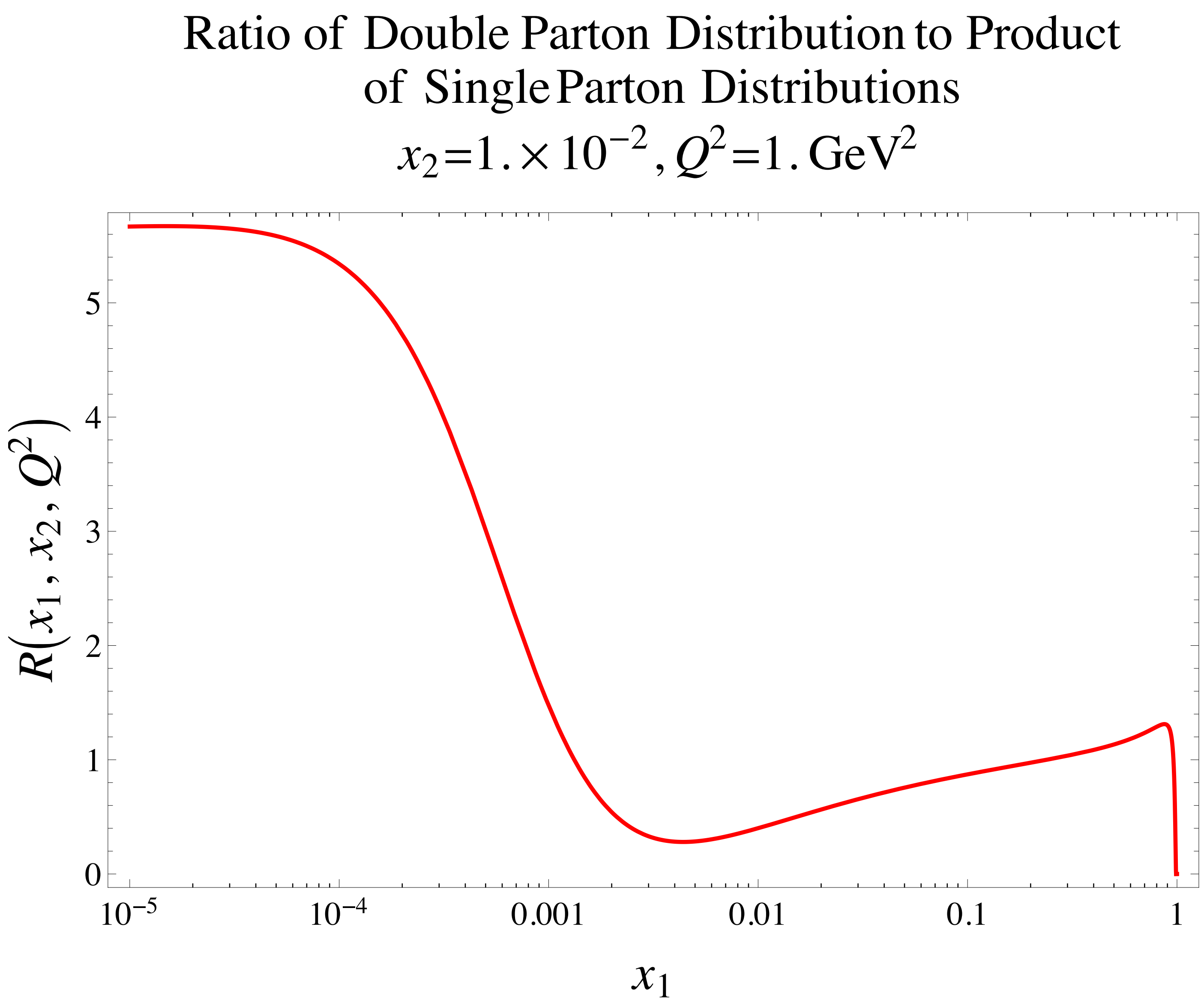


Initial conditions for dPDFs: ratios

Ratio of double distribution to product of single distributions:

$$R^{gg}\left(x_1,x_2,Q^2\right)=\frac{D_2^{gg}\left(x_1,x_2,Q^2\right)}{D_1^{g}\left(x_1,Q^2\right)D_1^{g}\left(x_2,Q^2\right)}$$

- Measure of the correlations at the initial scale.
- For this parametrization the correlations are very significant.
- Ratio different from unity over wide range of x.
- Factorization of powers at small x but different normalization.



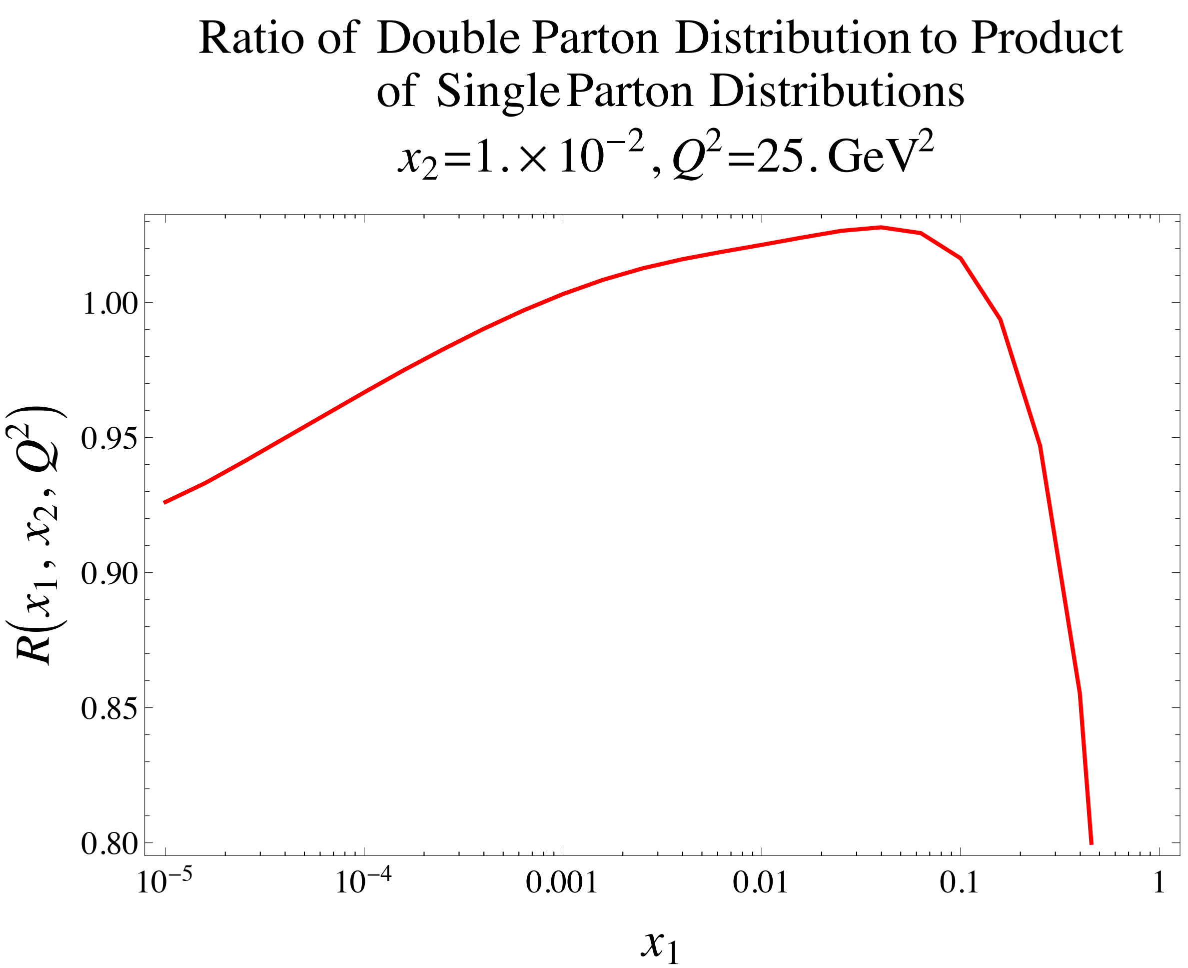
Evolution of single and double PDFs

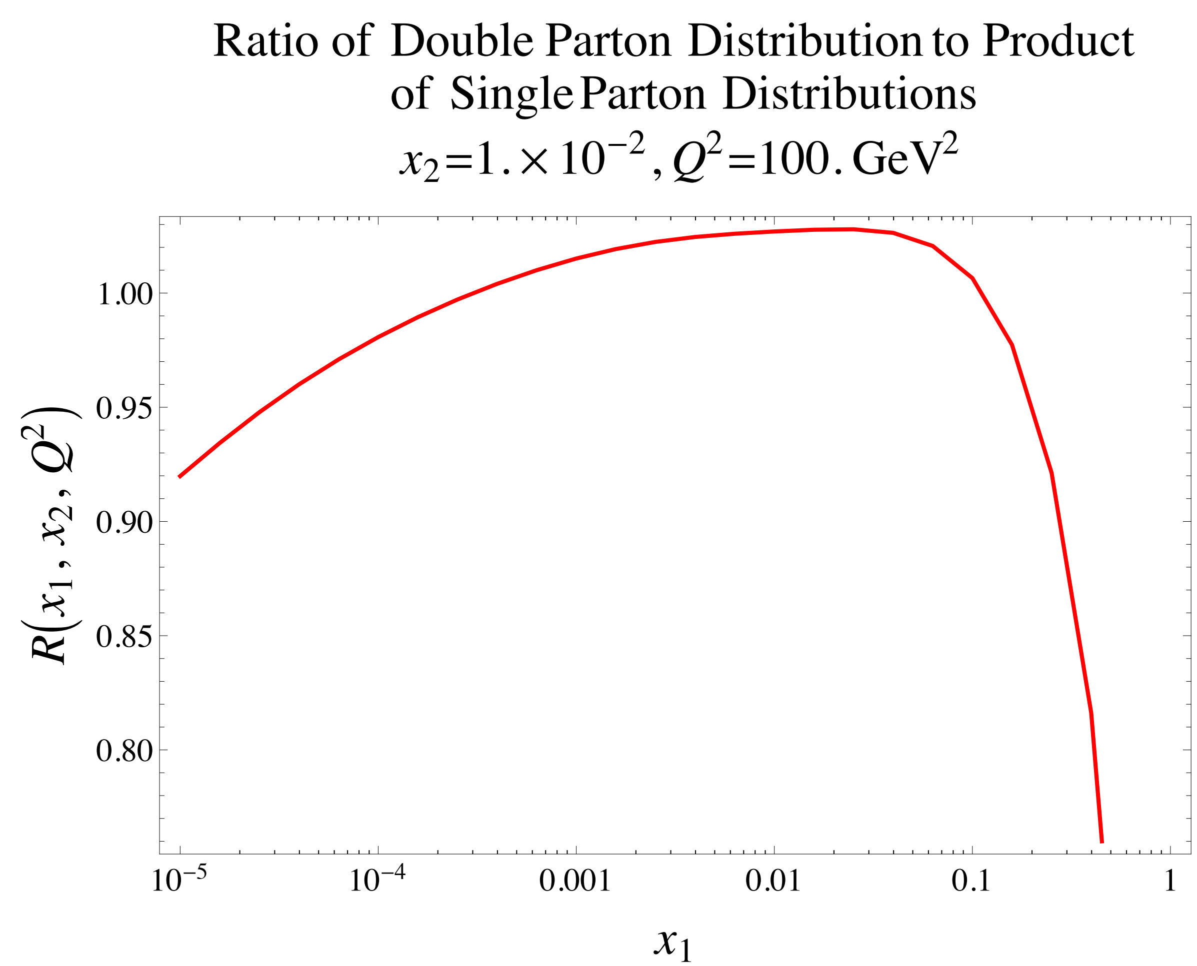
Evolve the dPDFs and sPDFs using DGLAP equations;

$$D_1^f(x, Q_0) \to D_1^f(x, Q)$$

$$D_2^{f_1 f_2}(x_1, x_2, Q_0) \to D_2^{f_1 f_2}(x_1, x_2, Q)$$

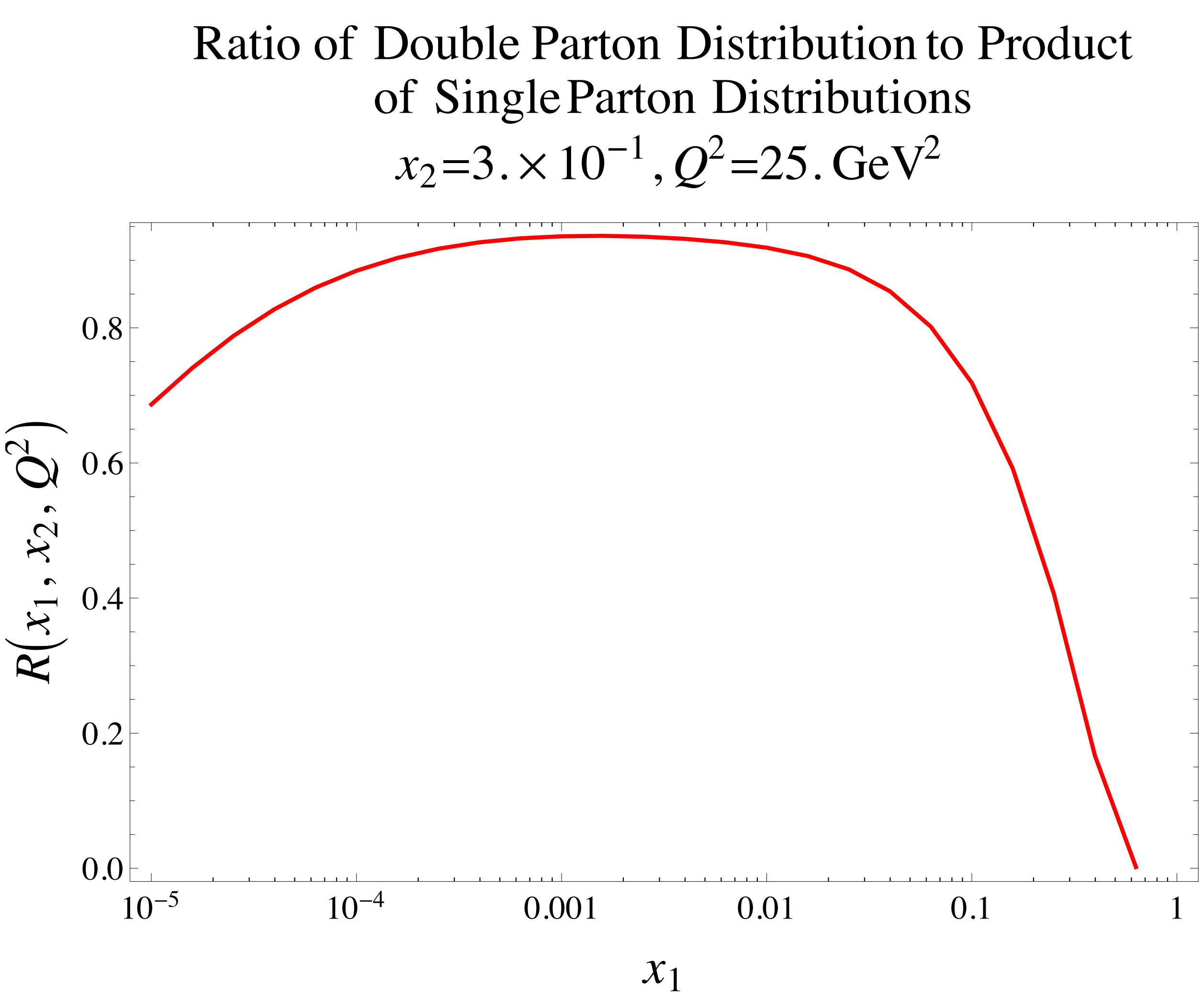
Solution found in the Mellin space and then numerically inverted to the momentum space.

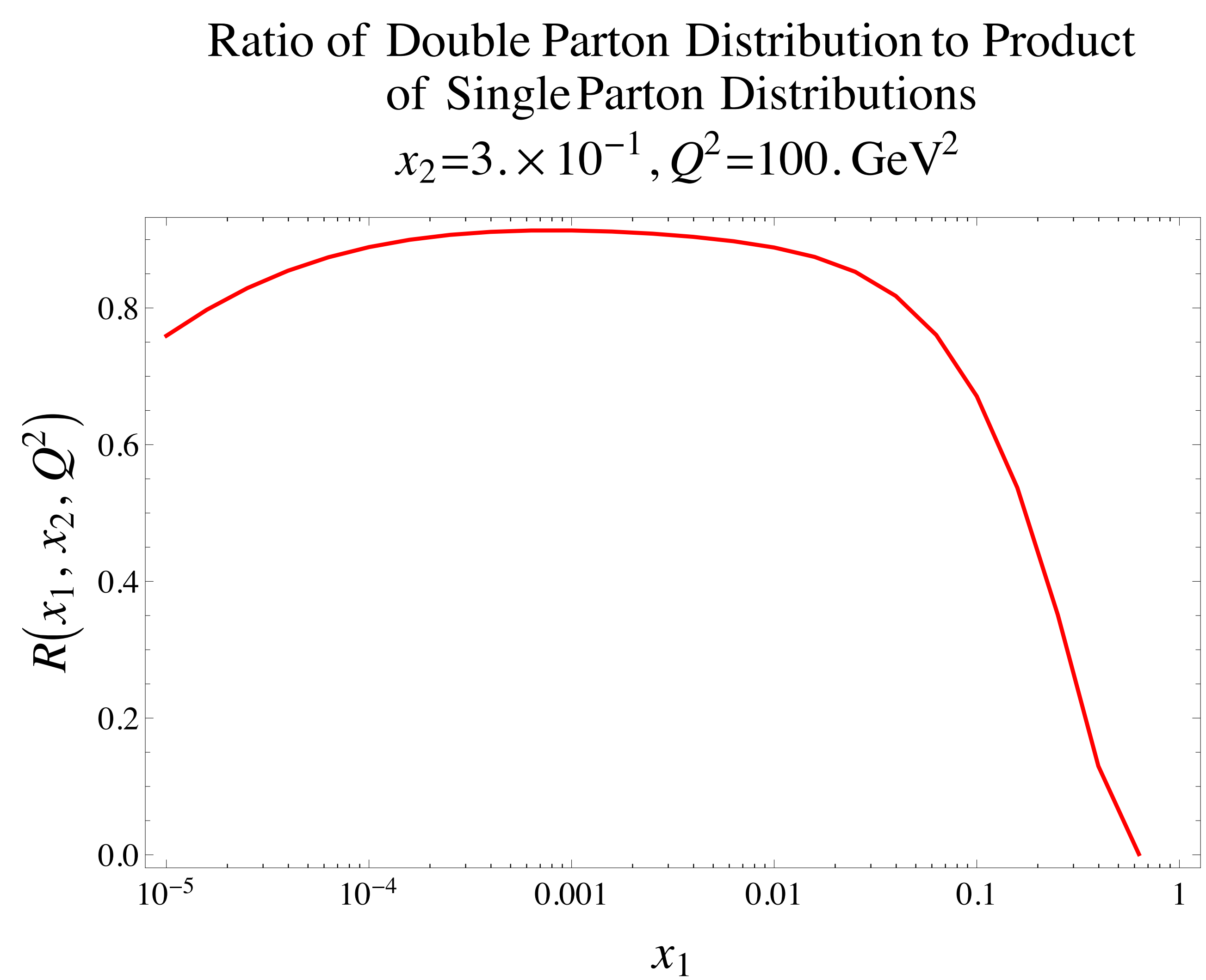




Correlation washed out by evolution except for large x.

Evolution of single and double PDFs





Correlations present for larger values of x

Very little change between two scales.

Summary and outlook

- Summary:
  - Double PDFs need consistent initial conditions for the evolution.
  - Beta functions for single PDF and Dirichlet distributions for double PDF with suitably matched powers and coefficients are good initial conditions.
  - The momentum sum rule and quark number sum rule are satisfied simultaneously.
  - Extending the formalism: expansion in terms of Dirichlet distributions. First numerical tests with gluons.
  - Sum rules provide relations between the powers at small and large x for single and double parton distributions.
  - Imply relations between powers of single parton distributions and therefore constraints on the form of single PDFs.
- Outlook:
  - Check the consistency of the Beta expansion for the system with quarks.
  - Perform expansion of the existing parametrizations in terms of the functions needed by the algorithm.
  - Is there any deeper physical meaning of the presented algorithm?